

Optical phonon production

The rate of emission of optical phonons with a wave vector \mathbf{q} and energy $\hbar\omega_{\mathbf{q}}$ (assumed at temperature T_{ph}) by electrons or holes at wavevector \mathbf{k} is

$$W[\mathbf{k} \rightarrow (\mathbf{k}-\mathbf{q})] = \frac{2\pi}{\hbar} |M|^2 [N_{\mathbf{q}}(T_{\text{ph}}) + 1] \delta(E_{\mathbf{k}-\mathbf{q}} - E_{\mathbf{k}} + \hbar\omega_{\mathbf{q}}), \quad (1.1)$$

where $N_{\mathbf{q}}(T) = [e^{\hbar\omega_{\mathbf{q}}/kT} - 1]^{-1}$ is the Planck function for phonons, and

$$|M|^2 = \frac{2\pi e^2 \hbar\omega_{\mathbf{q}}}{q^2} \left[\frac{1}{\epsilon_{\infty}} - \frac{1}{\epsilon_0} \right] \quad (1.2)$$

is the matrix element of Fröhlich's hamiltonian. For $\text{In}_{0.53}\text{Ga}_{0.47}\text{As}$, the optical and static permittivities are $(\epsilon_{\infty}, \epsilon_s) = (11.36, 13.73)$ and LO phonon energy is $\hbar\omega_{\mathbf{q}} = 32.7$ meV.

Assume that electrons are at a temperature T_e and Fermi distributed, $f_{\mathbf{k}} = f(\epsilon_{\mathbf{k}} - \epsilon_F)$, where

$$f(\epsilon) = \frac{1}{1 + e^{\epsilon}} \quad , \quad \epsilon_F \equiv \frac{E_F}{kT_e} \quad , \quad \epsilon_{\mathbf{k}} \equiv \frac{E_{\mathbf{k}}}{kT_e} \quad . \quad (1.3)$$

The rate of phonon emission is given by

$$R_{\text{em}}(q) = 2(2\pi)^{-3} \int d^3\mathbf{k} f_{\mathbf{k}} (1 - f_{\mathbf{k}-\mathbf{q}}) W[\mathbf{k} \rightarrow (\mathbf{k}-\mathbf{q})] \quad . \quad (1.4)$$

Denoting $\hat{\mathbf{k}} \cdot \hat{\mathbf{q}} = \cos\theta \equiv u$ and $q_{\omega} \equiv \sqrt{2m\omega_{\mathbf{q}}/\hbar}$, we have

$$\delta(E_{\mathbf{k}-\mathbf{q}} - E_{\mathbf{k}} + \hbar\omega_{\mathbf{q}}) = \frac{1}{q} \frac{dk}{dE_{\mathbf{k}}} \delta\left[\frac{q^2 + q_{\omega}^2}{2kq} - u\right]; \quad (1.5)$$

For a parabolic and isotropic band ($E_{\mathbf{q}} = \hbar^2 q^2/2m$) and an arbitrary function $F(E_{\mathbf{k}-\mathbf{q}})$

$$\begin{aligned} \int d^3\mathbf{k} F(E_{\mathbf{k}-\mathbf{q}}) \delta(E_{\mathbf{k}-\mathbf{q}} - E_{\mathbf{k}} + \hbar\omega_{\mathbf{q}}) &= \frac{2\pi m}{\hbar^2 q} \int_0^{\infty} k dk \int_{-1}^1 du F(k^2 + q^2 - 2kqu) \delta\left[\frac{q^2 + q_{\omega}^2}{2kq} - u\right] \\ &= \frac{2\pi m}{\hbar^2 q} \int_{k_{[+]}}^{\infty} k dk F(k^2 - q_{\omega}^2) \quad , \quad \text{where } k_{[+]}(q) \equiv \frac{q^2 + q_{\omega}^2}{2q} \quad . \end{aligned}$$

Using this equation and the identities

$$f(\epsilon) [1 - f(\epsilon - \beta)] = \frac{f(\epsilon - \beta) - f(\epsilon)}{e^{\beta} - 1}; \quad \int_{\gamma}^{\infty} f(\epsilon - \beta) d\epsilon = \ln\left[1 + e^{\beta - \gamma}\right], \quad (1.6)$$

we obtain the total emission rate in the form

$$R_{\text{em}}(q) = \frac{kT_e}{\hbar} \left[\frac{q_{\omega}}{q}\right]^3 [N_{\mathbf{q}}(T_{\text{ph}}) + 1] \alpha_{\omega} I_1(q), \quad (1.7a)$$

$$I_1(q) \equiv \int_{\epsilon_{[+]}}^{\infty} d\epsilon f(\epsilon - \epsilon_F) [1 - f(\epsilon - \epsilon_F - \beta)] = \frac{1}{e^{\beta} - 1} \ln\left[\frac{e^{\beta} + e^{\epsilon_{[+]} - \epsilon_F}}{e^{\epsilon_{[+]} - \epsilon_F} + 1}\right], \quad (1.7b)$$

$$\text{where} \quad \beta \equiv \frac{\hbar\omega_{\mathbf{q}}}{kT_e}, \quad \alpha_{\omega} \equiv \frac{q_{\omega} e^2}{2\hbar\omega_{\mathbf{q}}} \left[\frac{1}{\epsilon_{\infty}} - \frac{1}{\epsilon_0}\right], \quad (1.8)$$

$$\text{and} \quad E_{[+]}(q) \equiv \frac{\hbar^2 k_{[+]}^2}{2m}, \quad \epsilon_{[+]}(q) = \frac{(E_{\mathbf{q}} + \hbar\omega_{\mathbf{q}})^2}{4E_{\mathbf{q}} kT_e}, \quad (1.9)$$

Note that $k_{[+]}(q_{\omega}) = q_{\omega}$ and $E_{q_{\omega}} = \hbar\omega_{\mathbf{q}}$, so that $\epsilon_{[+]}(q_{\omega}) = \hbar^2 q_{\omega}^2/2mkT_e = \beta$.

Optical phonon absorption

The rate of absorption of optical phonons with a wave vector \mathbf{q} and energy $\hbar\omega_{\mathbf{q}}$ (assumed at temperature T_{ph}) by electrons or holes at wavevector \mathbf{k} is

$$W[\mathbf{k} \rightarrow (\mathbf{k} + \mathbf{q})] = \frac{2\pi}{\hbar} |M|^2 N_{\mathbf{q}}(T_{\text{ph}}) \delta(E_{\mathbf{k}+\mathbf{q}} - E_{\mathbf{k}} - \hbar\omega_{\mathbf{q}}), \quad (2.1)$$

For electrons in the Fermi-Dirac ensemble the total rate of absorption is given by

$$R_{\text{ab}}(q) = 2(2\pi)^{-3} \int d^3\mathbf{k} f_{\mathbf{k}} (1 - f_{\mathbf{k}+\mathbf{q}}) W[\mathbf{k} \rightarrow (\mathbf{k} + \mathbf{q})]. \quad (2.2)$$

The δ -function identity replacing (1.5) is now

$$\delta(E_{\mathbf{k}+\mathbf{q}} - E_{\mathbf{k}} - \hbar\omega_{\mathbf{q}}) = \frac{1}{q} \frac{dk}{dE_{\mathbf{k}}} \delta\left[\frac{q^2 - q_{\omega}^2}{2kq} - u\right]. \quad (2.3)$$

Integration over u leads to the following lower limit for the integration over k :

$$k_{[-]}(q) = \frac{|q^2 - q_{\omega}^2|}{2q}, \quad E_{[-]}(q) = \frac{(E_{\mathbf{q}} - \hbar\omega_{\mathbf{q}})^2}{4E_{\mathbf{q}}} \quad (2.4)$$

and integral (2.2) reduces to

$$R_{\text{ab}}(q) = \frac{kT_e}{\hbar} \left[\frac{q_{\omega}}{q}\right]^3 \alpha_{\omega} N_{\mathbf{q}}(T_{\text{ph}}) I_2(q), \quad (2.5a)$$

$$I_2(q) \equiv \int_{\varepsilon_{[-]}}^{\infty} d\varepsilon f(\varepsilon - \varepsilon_{\text{F}}) [1 - f(\varepsilon - \varepsilon_{\text{F}} + \beta)] = \frac{e^{\beta}}{e^{\beta} - 1} \ln \left[\frac{e^{\varepsilon_{[-]} - \varepsilon_{\text{F}}} + 1}{e^{\varepsilon_{[-]} - \varepsilon_{\text{F}}} + e^{-\beta}} \right], \quad (2.5b)$$

where $\varepsilon_{[-]}(q) \equiv \frac{E_{[-]}(q)}{kT_e}$. Note that $\varepsilon_{[+]}(q) - \varepsilon_{[-]}(q) = \frac{\hbar\omega_{\mathbf{q}}}{kT_e}$ (2.6)

Using the identities $[e^{\beta} - 1]^{-1} \equiv N_{\mathbf{q}}(T_e)$ and $e^{\beta} [e^{\beta} - 1]^{-1} \equiv 1 + N_{\mathbf{q}}(T_e)$, we can rewrite Eqs. (1.7) and (2.5) in the form:

$$R_{\text{em}}(q) = \frac{kT_e}{\hbar} \left[\frac{q_{\omega}}{q}\right]^3 \alpha_{\omega} [N_{\mathbf{q}}(T_{\text{ph}}) + 1] N_{\mathbf{q}}(T_e) \ln \left[\frac{e^{\beta} + e^{\varepsilon_{[+]} - \varepsilon_{\text{F}}}}{e^{\varepsilon_{[+]} - \varepsilon_{\text{F}}} + 1} \right], \quad (2.7a)$$

$$R_{\text{ab}}(q) = \frac{kT_e}{\hbar} \left[\frac{q_{\omega}}{q}\right]^3 \alpha_{\omega} N_{\mathbf{q}}(T_{\text{ph}}) [N_{\mathbf{q}}(T_e) + 1] \ln \left[\frac{e^{\varepsilon_{[-]} - \varepsilon_{\text{F}}} + 1}{e^{\varepsilon_{[-]} - \varepsilon_{\text{F}}} + e^{-\beta}} \right]. \quad (2.7b)$$

A sanity check: for $T_e = T_{\text{ph}}$ the rates must balance, $R_{\text{ab}}(q) = R_{\text{em}}(q)$. This is indeed the case.
 ☛ Check that arguments of the logarithms are identical (note $\beta = \varepsilon_{[+]} - \varepsilon_{[-]}$). ☚

The net rate $R(q) \equiv R_{\text{em}} - R_{\text{ab}}$ is of the form:

$$R(q, n, T_e, T_{\text{ph}}) = [N_{\mathbf{q}}(T_e) - N_{\mathbf{q}}(T_{\text{ph}})] R_{\text{sp}}, \quad (2.8a)$$

$$R_{\text{sp}}(q, n, T_e) \equiv \frac{kT_e}{\hbar} \left[\frac{q_{\omega}}{q}\right]^3 \alpha_{\omega} \ln \left[1 + \frac{e^{\beta} - 1}{e^{\varepsilon_{[+]} - \varepsilon_{\text{F}}} + 1} \right]. \quad (2.8b)$$

where R_{sp} is the *spontaneous* emission rate.

Steady-state heating power

The balance equation in an optical phonon mode \mathbf{q} is of the form:

$$\frac{dN_{\mathbf{q}}}{dt} = R(q, n, T_e, T_{\text{ph}}) - \frac{N_{\mathbf{q}}(T_{\text{ph}}) - N_{\mathbf{q}}(T)}{\tau_{\text{op} \rightarrow \text{ac}}}, \quad (3.1)$$

where $\tau_{\text{op} \rightarrow \text{ac}} = \tau_{\text{op} \rightarrow \text{ac}}(T)$ is the rate of decay of optical phonons into acoustic phonons. Its dependence on the lattice temperature T results from the stimulated emission and absorption processes¹ of two acoustic phonons of energy $\hbar\omega_{\mathbf{q}}^{(1)}$ and $\hbar\omega_{\mathbf{q}}^{(2)}$, (where $\hbar\omega_{\mathbf{q}}^{(1)} + \hbar\omega_{\mathbf{q}}^{(2)} = \hbar\omega_{\mathbf{q}}$),

$$\frac{1}{\tau_{\text{op} \rightarrow \text{ac}}(T)} = \frac{1 + N_{\mathbf{q}}^{(1)} + N_{\mathbf{q}}^{(2)}}{\tau_{\text{op} \rightarrow \text{ac}}(0)} \approx \frac{1}{\tau_{\text{op} \rightarrow \text{ac}}(0)} \left[1 + \frac{2}{e^{\hbar\omega_{\mathbf{q}}/2kT} - 1} \right], \quad (3.2)$$

where it is assumed that the energy of the decay products is approximately $\hbar\omega_{\mathbf{q}}/2$.

Using Eq. (2.8a) we have in a steady-state:

$$R_{\text{sp}} [N_{\mathbf{q}}(T_e) - N_{\mathbf{q}}(T_{\text{ph}})] = \frac{N_{\mathbf{q}}(T_{\text{ph}}) - N_{\mathbf{q}}(T)}{\tau_{\text{op} \rightarrow \text{ac}}(T)}, \quad (3.3)$$

whence we find the temperature $T_{\text{ph}}(q, n, T_e, T)$ of optical phonons at wavevector \mathbf{q} interacting with electrons of density n and effective temperature T_e . The excess number $N_{\mathbf{q}}^{\text{exc}} \equiv N_{\mathbf{q}}(T_{\text{ph}}) - N_{\mathbf{q}}(T)$ of these phonons is given by:

$$N_{\mathbf{q}}^{\text{exc}} = \frac{[N_{\mathbf{q}}(T_e) - N_{\mathbf{q}}(T)] R_{\text{sp}} \tau_{\text{op} \rightarrow \text{ac}}}{1 + R_{\text{sp}} \tau_{\text{op} \rightarrow \text{ac}}}, \quad (3.4)$$

where $R_{\text{sp}}(q, n, T_e)$ is given by Eq. (2.8b) and $\tau_{\text{op} \rightarrow \text{ac}} = \tau_{\text{op} \rightarrow \text{ac}}(T)$ by Eq. (3.2).

In a steady state the total power, supplied to the electronic system per cm^3 , equals the power $P(n, T_e, T)$ dissipated to the lattice:

$$P = \frac{\hbar\omega_{\mathbf{q}}}{\tau_{\text{op} \rightarrow \text{ac}}} \sum_{\mathbf{q}} N_{\mathbf{q}}^{\text{exc}} = \frac{\hbar\omega_{\mathbf{q}}}{2\pi^2 \tau_{\text{op} \rightarrow \text{ac}}} \int_0^{\infty} q^2 N_{\mathbf{q}}^{\text{exc}} dq. \quad (3.5)$$

Equation (3.5) gives the power necessary to maintain electrons at a temperature T_e , given their density n and the lattice temperature T . We can use (3.5) to construct the dependence $P(T_e, T, n)$ or, equivalently, $T_e(P; n, T)$.

It may be convenient to define an effective energy relaxation time τ_{ε} for carriers:

$$\frac{P}{n} \equiv \frac{k(T_e - T)}{\tau_{\varepsilon}}, \quad (3.6)$$

$$\begin{aligned} \frac{1}{\tau_{\varepsilon}} &= \frac{1}{\tau_{\text{op} \rightarrow \text{ac}}} \frac{\hbar\omega_{\mathbf{q}}}{k(T_e - T)} \frac{1}{n} \sum_{\mathbf{q}} N_{\mathbf{q}}^{\text{exc}} = \\ &= \sum_{\mathbf{q}} \frac{N_{\mathbf{q}}(T_e) - N_{\mathbf{q}}(T)}{n k (T_e - T)} \frac{\hbar\omega_{\mathbf{q}} R_{\text{sp}}}{1 + R_{\text{sp}} \tau_{\text{op} \rightarrow \text{ac}}}. \end{aligned} \quad (3.7)$$

In general, τ_{ε} may depend on n , T_e , and T . The concept of τ_{ε} becomes quite useful at high lattice temperatures, when for a fixed T and in a wide range of T_e the energy relaxation time is practically constant. When both type of carriers are present (at the same T_e) we may define an "average" $\bar{\tau}_{\varepsilon}$ in terms of the total dissipated power: $P/(n+p) = k(T_e - T)/\bar{\tau}_{\varepsilon}$. Because of their larger mass, holes are on speaking terms with phonons of higher wavenumber. Therefore, $\tau_{\varepsilon}^{\text{h}} \ll \tau_{\varepsilon}^{\text{e}}$ and for $n = p$ we have $2/\bar{\tau}_{\varepsilon} = 1/\tau_{\varepsilon}^{\text{e}} + 1/\tau_{\varepsilon}^{\text{h}} \approx 1/\tau_{\varepsilon}^{\text{h}}$.

1. P. G. Klemens, "Anharmonic decay of optical phonons", *Phys. Rev.* **148**, pp. 845-848 (1966).

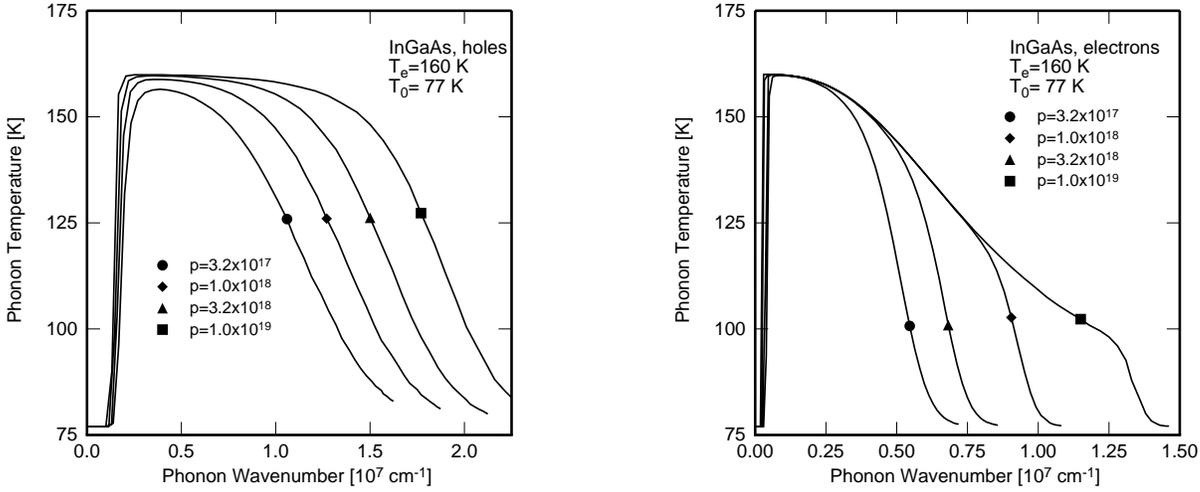
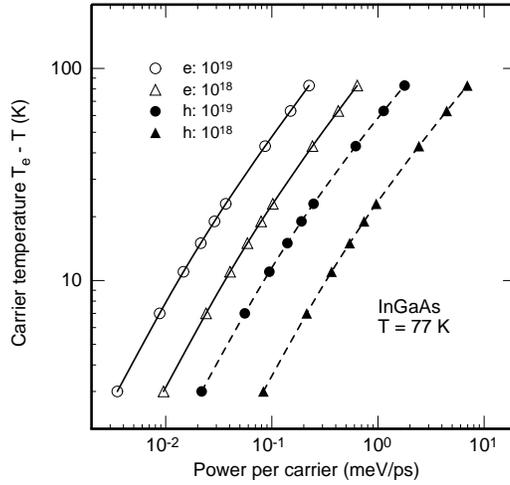


Figure 1. Interaction of hot carriers with the lattice at 77 K. Top two figures show the optical phonon temperature T_{ph} resulting from interaction with holes (left) and electrons (right) at $T_e = 160 \text{ K}$. Assumed $\tau_{op \rightarrow ac}(0) = 20 \text{ ps}$, corresponding to $\tau_{op \rightarrow ac}(77 \text{ K}) = 16.86 \text{ ps}$.

Figure on the right gives the power per carrier necessary to maintain a steady state at a given T_e . Evidently, relaxation times τ_e^e and τ_e^h themselves depend on T_e .



For high enough lattice temperature and not too high overheating, we can use the approximate relation

$$N_{\mathbf{q}}(T_e) - N_{\mathbf{q}}(T) \approx \frac{(T_e - T)}{\hbar\omega_{\mathbf{q}}}, \quad \text{for } k(T_e - T) \leq kT \sim \hbar\omega_{\mathbf{q}}, \quad (4.1)$$

which permits us to write down the phonon temperature in an explicit form

$$T_{ph}(\mathbf{q}) = T + \frac{(T_e - T) R_{sp} \tau_{op \rightarrow ac}}{1 + R_{sp} \tau_{op \rightarrow ac}}. \quad (4.2)$$

The effective energy relaxation time τ_e for carriers defined by (3.6) becomes

$$\frac{1}{\tau_e} = \frac{1}{2\pi^2 n} \int_0^\infty \frac{q^2 dq R_{sp}}{1 + R_{sp} \tau_{op \rightarrow ac}} = \frac{1}{\tau_{op \rightarrow ac}} \frac{3}{2k_F^3} \int_0^\infty \frac{q^2 dq R_{sp} \tau_{op \rightarrow ac}}{1 + R_{sp} \tau_{op \rightarrow ac}}, \quad (4.3)$$

where $k_F = (3\pi^2 n)^{1/3}$. If we could clearly identify a phonon wave vector \mathbf{q}_{eff} such that $R_{sp} \tau_{op \rightarrow ac} \gg 1$ for $q < q_{eff}$ and $R_{sp} \tau_{op \rightarrow ac} \ll 1$ for $q > q_{eff}$, then Eq. (4.3) could be further simplified:

$$\frac{1}{\tau_e} = \frac{1}{\tau_{op \rightarrow ac}} \frac{q_{eff}^3}{2k_F^3}. \quad (4.4)$$

The wavevector q_{eff} determines the volume of the Brillouin zone where optical phonons are "on speaking terms" with the electron/hole gas. It would be nice to have an approximate expression for q_{eff} in terms of (k_F, q_ω, q_T) , where $\hbar^2 q_T^2 / 2m = kT_e$.

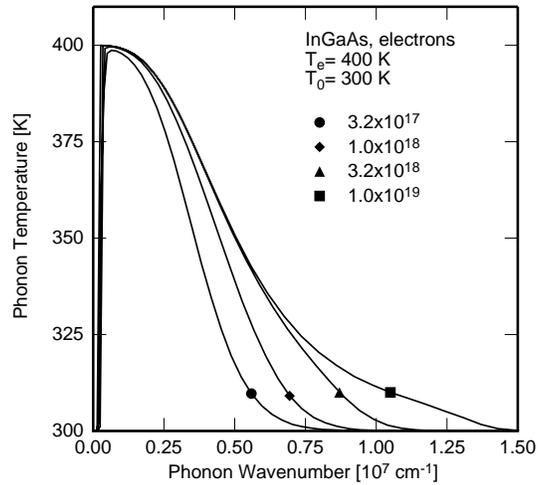
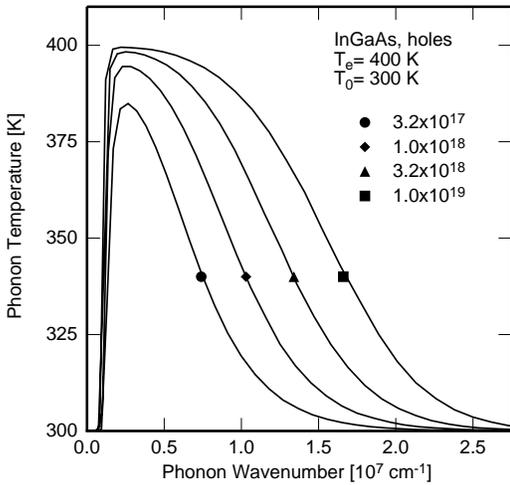


Figure 2. Interaction of hot carriers with the lattice at 300 K. Top two figures show the optical phonon temperature T_{ph} resulting from interaction with holes (left) and electrons (right) at $T_e = 400$ K. Assumed $\tau_{op \rightarrow ac}(0) = 20$ ps, corresponding to $\tau_{op \rightarrow ac}(300 \text{ K}) = 6.12$ ps.

Figure on the right gives the power per carrier necessary to maintain a steady state at a given T_e . The dependence $P(T_e - T)$ is to a high degree linear in the entire range $300 < T_e < 600$ K. This enables the definition of energy relaxation times that depend only on the carrier concentration.

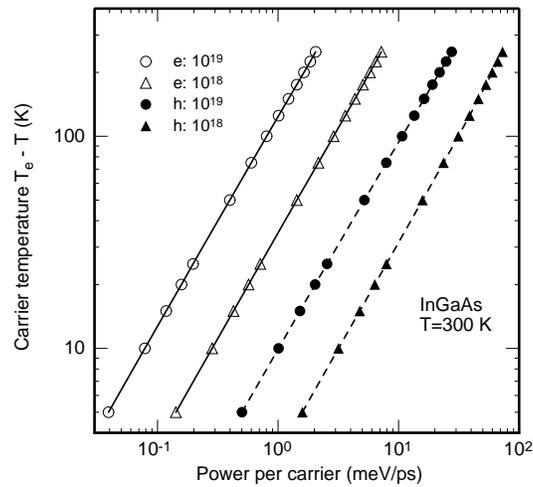


Figure 3. Energy relaxation times, τ_{ϵ}^e , τ_{ϵ}^h , and $\bar{\tau}_{\epsilon}$, calculated according to Eq. (3.7) in the carrier temperature range $350 < T_e < 550$ K. The values of τ_{ϵ} are practically independent of T_e in this range.

