Rao Transforms: Application to the Restoration of Shift-Variant Blurred Images

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Introduction

We describe concrete one-dimensional (1D) and two-dimensional (2D) examples of the practical application of *Rao Transforms* (RTs) [1]. The one-dimensional example is relevant to the restoration or deblurring of shift-variant motion blurred images. When a photograph is captured by a moving camera with a finite exposure period, say 0.1 second, objects nearer to the camera will have larger motion blur than farther objects. This situation can arise when the camera is on a moving platform such as a car or a robot. A similar situation, i.e. a shift-variant defocus blur, can arise in a laser barcode scanner for one-dimensional barcodes. When the plane of the barcode is slanted instead of being perpendicular to the direction of view, the barcode image will be blurred by a shift-variant point spread function (SV-PSF). The two-dimensional example is related to the restoration of images (or signals) degraded by a SV-PSF, such as blurred images of a slanted plane or a curved surface produced by a defocused camera system.

One–Dimensional case

Let the original focused image be $f(x)$, and the corresponding blurred image be $g(x)$. The conventional image blurring model in this case uses a shift-variant point spread function or kernel $k(x, \alpha)$. The blurred image $g(x)$ and the kernel $k(x, \alpha)$ are assumed to be given and the problem is to solve for the focused image $f(x)$. The conventional blurring model is an integral equation of the form:

$$g(x) = \int_{-\infty}^{\infty} k(x,\alpha)\, f(\alpha)\, d\alpha \qquad (1)$$

It is a one-dimensional Fredholm Integral Equation of the First Kind. The above model of blurring has two problems. First, it is difficult to find a closed-form inversion formula that is explicit and numerically stable. For example, the well-known Singular Value Decomposition (SVD) technique is computationally expensive and unstable. Second, the above model does not capture the physical blurring process in a natural way. It seems to impose mathematical simplicity at the cost of direct natural modeling of the physical blurring process.

We model the blurred image $g(x)$ measured at x as the sum or integral over all possible point sources of the contribution due to each point source located at x-α. The contribution is given by the product of the strength of the signal point source $f(x$-$\alpha)$ and the value of a new *localized* shift-variant point spread function $h(x$-$\alpha,\alpha)$. The new model of blurring is:

$$g(x) = \int_{-\infty}^{\infty} h(x-\alpha,\alpha)\, f(x-\alpha)\, d\alpha \qquad \text{(RT)} \qquad (2)$$

where

$$h(x,\alpha) = k(x+\alpha, x) \text{ and} \qquad \text{(RLT)} \qquad (3)$$

$$k(x,\alpha) = h(\alpha, x-\alpha) \qquad \text{(IRLT)} \qquad (4)$$

The new model above is an integral equation that is exactly equivalent to the original integral equation (1) (see [1] for proof). Equation (2) is referred to as the *Rao Integral Equation* (RIE) and defines the *Rao Transform* (RT). Equation (3) defines the *Rao Localization Transform* (RLT).

Now the *m*-th order partial derivative of $h(x,\alpha)$ at (x,α) with respect to x is denoted by

$$h^{(m)} = h^{(m)}(x,\alpha) = \frac{\partial^m h(x,\alpha)}{\partial x^m}. \qquad (5)$$

The *n*-th derivative of $f(x)$ at x with respect to x will be denoted by

$$f^{(n)} = f^{(n)}(x) = \frac{d^n f(x)}{dx^n}. \qquad (6)$$

The *n*-th *moment* of the *m*-th derivative of h is defined by

$$h_n^{(m)} = h_n^{(m)}(x) = \int_{-\infty}^{\infty} \alpha^n \frac{\partial^m h(x,\alpha)}{\partial x^m} d\alpha \qquad (7)$$

Note that the derivative is with respect to x and the moment is with respect to α. The original signal $f(x)$ will be taken to be smooth or analytic so that it can be expanded in a Taylor series. The Taylor series expansion of $f(x-\alpha)$ around the point x up to order N is

$$f(x-\alpha) = \sum_{n=0}^{N} a_n \, \alpha^n \, f^{(n)}(x) \qquad (8)$$

where

$$a_n = \frac{(-1)^n}{n!} \quad . \qquad (9)$$

The above equation is exact and free of any approximation error when f itself is a polynomial of degree less than or equal to N. In this case, the derivatives of f of order greater than N are all zero. When f has non-zero derivatives of order greater than N, then the above equation will have an approximation error corresponding to the residual term of the Taylor series expansion. This approximation error usually converges rapidly to zero as N increases. In the limit as N tends to infinity, the above series expansion becomes exact and complete.

Similarly, the Taylor series expansion of $h(x-\alpha, \alpha)$ around the point (x,α) up to order M is

$$h(x-\alpha,\alpha) = \sum_{m=0}^{M} a_m \, \alpha^m \, h^{(m)}(x,\alpha) \qquad (10)$$

where a_m are as in Eq. (9). Using a truncated Taylor series expansion as above gives very accurate approximations in many practical applications such as image deblurring as the kernel function usually changes smoothly and slowly with respect to x.

Example(1):
We conclude our one-dimensional discussion with a specific example where we let N = 2, M = 1, and let the original kernel *k* be a Gaussian, that is

$$k(x,\alpha)=\frac{1}{\sqrt{2\pi}\sigma(\alpha)}\exp\left(-\frac{(x-\alpha)^2}{2\sigma^2(\alpha)}\right) \tag{11}$$

where $\exp(x)=e^x$. The kernel above is a "global" kernel. It is localized using the *Rao Localization Transform* (RLT) to define a new "local" kernel *h* as in Eq. (3), i.e. $h(x,\alpha)=k(x+\alpha,x)$, which becomes

$$h(x,\alpha)=\frac{1}{\sqrt{2\pi}\sigma(x)}\exp\left(-\frac{\alpha^2}{2\sigma^2(x)}\right) \tag{12}$$

For notational convenience, we denote

$$\rho(x)=\frac{1}{\sigma(x)}. \tag{13}$$

Therefore

$$h(x,\alpha)=\frac{\rho(x)}{\sqrt{2\pi}}\exp\left(-\frac{\alpha^2\rho^2(x)}{2}\right) \tag{14}$$

The Taylor series expansion of $h(x-\alpha,\alpha)$ around the point (x,α) up to order $M=1$ is

$$h(x-\alpha,\alpha)=h^{(0)}(x,\alpha)+h^{(1)}(x,\alpha)\,(-\alpha)\;. \tag{15}$$

It can be shown that, when *h* is as defined in Eq. (14),

$$h^{(1)}(x,a)=\frac{\partial h(x,\alpha)}{\partial x}=h(x,\alpha)\,\frac{\rho_x(x)}{\rho(x)}\left(1-\alpha^2\rho^2(x)\right) \tag{16}$$

where $\rho_x(x)$ is the derivative of $\rho(x)$ with respect to *x*.

Note that the above function is an *even* function of α as it involves only α^2. This function is symmetric with respect to α, i.e. $h^{(1)}(x,\alpha)=h^{(1)}(x,-\alpha)$. Therefore, all odd moments of $h^{(1)}$ with respect to α will be zero, and with $M=1$ and $N=2$, the RT becomes

$$g(x)=\int_{-\infty}^{\infty}\left(f^{(0)}-\alpha f^{(1)}+\frac{\alpha^2}{2}f^{(2)}\right)\left(h^{(0)}-\alpha\, h^{(1)}\right)d\alpha. \tag{17}$$

Simplifying, we get

$$g^{(0)}=f^{(0)}(h_0^{(0)}-h_1^{(1)})+f^{(1)}(h_2^{(1)}-h_1^{(0)})+f^{(2)}\frac{1}{2}(h_2^{(0)}-h_3^{(1)}) \tag{18}$$

Since all odd moments of $h^{(0)}$ and $h^{(1)}$ are zero, we set $h_1^{(0)}=h_1^{(1)}=h_3^{(1)}=0$. This simplifies the problem. Further, we have for this case, $h_0^{(0)}=1$ and both first $h_0^{(1)}$ and second $h_0^{(2)}$ (and all higher) derivatives of $h_0^{(0)}$ are always zero. Also the first and higher derivatives with respect to *x* of $h_1^{(0)}$, $h_1^{(1)}, h_2^{(1)}$, and $h_3^{(1)}$ are all zero. Only the first derivative of $h_2^{(0)}$ may not be zero. It will be denoted by $h_2^{(1)}$. Significant simplification of a mathematical problem such as here is likely in many practical applications. With the above simplifications, we get

$$g^{(0)} = f^{(0)} + f^{(1)} h_2^{(1)} + f^{(2)} \frac{1}{2} h_2^{(0)} \qquad (19)$$

Taking derivatives of the above equation once and twice, we get

$$g^{(1)} = f^{(1)} + f^{(2)} h_2^{(1)} + f^{(2)} \frac{1}{2} h_2^{(1)} \qquad (20)$$

and

$$g^{(2)} = f^{(2)}. \qquad (21)$$

We treat the above equations (19 to 21) as three algebraic equations in the three unknowns $f^{(0)}$, $f^{(1)}$, and $f^{(2)}$. They can be easily solved through successive elimination and back substitution. In this particular example, which corresponds to a typical practical application, the process of solving becomes trivial. We solve for $f^{(0)} = f(x)$ to obtain

$$f(x) = f^{(0)} = g^{(0)} - g^{(1)} h_2^{(1)} - g^{(2)} \frac{1}{2} (h_2^{(0)} - 3h_2^{(1)} \cdot h_2^{(1)}) \qquad (22)$$

The solution above can be further simplified by noting that

$$h_2^{(0)} = \sigma^2(x) \text{ and } h_2^{(1)} = \frac{\rho_x(x)}{\rho(x)} \left(\sigma^2(x) - \rho^2(x)\, 3\, \sigma^4(x)\right) = -2 \frac{\rho_x(x)}{\rho^3(x)} \qquad (23)$$

Thus we have obtained (in Eq. 22) the *Inverse Rao Transform* (IRT) for a case that is useful in practical applications. It is a closed-form solution up to second order terms. Solution up to any order N can be obtained similarly. A solution for *f(x)* is given in terms of the derivatives of *g(x)* at *x*, and moments of derivatives of the localized kernel *h*. In all our search of relevant research literature, we have never seen such a closed-from solution for the Fredholm Integral Equation of the First Kind. This is a "local" solution and converges rapidly for this particular example. A few other more complicated examples are presented in the book [1], but the simple example here illustrates the potential power of RTs.

In matrix notation, the forward and inverse RT for this case can be written as

$$\begin{bmatrix} g^{(0)} \\ g^{(1)} \\ g^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & h_2^{(1)} & (1/2)h_2^{(0)} \\ 0 & 1 & (3/2)h_2^{(1)} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f^{(0)} \\ f^{(1)} \\ f^{(2)} \end{bmatrix} \qquad (44)$$

$$\begin{bmatrix} f^{(0)} \\ f^{(1)} \\ f^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & -h_2^{(1)} & -(1/2)(h_2^{(0)} - 3h_2^{(1)} \cdot h_2^{(1)}) \\ 0 & 1 & -(3/2)h_2^{(1)} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g^{(0)} \\ g^{(1)} \\ g^{(2)} \end{bmatrix} \qquad (45)$$

If the input function is a polynomial, and the value of N is chosen to be the same as the degree of the polynomial, then from the theory one sees that the reconstruction should be perfect. This has been verified in simulation experiments.

Two-Dimensional case

A shift-variant defocused image will be denoted by *g(x,y)*. The shift-variant point spread function (SV-PSF) will be denoted by *h(x,y,α ,β)* where x and y are shift-variance variables and α and β are spread function variables. The original uncorrupted focused input image will be denoted by *f(x,y)* . The *Rao Transform* in this case is

$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\alpha,y-\beta,\alpha,\beta)\, f(x-\alpha,y-\beta)\, d\alpha\, d\beta. \tag{48}$$

The following notation will be used to represent partial derivatives of $g(x,y)$, $f(x,y)$, and the moments of $h(x,y)$:

$$g^{(m,n)}=\frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial y^n}g(x,y) \tag{49}$$

$$f^{(m,n)}=\frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial y^n}f(x,y) \tag{50}$$

$$h^{(m,n)}=h^{(m,n)}(x,y,\alpha,\beta)=\frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial y^n}h(x,y,\alpha,\beta) \tag{51}$$

$$h_{i,k}^{(m,n)}=h_{i,k}^{(m,n)}(x,y,\alpha,\beta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha^i\beta^k\,\frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial y^n}h(x,y,\alpha,\beta)\,d\alpha\,d\beta \tag{52}$$

$$\text{for}\quad m,n=0,1,2,\cdots.$$

Using the above notation, the Taylor series expansion of $f(x-\alpha,y-\beta)$ around (x,y) up to order N and $h(x-\alpha,y-\beta,\alpha,\beta)$ around the point (x,y,α,β) up to order M are given by

$$f(x-\alpha,y-\beta)=\sum_{n=0}^{N}a_n\sum_{i=0}^{n}C_i^n\alpha^{n-i}\beta^i f^{(n-i,i)} \tag{53}$$

$$h(x-\alpha,y-\beta,\alpha,\beta)=\sum_{m=0}^{M}a_m\sum_{j=0}^{m}C_j^m\alpha^{m-j}\beta^j h^{(m-j,j)} \tag{54}$$

where C_i^n and C_j^m denotes the binomial coefficients defined by

$$C_p^k=\frac{k!}{p!(k-p)!}$$

and a_m and a_n are constants as defined in Eq. (9). Substituting the above expressions into the Rao Transform of Eq. (48) and simplifying, we get

$$g(x,y)=\sum_{n=0}^{N}a_n\sum_{i=0}^{n}C_i^n f^{(n-i,i)}\sum_{m=0}^{M}a_m\sum_{j=0}^{m}C_j^m h_{m+n-i-j,i+j}^{(m-j,j)} \tag{55}$$

The above equation can be rewritten as

$$g(x,y)=\sum_{n=0}^{N}\sum_{i=0}^{n}S_{n,i}\ f^{(n-i,i)} \tag{56}$$

where

$$S_{n,i}=a_n C_i^n \sum_{m=0}^{M} a_m \sum_{j=0}^{m} C_j^m h_{m+n-i-j,i+j}^{(m-j,j)} \tag{57}$$

We can now write expressions for the various partial derivatives of g as

$$g^{(p,q)}=\sum_{n=0}^{N}\sum_{i=0}^{n}\frac{\partial^p}{\partial x^p}\frac{\partial^q}{\partial y^q}\left[S_{n,i}f^{(n-i,i)}\right]. \tag{58}$$

for $p+q=0,1,2,\cdots,N$. Note that

$$S_{n,i}^{(p,q)}=\frac{\partial^p}{\partial x^p}\frac{\partial^q}{\partial y^q}S_{n,i}=a_n C_i^n \sum_{m=0}^{M-(p+q)} a_m \sum_{j=0}^{m} C_j^m h_{m+n-i-j,i+j}^{(m-j+p,j+q)} \tag{59}$$

and

$$\frac{\partial^p}{\partial x^p}\frac{\partial^q}{\partial y^q}f^{(n-i,i)}=f^{(n-i+p,i+q)}. \tag{60}$$

The above equation for $g^{(p,q)}$ for $p,q=0,1,2,\cdots,N$ and $0\le p+q\le N$ constitute $(N+1)(N+2)/2$ equations in as many unknowns $f^{(p,q)}$. The system of equations for $g^{(p,q)}$ can be expressed in matrix form with a suitable RT coefficient matrix of size $(N+1)(N+2)/2$ rows and columns. These equations can be solved either numerically or algebraically to obtain $f^{(p,q)}$, and in particular, $f^{(0,0)}$. The solution for $f^{(0,0)}$ can be expressed as

$$f(x,y)=f^{(0,0)}=\sum_{n=0}^{N}\sum_{i=0}^{n}S'_{n,i}\ g^{(n-i,i)} \tag{61}$$

where $S'_{n,i}$ are the inverse RT coefficients for the 2-dimensional case.

Example(2):
We present a solution for the case of $N=2$ and $M=1$ for the case of a 2-D Gaussian SV-PSF given by

$$h(x,y,\alpha,\beta)=\frac{1}{2\pi\sigma^2(x,y)}\exp\left(-\frac{\alpha^2+\beta^2}{2\sigma^2(x,y)}\right) \tag{62}$$

We will define a new parameter $\rho(x,y)$ as

$$\rho(x,y)=\frac{1}{\sigma^2(x,y)}. \tag{63}$$

Therefore the SV-PSF can be written as

$$h(x,y,\alpha,\beta)=\frac{\rho(x,y)}{2\pi}\ \exp\left(-\ \frac{\rho(x,y)(\alpha^2+\beta^2)}{2}\right) \tag{64}$$

For this case, as in the 1-D case, many moment parameters and their derivatives become zero. Specifically,

$$h^{(1,0)}=\frac{\partial}{\partial x}h(x,y,\alpha,\beta)=\frac{\rho^{(1,0)}}{\rho}\,h(x,y,\alpha,\beta)\left(1\ -\ \frac{\rho\,(\alpha^2+\beta^2)}{2}\right). \tag{65}$$

Similarly

$$h^{(0,1)}=\frac{\partial}{\partial y}h(x,y,\alpha,\beta)=\frac{\rho^{(0,1)}}{\rho}h(x,y,\alpha,\beta)\left(1-\frac{\rho(\alpha^2+\beta^2)}{2}\right). \tag{66}$$

We see that $h^{(1,0)}$ and $h^{(0,1)}$ are both rotationally symmetric with respect to α and β. Therefore all odd moments are zero, i.e.

$$h_{i,j}^{(1,0)}=h_{i,j}^{(0,1)}=0 \text{ if } \ i \ \text{ is odd or } \ j \ \text{ is odd.} \tag{67}$$

Also,

$$h_{0,0}^{(0,0)}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(x,y,\alpha,\beta)\,d\alpha\,d\beta=1 \tag{68}$$

for all (x,y) and therefore, all derivatives of $h_{0,0}^{(0,0)}$ with respect to x and y are zero. Also, since $M=1$, all derivatives of h of order more than 1 with respect to x and y are zero. In summary,

$h_{0,0}^{(0,0)}=1$, $h_{1,0}^{(0,0)}=h_{0,1}^{(0,0)}=h_{1,1}^{(0,0)}=0$,

$h_{1,0}^{(1,0)}=h_{1,1}^{(1,0)}=h_{2,1}^{(1,0)}=h_{1,2}^{(1,0)}=h_{3,0}^{(1,0)}=0$,

$h_{0,1}^{(0,1)}=h_{1,1}^{(0,1)}=h_{2,1}^{(0,1)}=h_{1,2}^{(0,1)}=h_{0,3}^{(0,1)}=0$.

Therefore we get RT to be

$$g^{(0,0)}=f^{(0,0)}+f^{(1,0)}\ h_{2,0}^{(1,0)}+f^{(0,1)}\ h_{0,2}^{(0,1)}+\frac{1}{2}f^{(2,0)}\ h_{2,0}^{(0,0)}+\frac{1}{2}f^{(0,2)}\ h_{0,2}^{(0,0)} \tag{69}$$

The above equation gives a method of computing the output signal *g(x,y)* given the input signal *f(x,y)*. It can be written in a form similar to Eq. (56) to obtain the RT coefficients $S_{n,i}$.

We can derive the inverse RT for this case using Equation (69). As in the 1-dimensional case, we consider the various derivatives of g in Eq. (69) and solve for the derivatives of f as unknowns. In this particular example, we first solve for $f^{(0,0)}$ in terms of other terms using Eq. (69). Then, we take the derivative of the expression for $f^{(0,0)}$ with respect to x

and solve for $f^{(0,1)}$. Next we take the derivative of $f^{(0,0)}$ with respect to y and solve for $f^{(0,1)}$. Then we take the derivative with respect to x of $f^{(1,0)}$ and $f^{(0,1)}$ and solve for $f^{(2,0)}$ and $f^{(1,1)}$ respectively. Similarly we take derivatives with respect to y of $f^{(0,1)}$ and $f^{(1,0)}$ and solve for $f^{(0,2)}$ and $f^{(1,1)}$ respectively. Finally, we back substitute these results and eliminate $f^{(1,0)}$, $f^{(0,1)}$, $f^{(2,0)}$, $f^{(1,1)}$, and $f^{(0,2)}$ to get the following explicit solution for $f^{(0,0)}$ in terms of the derivatives of g and moments of the derivatives of h as below:

$$f^{(0,0)} = g^{(0,0)} - g^{(1,0)}\, h_{2,0}^{(1,0)} - g^{(0,1)}\, h_{0,2}^{(0,1)} + g^{(2,0)}\left(\frac{3}{2}(h_{2,0}^{(1,0)})^2 + \frac{1}{2}h_{0,2}^{(0,1)}\, h_{2,0}^{(0,1)} - \frac{1}{2}h_{2,0}^{(0,0)}\right)$$

$$+g^{(0,2)}\left(\frac{3}{2}(h_{0,2}^{(0,1)})^2 + \frac{1}{2}h_{2,0}^{(1,0)}\, h_{0,2}^{(1,0)} - \frac{1}{2}h_{0,2}^{(0,0)}\right) \qquad (70)$$

Further simplification of the above equation is possible due to rotational symmetry (e.g. $h_{2,0}^{(1,0)} = h_{0,2}^{(0,1)}$, and $h_{2,0}^{(0,0)} = h_{0,2}^{(0,0)}$). The above equation gives an explicit, closed-form, formula for restoring an image blurred by a shift-variant Gaussian point spread function. The above equation can be written in a form similar to Equation (61) for inverse RT to obtain the inverse RT coefficients. It is clear from the above discussion that the method for implementing the forward and inverse RT for the two-dimensional case is similar to the one-dimensional case explained earlier.

Experiments:

Several simulation experiments were done to verify the theory above. The experiments consisted of both 1D and 2D cases. First the unknown function f was chosen to be a polynomial of a certain order (e.g. $7x^5 + 6x^4 - 2x^3 - 6x^2 5x - 1$) or a sine function of a certain period (e.g. Sin(x)), then the kernel h was chosen to be one of Gaussian or rect (and a Cylindrical in the 2D case) with a Taylor series expansion up to order M=1 or 2. The order N of the polynomial was varied (3 to 8) and the period of the sine function was varied (sin(x) to sin(2x)) in different experiments. The spread parameter sigma of the SV-PSF in the different cases was varied linearly (e.g. s=0.5+0.1x). The analytic expressions for the blurred image and the restored image were plotted in an interval (e.g. x_min=-10 to x_max= 10 with 200 sampled points). As expected, when the unknown function was a polynomial, the solution for f(x) was exact. However, in the case of sine functions, due to truncation of the series expansion, as expected, the solution had small errors. This error increased when the ratio of the parameter sigma to the period of the sine wave increased. The error was small up to a ratio of 0.2.

Two examples of 2D input functions are as below

$f(x,y) = 0.2x^3 + 0.13x^2y - 0.1xy^2 + 0.5y^3, \;\; N = 3, \; M = 1.$

$f(x,y) = \sin(1.5x) + \cos(1.5y), \; N = 4, \; M = 1, \; x_{\min}, y_{\min} = -5,$ and $x_{\max}, y_{\max} = 5.$

Conclusion
Equation (70) gives a closed-form solution for the restoration of a shift-variant defocused image. It has been verified experimentally through simulations. This localized solution to shift-variant image restoration can be easily extended to higher order local polynomial approximations, and many different models of SV-PSF (Gaussian, cylindrical, rect, etc.). The resulting computational approach is 2 to 3 orders of magnitude faster than the classical SVD approach [1,2,3,4]. The method here has been implemented on actual images with simulated blur and verified. This method holds much promise in many applications.

Appendix
In this section we present closed-form explicit expressions for the moments of the derivatives of the SV-PSF h for different cases.

1D Gaussian
The PSF has the following form :

$$h(x,\alpha) = \frac{1}{\sqrt{2\pi}\sigma(x)} \exp\left(- \frac{\alpha^2}{2\sigma^2(x)} \right) \qquad \text{(A1)}$$

Therefore the n^{th} moment, $h_n(x)$ is expressed as the following integral,

$$h_n(x) = \frac{1}{\sqrt{2\pi}\sigma(x)} \int_{-\infty}^{\infty} \alpha^n \exp\left(- \frac{\alpha^2}{2\sigma^2(x)} \right) d\alpha \qquad \text{(A2)}$$

Since the limits of integration do not depend on the variable x, we can interchange the integration with respect to α and differentiation with respect to x. Therefore, to get the derivatives of the moments, that is $h_n^{(m)}(x)$ we compute the integral (A2) and then differentiate them. The following steps lead us to a general formula for $h_n(x)$.

$$h_n(x) = \frac{1}{\sqrt{2\pi}\sigma(x)} \int_{-\infty}^{\infty} \alpha^n \exp\left(- \frac{\alpha^2}{2\sigma^2(x)} \right) d\alpha$$

$$= \frac{(1+(-1)^n)}{\sqrt{2\pi}\sigma(x)} \int_{0}^{\infty} \alpha^n \exp\left(- \frac{\alpha^2}{2\sigma^2(x)} \right) d\alpha$$

$\alpha^n \exp\left(- \frac{\alpha^2}{2\sigma^2(x)} \right)$ is an odd(even) function if n is odd(even). We make the following substitution : $\frac{\alpha^2}{2\sigma^2} = t$, therefore $\alpha = \sigma\sqrt{2t}$ and $d\alpha = \frac{\sigma dt}{\sqrt{2t}}$. With this substitution the above equation becomes:

$$h_n(x) = \frac{(1+(-1)^n)(\sigma^{n+1}(\sqrt{2})^{n-1})}{\sqrt{2\pi}\sigma(x)} \int_{0}^{\infty} t^{\frac{n-1}{2}} \exp(-t) dt$$

$$= \frac{(1+(-1)^n)(\sigma^n(\sqrt{2})^{n-2})}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right) \qquad \text{(A3)}$$

In the experiments, we have chosen σ to vary linearly as $\sigma = 0.5 + 0.1x$, and N = 2.

Therefore $h_0(x) = 1$, $h_1(x) = 0$ and $h_2(x) = \sigma^2$, therefore,

$$h_2^{(0)}(x) = h_2(x) = (0.5 + 0.1x)^2$$
$$h_2^{(1)}(x) = 2\sigma\sigma' = 0.2(0.5 + 0.1x) = 0.1 + 0.02x$$
$$h_2^{(2)}(x) = 0.02 \;, \qquad h_2^{(3)}(x) = 0$$

1D Rectangular PSF

The PSF for the rectangular case, has the following form

$$h(x,\alpha) = \frac{1}{T(x)} \quad \text{for} \quad \frac{-T}{2} \le x \le \frac{T}{2} \qquad \text{(A4)}$$
$$= 0 \quad \text{Otherwise}$$

Therefore the n^{th} moment can be expressed as:

$$h_n(x) = \frac{1}{T(x)}\int_{-\frac{T}{2}}^{\frac{T}{2}} \alpha^n d\alpha \qquad \text{(A5)}$$

As seen from the formula (A5), the limits of integration now do depend on the variable x, So we cannot take derivatives of (A5), to get $h_n^{(m)}(x)$ ‘s. Below we derive a general formula for $h_n^{(m)}(x)$, under the assumption that T(x) is linearly varying, that is second and higher order derivatives of T(x) are all zero.

$$h_n^{(m)}(x) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \alpha^n \frac{d^{(m)}}{dx^m}\left(\frac{1}{T(x)}\right) d\alpha \qquad \text{(A6)}$$

Now, under the assumption that T(x) is linear, we have, for the derivative:

$$\frac{d^{(m)}}{dx^m}\left(\frac{1}{T(x)}\right) = \frac{(-1)^m (m!)(T')^m}{T^{m+1}} \qquad \text{(A7)}$$

Substituting (A7) to (A6), we get,

$$h_n^{(m)}(x) = \frac{(-1)^m (m!)(T')^m}{T^{m+1}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \alpha^n d\alpha$$

$$= \frac{(-1)^m (1+(-1)^n)(m!)(T')^m T^{n-m}}{(n+1)2^{n+1}} \qquad \text{(A8)}$$

2D Gaussian

The PSF has the following form :

$$h(x,y,\alpha,\beta)=\frac{1}{2\pi\sigma^2(x,y)}\exp\left(-\frac{\alpha^2+\beta^2}{2\sigma^2}\right) \qquad \text{(A9)}$$

Therefore the $(m,n)^{th}$ moment, $h_{m,n}(x)$ is expressed as the following integral,

$$h_{m,n}(x,y) = \frac{1}{2\pi\sigma^2(x,y)}\int_{-\infty}^{\infty}d\beta\int_{-\infty}^{\infty}\alpha^m\beta^n\exp\left(-\frac{\alpha^2+\beta^2}{2\sigma^2(x,y)}\right)d\alpha \qquad \text{(A10)}$$

We split the double integral as a product of two single integrals,

$$h_{m,n}(x,y) = \frac{1}{2\pi\sigma^2(x,y)}\int_{-\infty}^{\infty}\alpha^m\exp\left(-\frac{\alpha^2}{2\sigma^2(x,y)}\right)d\alpha\int_{-\infty}^{\infty}\beta^n\exp\left(-\frac{\beta^2}{2\sigma^2(x,y)}\right)d\beta$$

We integrate each, as we did for the one dimnensional case. The result is as follows:

$$h_{m,n}(x,y)=\frac{(\sigma\sqrt{2})^{m+n}\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\left[(1+(-1)^m)(1+(-1)^n)\right]}{4\pi} \qquad \text{(A11)}$$

We differentiate (A11) to get the derivatives.

2D Rectangular PSF

The PSF for the rectangular case has the following form

$$h(x,y\alpha,\beta)=\frac{1}{T^2(x,y)} \qquad \text{for } \frac{-T}{2}\le\alpha,\beta\le\frac{T}{2} \qquad \text{(A12)}$$

$$= 0 \qquad \text{Otherwise}$$

Therefore the $(m,n)^{th}$ moment can be expressed as:

$$h_{m,n}(x,y)=\frac{1}{T^2(x,y)}\int_{-\frac{T}{2}}^{\frac{T}{2}}d\beta\int_{-\frac{T}{2}}^{\frac{T}{2}}\alpha^m\beta^n d\alpha \qquad \text{(A13)}$$

As in the One-Dimensional case, the limits of integration depend on x,y. Therefore to get the derivatives of the moments we need to differentiate under the integral. The formula for the derivatives of the moments is:

$$h_{m,n}^{(i,j)}(x,y)=\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}}\alpha^m\beta^n\frac{\partial^{i+j}}{\partial^i x\partial^j y}\left(\frac{1}{T^2(x,y)}\right)d\alpha d\beta \qquad \text{(A14)}$$

Under the assumption that T(x,y) varies linearly with x and y,

$$\frac{\partial^{i+j}}{\partial^i x\partial^j y}\left(\frac{1}{T^2(x,y)}\right) = (-1)^{i+j}\frac{(i+j+1)!(T_x)^i(T_y)^j}{T^{i+j+2}} \qquad \text{(A15)}$$

Here T_x is the first partial derivative of T with respect to x, and T_y is the first partial derivative of T with respect to y.

Therefore, putting (A15) into (A14) and integrating, we get,

$$h_{m,n}^{(i,j)}(x,y) = (-1)^{i+j}\left[\frac{(i+j+1)!(T_x)^i(T_y)^j}{T^{i+j+2}}\right]\left[\frac{T^{m+n+2}}{(m+1)(n+1)2^{m+n+2}}\right]\left[(1+(-1)^m)(1+(-1)^n)\right]$$

(A16)

2D Cylindrical PSF:

The derivation is very similar to the Rectangular case. So we will skip most of the steps here. The PSF for the cylindrical case, has the following form

$$h(x,y\alpha,\beta) = \frac{1}{\pi R^2(x,y)} \qquad (\alpha,\beta)\in B(0,R) \qquad \text{(A17)}$$

$$= \quad 0 \qquad \text{Otherwise}$$

The moments and their derivatives can be obtained by evaluating the following integral,

$$h_{m,n}^{(i,j)}(x,y) = \iint_{B(0,R)} \alpha^m \beta^n \frac{\partial^{i+j}}{\partial^i x \partial^j y}\left(\frac{1}{\pi R^2(x,y)}\right)\mathrm{dA} \qquad \text{(A18)}$$

We make the assumption that R varies linearly with x and y. Therefore the partial derivatives of h are as follows:

$$\frac{\partial^{i+j}}{\partial^i x \partial^j y}\left(\frac{1}{\pi R^2(x,y)}\right) = (-1)^{i+j}\frac{(i+j+1)!(R_x)^i(R_y)^j}{\pi R^{i+j+2}} \qquad \text{(A19)}$$

To compute (A18), we switch to polar co-ordinates, that is we replace α by $r\cos\theta$ and β by $r\sin\theta$, and the area element dA by $r\,dr\,d\theta$. Therefore the integral (A18) becomes,

$$h_{m,n}^{(i,j)}(x,y) = (-1)^{i+j}\frac{(i+j+1)!(R_x)^i(R_y)^j}{\pi R^{i+j+2}}\int_0^{2\pi}\int_0^{R} r^{m+n+1}\cos^m\theta\sin^n\theta\, dr\, d\theta$$

$$= (-1)^{i+j}\frac{(i+j+1)!(R_x)^i(R_y)^j}{\pi(m+n+2)R^{i+j+2}} \quad R^{m+n+2}\,Trig(m,n) \qquad \text{(A20)}$$

where $Trig(m,n) = \int_0^{2\pi}\cos^m\theta\sin^n\theta\, d\theta$, can be computed separately.

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