Introduction

We shall introduce a complete set of localized states for a 2D electron gas. These states would be stationary in a normal magnetic field of appropriate intensity.

Electron's dynamical variables are two components of the 2D coordinate $\hat{\mathbf{r}}$ and two components of the momentum $\hat{\mathbf{p}} = -i\hbar\nabla$:

$$\hat{x}\ , \quad \hat{p}_x \ , \text{ where } \quad [\hat{x}\ , \hat{p}_x] = i\hbar \tag{1.1}$$

$$\hat{y}\ , \quad \hat{p}_y \ , \text{ where } \quad [\hat{y}\ , \hat{p}_y] = i\hbar \tag{1.2}$$

Form the canonical momentum operators, corresponding to a magnetic field $\mathbf{B}$:

$$\hat{\pi} \equiv \hat{\mathbf{p}} - \frac{e\hat{\mathbf{A}}}{c}\ , \tag{1.3}$$

where $\hat{\mathbf{A}} = \mathbf{A}(\hat{x}\ , \hat{y})$ is a vector potential:

$$\nabla\times\mathbf{A} = \mathbf{B}\ , \quad \text{(vector } \mathbf{B} \text{ is along the } z \text{ axis)} \tag{1.4}$$

Next, introduce the Larmor coordinates (center of the classical cyclotron orbit):

$$\hat{\xi}_x = \hat{x} + \frac{l^2}{\hbar}\,\hat{\pi}_y \tag{1.5}$$

$$\hat{\xi}_y = \hat{y} - \frac{l^2}{\hbar}\,\hat{\pi}_x \tag{1.6}$$

Definitions:

$$l^2 \equiv \frac{\hbar c}{eB} = \frac{\hbar}{m\omega}\ , \quad l = 81.13\,\text{Å} \tag{1.7}$$

$$\omega = \frac{eB}{mc} = 2.625\times10^{13}\,\text{rad / sec}\ , \quad \frac{\omega}{2\pi} = 4.178\times10^{12}\ \text{Hz} \tag{1.8}$$

$$\hbar\omega = 2.768\times10^{-14}\,\text{erg} = 17.28\,\text{meV} \tag{1.9}$$

$$\frac{1}{2\pi l^2} = \frac{eB}{hc} = 2.418\times10^{11}\ \text{cm}^{-2} = \frac{B}{\Phi_0}\ , \quad \Phi_0 \equiv \frac{hc}{e} = 4.136\times10^{-7}\ \text{Gauss}\cdot\text{cm}^2 \tag{1.10}$$

The above characteristic numbers assume $B = 10\,\text{T} = 10^5$ Gauss and $m = 0.067\,m_0$ (GaAs).

Commutation relations:

$$[\hat{\pi}_x\ , \hat{\pi}_y] = i\hbar^2/l^2 = i\hbar\omega m \tag{1.11}$$

$$[\hat{\xi}_x\ , \hat{\xi}_y] = -i\,l^2 \tag{1.12}$$

$$[\hat{\pi}_x\ , \hat{\xi}_x] = [\hat{\pi}_y\ , \hat{\xi}_x] = [\hat{\pi}_x\ , \hat{\xi}_y] = [\hat{\pi}_y\ , \hat{\xi}_y] = 0 \tag{1.13}$$

Note:

We could obtain the same commutation relations defining

$$\hat{\pi}'_x = \hat{p}_x + \frac{e}{c}\hat{A}_y \tag{1.14}$$

$$\hat{\pi}'_y = \hat{p}_y - \frac{e}{c}\hat{A}_x \tag{1.15}$$

and choosing $\hat{\mathbf{A}}$ as a longitudinal field $\nabla\cdot\hat{\mathbf{A}} = \mathbf{B}$, $\nabla\times\hat{\mathbf{A}} = 0$. Expressions of $\hat{\xi}_x$ and $\hat{\xi}_y$ in terms of $\hat{\pi}'_x$ and $\hat{\pi}'_y$ are the same as those above in terms of $\hat{\pi}_x$ and $\hat{\pi}_y$. Of course, $(\hat{\pi}'_x, \hat{\pi}'_y)$ is no longer a canonical momentum. It is not even a vector [but neither is $(\hat{\xi}_x\ , \hat{\xi}_y)$]

Creation and destruction operators.

Define:

$$\hat{a} = \frac{1}{l\sqrt{2}}\left[\hat{\xi}_y + \mathrm{i}\,\hat{\xi}_x\right] \qquad \hat{\xi}_y = \frac{l}{\sqrt{2}}\left[\hat{a}^\dagger + \hat{a}\right]$$
$$\hat{a}^\dagger = \frac{1}{l\sqrt{2}}\left[\hat{\xi}_y - \mathrm{i}\,\hat{\xi}_x\right] \qquad \hat{\xi}_x = \frac{\mathrm{i}\,l}{\sqrt{2}}\left[\hat{a}^\dagger - \hat{a}\right] \tag{2.1a}$$

$$\hat{b} = \frac{l}{\hbar\sqrt{2}}\left[\hat{\pi}_x + \mathrm{i}\,\hat{\pi}_y\right] \qquad \hat{\pi}_x = \frac{\hbar}{l\sqrt{2}}\left[\hat{b}^\dagger + \hat{b}\right]$$
$$\hat{b}^\dagger = \frac{l}{\hbar\sqrt{2}}\left[\hat{\pi}_x - \mathrm{i}\,\hat{\pi}_y\right] \qquad \hat{\pi}_y = \frac{\mathrm{i}\,\hbar}{l\sqrt{2}}\left[\hat{b}^\dagger - \hat{b}\right] \tag{2.1b}$$

Commutation relations:

$$[\hat{a}\,,\,\hat{a}^\dagger] = 1 = [\hat{b}\,,\,\hat{b}^\dagger] \tag{2.2}$$
$$[\hat{a}\,,\,\hat{b}] = [\hat{a}\,,\,\hat{b}^\dagger] = [\hat{a}^\dagger,\,\hat{b}] = [\hat{a}^\dagger,\,\hat{b}^\dagger] = 0$$

Number operators:

$$\hat{a}^\dagger\hat{a} = \frac{1}{2l^2}\left[\hat{\xi}_x^2 + \hat{\xi}_y^2\right] - \frac{1}{2} \equiv \hat{m} \quad \text{(eigenvalues } 0, 1, 2, \cdots) \tag{2.3a}$$
$$\hat{b}^\dagger\hat{b} = \frac{l^2}{2\hbar^2}\left[\hat{\pi}_x^2 + \hat{\pi}_y^2\right] - \frac{1}{2} \equiv \hat{n} \quad \text{(eigenvalues } 0, 1, 2, \cdots) \tag{2.3b}$$

Because the $\hat{a}$'s and the $\hat{b}$'s commute, we can specify states by the eigenvalues of the hermitean operators $\hat{n}$ and $\hat{m}$. In this way, we find a complete orthonormal set of states $|nm>$. Degeneracy of each level n with respect to the quantum number m is $1/(2\pi l^2)$. Conversely, degeneracy of each level m with respect to the quantum number n is $l^2/(2\pi\hbar^2)$.

The Landau level number operator $\hat{n}$ describes quantization of the Larmor radius $\hat{\rho}_L^2$:

$$\hat{\rho}_L^2 \equiv (\hat{v}_x^2 + \hat{v}_y^2)/\omega^2 = \frac{l^4}{\hbar^2}\left[\hat{\pi}_x^2 + \hat{\pi}_y^2\right] = 2l^2\,(\hat{n} + ½)\,. \tag{2.4}$$

On the other hand, $\hat{m}$ describes quantization of the Larmor orbit *center* $\hat{\rho}_0^2$:

$$\hat{\rho}_0^2 \equiv \hat{\xi}_x^2 + \hat{\xi}_y^2 = 2l^2\,(\hat{m} + ½)\,. \tag{2.5}$$

The number operators $\hat{n}$ and $\hat{m}$ can be used to represent the angular momentum operator,

$$\hat{L}_z \equiv [\hat{\mathbf{r}}\times\hat{\mathbf{p}}]_z = [\hat{\mathbf{r}}\times\hat{\pi}]_z + \frac{e}{c}\,[\hat{\mathbf{r}}\times\hat{\mathbf{A}}]_z \tag{2.6}$$

In the symmetric gauge $\mathbf{A} = \dfrac{1}{2}\mathbf{B}\times\mathbf{r}$ this reduces to the following explicit expression:

$$\frac{2l^2}{\hbar}\,\hat{L}_z = \hat{\rho}_0^2 - \hat{\rho}_L^2 \tag{2.7}$$

☛ $$\hat{L}_z = \hbar\,(\hat{m} - \hat{n})\,. \tag{2.8}$$

Coherent states

Define normalized eigenstates of the destruction operators (arbitrary complex ν and μ):

$$\hat{a}\ |n\,\mu\rangle = \mu\ |n\,\mu\rangle \quad ☚ \quad |n\,\mu\rangle = e^{-\frac{1}{2}|\mu|^2}\sum_{m=0}^{\infty}\frac{\mu^m}{\sqrt{m!}}\,|n\,m\rangle \tag{3.1a}$$

$$\hat{b}\ |\nu\,m\rangle = \nu\ |\nu\,m\rangle \quad ☚ \quad |\nu\,m\rangle = e^{-\frac{1}{2}|\nu|^2}\sum_{n=0}^{\infty}\frac{\nu^n}{\sqrt{n!}}\,|n\,m\rangle \tag{3.1b}$$

States $|n\,\mu\rangle$ are complete in the nth subspace (n-th Landau level) of the Hilbert space H that we shall denote by $M_n \subset H$. Similarly, states $|\nu\,m\rangle$ are complete in the mth subspace $N_m \subset H$. In fact, these sets of states are vastly overcomplete (see p. 11).

In a given M_n, states $|n\,\mu\rangle$ (ditto states $|\nu\,m\rangle$ in a given N_m) are the usual coherent states discussed by Glauber. All individual spaces M_n and N_m are isomorphic because of the identical algebraic properties of $\hat{a}$'s and $\hat{b}$'s (this establishes the correspondence $M \longleftrightarrow N$) and because all subspaces M_n of different index n (ditto N_m of different index m) are related:

$$\sqrt{n}\,|n\,\mu\rangle = \hat{b}^{\dagger}\,|(n-1)\,\mu\rangle \quad ☚ \quad |n\,\mu\rangle = \frac{(\hat{b}^{\dagger})^n}{\sqrt{n!}}\,|0\,\mu\rangle \tag{3.2a}$$

$$\sqrt{m}\,|\nu\,m\rangle = \hat{a}^{\dagger}\,|\nu\,(m-1)\rangle \quad ☚ \quad |\nu\,m\rangle = \frac{(\hat{a}^{\dagger})^m}{\sqrt{m!}}\,|\nu\,0\rangle \tag{3.2b}$$

Thus, so long as we are staying within any of the isomorphic subspaces, we may not need to specify the index that does not vary and use the same symbols to denote coherent states in either M_n or N_m. In other words, $|\alpha\rangle$ may stand for either $|n\,\alpha\rangle$ or $|\alpha\,m\rangle$.

Scalar product:

$$\langle\beta|\alpha\rangle = e^{-\frac{1}{2}\left[\,|\alpha|^2 + |\beta|^2\right] + \alpha\beta^*} = e^{-\frac{1}{2}\left[\,|\alpha-\beta|^2\right] + i\,\mathrm{Im}(\alpha\beta^*)} \tag{3.3}$$

$$\langle\nu_2\,\mu_2\,|\,\nu_1\,\mu_1\rangle = e^{-\frac{1}{2}\left[\,|\nu_1|^2 + |\nu_2|^2 + |\mu_1|^2 + |\mu_2|^2\right]}\,e^{\mu_1\mu_2^* + \nu_1\nu_2^*} \tag{3.4}$$

The phase $\mathrm{Im}(\alpha\beta^*)$ of the scalar product has a simple geometric meaning: its magnitude equals the area of the parallelogram spanned by vectors α and β. The sign of the phase is that of the sine of the angle between α and β counted from α in the clockwise direction. If the area of the parallelogram is a multiple of π then the overlap $\langle\beta|\alpha\rangle$ is real. Thus, all scalar products on a Neumann lattice (see p. 5) are real.

Averages. The following relations hold in every subspace M_n :

$$\langle\mu|\hat{\xi}_x|\mu\rangle = \langle\mu|\hat{x}|\mu\rangle = l\sqrt{2}\,\mathrm{Im}(\mu) \tag{3.5x}$$

$$\langle\mu|\hat{\xi}_y|\mu\rangle = \langle\mu|\hat{y}|\mu\rangle = l\sqrt{2}\,\mathrm{Re}(\mu) \tag{3.5y}$$

The fact that $\hat{x}$ and $\hat{\xi}_x$ have the same expectation value follows from the fact that $\langle\hat{\pi}\rangle = 0$ within the same M_n. Moreover, $\langle n_1\,\mu_1|\hat{\pi}|n_2\mu_2\rangle = 0$ for n_1 and n_2 of the same parity (and arbitrary μ_1 and μ_2).

<u>Dispersions:</u> Coherent states minimize the uncertainty product, $\Delta\xi_x \cdot \Delta\xi_y = \frac{1}{2}\,l^2$.

$$\langle\mu|\hat{\xi}_x^2|\mu\rangle = 2l^2[\mathrm{Im}(\mu)]^2 + \tfrac{1}{2}\,l^2 = \langle\mu|\hat{\xi}_x|\mu\rangle^2 + \tfrac{1}{2}\,l^2 \tag{3.6x}$$

$$\langle\mu|\hat{\xi}_y^2|\mu\rangle = 2l^2[\mathrm{Re}(\mu)]^2 + \tfrac{1}{2}\,l^2 = \langle\mu|\hat{\xi}_y|\mu\rangle^2 + \tfrac{1}{2}\,l^2 \tag{3.6y}$$

$$\Delta\xi_x \equiv \langle(\hat{\xi}_x - \langle\hat{\xi}_x\rangle)^2\rangle^{1/2} = \frac{l}{\sqrt{2}} = \langle(\hat{\xi}_y - \langle\hat{\xi}_y\rangle)^2\rangle^{1/2} \equiv \Delta\xi_y \tag{3.6xy}$$

Displacement:

We shall distinguish the displacement operators acting in the subspaces M and N:

$$\hat{D}_M(\mu)\ |0> = |0\mu> \quad ☚ \quad \hat{D}_M(\mu) = e^{\mu\hat{a}^\dagger - \mu^*\hat{a}} \tag{4.1M}$$

$$\hat{D}_N(\nu)\ |0> = |\nu 0> \quad ☚ \quad \hat{D}_N(\nu) = e^{\nu\hat{b}^\dagger - \nu^*\hat{b}} \tag{4.1N}$$

The "vacuum" ket $|0> \equiv |00>$ is uniquely defined by $\hat{b}\ |0>=0$ and $\hat{a}\ |0>=0$. Such a ket exists; in the coordinate representation it is given by the wave function (7.6), see p. 7.

Since $\hat{D}_M$ commutes with $\hat{b}^\dagger$ and $\hat{D}_N$ with $\hat{a}^\dagger$, the form of these operators does not depend on the index (n or m) of the particular subspace M_n or N_m. Moreover, we have

$$\hat{D}_N(\nu)\ \hat{D}_M(\mu)\ |0> = |\nu\mu> = e^{-½(|\mu|^2 + |\nu|^2)} \sum_{m,\,n=0}^{\infty} \frac{\mu^m}{\sqrt{m!}} \frac{\nu^n}{\sqrt{n!}} |n\,m> \tag{4.2}$$

Operators $\hat{D}$ are unitary:

$$\hat{D}^{-1}(\alpha) = \hat{D}^\dagger(\alpha) = \hat{D}(-\alpha) \tag{4.3}$$

A great convenience is provided by the Hausdorff identity which allows to bring $\hat{D}$ into a normally ordered form:

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} = e^{-½|\alpha|^2}\ e^{\alpha\hat{a}^\dagger}\ e^{-\alpha^*\hat{a}} \tag{4.4}$$

The fact that $\hat{D}$ effects displacements in the complex plane follows from the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ and the operator identity

$$\exp(\hat{P})\ \hat{Q}\ \exp(-\hat{P}) = \hat{Q} + [\hat{P}, \hat{Q}] + \frac{1}{2!}[\hat{P}, [\hat{P}, \hat{Q}]] + \cdots , \tag{4.5}$$

valid for any pair of operators. In our case the series terminates after the second term:

$$\hat{D}^{-1}(\beta)\ \hat{a}\ \hat{D}(\beta) = \hat{a} + \beta \qquad \hat{D}^{-1}(\beta)\ \hat{a}^\dagger\ \hat{D}(\beta) = \hat{a}^\dagger + \beta^* \tag{4.6}$$

$$☚ \qquad \hat{a}\ \left[\hat{D}^{-1}(\beta)\ |\alpha>\right] = (\alpha-\beta)\ \left[\hat{D}^{-1}(\beta)\ |\alpha>\right] \tag{4.7}$$

Equation (4.7) implies that $\hat{D}^{-1}(\beta)\ |\alpha>$ can be interpreted as ket $|(\alpha-\beta)>$. However, the use of displacements $\hat{D}(\alpha_{j+1}-\alpha_j)$ to arrive at the same ket via different routes in the complex plane, $\alpha_1 \to \alpha_2 \to \cdots \to \alpha$, will produce, generally, a different phase of the wave function representing ket $|\alpha>$. The reason for this is that the $\hat{D}$ operators in general do not commute. Their multiplication law is of the form:

$$\hat{D}(\alpha)\ \hat{D}(\beta) = e^{i\,\mathrm{Im}(\alpha\beta^*)}\ \hat{D}(\alpha+\beta) \qquad ☚ \qquad \hat{D}(\alpha)\ |\beta> = e^{i\,\mathrm{Im}(\alpha\beta^*)}\ |\alpha+\beta> \tag{4.8}$$

The "standard" phase of the coherent states [Eqs. (3.1)] is obtained by displacing the vacuum ket, as in Eqs. (4.1).

The unitary displacement operators $\hat{D}$ form a group, called the Weyl group. Note that these operators commute if and only if the parallelogram spanned by vectors α and β is a multiple of π. Any Neumann lattice (p. 5) generates an Abelian subgroup of the Weyl group.

The following relationships are valid for either $\hat{\xi} = \hat{\xi}_x$ or $\hat{\xi} = \hat{\xi}_y$ and either $\hat{\pi} = \hat{\pi}_x$ or $\hat{\pi} = \hat{\pi}_y$:

$$\hat{D}_M^{-1}(\mu)\ \hat{\xi}\ \hat{D}_M(\mu) = \hat{\xi} + <\mu|\hat{\xi}|\mu> , \qquad \hat{D}_N^{-1}(\nu)\ \hat{\pi}\ \hat{D}_N(\nu) = \hat{\pi} + <\nu|\hat{\pi}|\nu> \tag{4.10}$$

Also, since $\hat{D}_M$ commutes with $\hat{\pi}_x$ and $\hat{\pi}_y$, we have

$$\hat{D}_M^{-1}(\mu)\ \hat{\mathbf{r}}\ \hat{D}_M(\mu) = \hat{\mathbf{r}} + <\mu|\hat{\mathbf{r}}|\mu> \tag{4.11}$$

Neumann lattice

The Neumann lattice (NL) is a periodic array of coherent states in the complex plane (corresponding to any particular M_n or N_m) with the unit cell area equal π. For the M_n case, in the coordinates $\bar{x}$, $\bar{y}$ (where $\bar{x} = l\sqrt{2}\,\mathrm{Im}\,\alpha$ and $\bar{y} = l\sqrt{2}\,\mathrm{Re}\,\alpha$) the NL is a 2D crystal of arbitrary symmetry with the unit cell area equal $2\pi l^2$. This peculiar density corresponds to one state per single flux quantum area. For the conventional coherent state, built out of $\hat{x}$ and $\hat{p}$ operators, the density of states in the NL corresponds to 1 state per Planck cell, i.e. the unit cell area in the phase space equals $2\pi\hbar$. *The states of a Neumann lattice form a complete set* (see p. 11), as was first stated by von Neumann and proven by Perelomov [*Theor. Math. Phys.* **6**, 156 (1971)]. **Examples**:

Square lattice

$$\alpha_{mn} = \sqrt{\pi}\,(m + \mathrm{i}n)\,, \qquad m\,,\, n = 0, \pm 1, \pm 2, \cdots \tag{5.1}$$

Consider the scalar product $<\alpha_{m_1 n_1} | \alpha_{m_2 n_2}>$. A little algebra first:

$$|\,\alpha_{m_1 n_1} - \alpha_{m_2 n_2}\,|^2 = \pi\,(M^2 + N^2) \tag{5.2}$$

$$\alpha^*_{m_1 n_1}\alpha_{m_2 n_2} = \pi\,[\,(m_1 m_2 + n_1 n_2) + \mathrm{i}\,(m_1 n_2 - m_2 n_1)\,] \tag{5.3}$$

where $M \equiv m_1 - m_2$ and $N \equiv n_1 - n_2$. Whence we have

$$<\alpha_{m_1 n_1} | \alpha_{m_2 n_2}> = (-)^{m_1 n_2 - m_2 n_1}\; e^{-\frac{\pi}{2}\left[M^2 + N^2\right]} \tag{5.4}$$

Hexagonal lattice

$$\alpha_{mn} = m\,\omega^{(1)} + n\,\omega^{(2)}\,, \qquad m\,,\, n = 0, \pm 1, \pm 2, \cdots \tag{5.5}$$

where

$$\omega^{(1)} = \left[\frac{2\pi}{\sqrt{3}}\right]^{1/2} \quad \text{and} \quad \omega^{(2)} = \left[\frac{2\pi}{\sqrt{3}}\right]^{1/2} e^{\mathrm{i}\pi/3} \tag{5.6}$$

The unit cell area equals $|\omega^{(1)}|\,|\omega^{(2)}|\,\sin(\pi/3) = \pi$. Proceeding as above, we find:

$$|\,\alpha_{m_1 n_1} - \alpha_{m_2 n_2}\,|^2 = (2\pi/\sqrt{3})\,(M^2 + N^2 + MN) \tag{5.7}$$

$$\mathrm{Im}\,(\alpha^*_{m_1 n_1}\alpha_{m_2 n_2}) = \pi\,(m_1 n_2 - m_2 n_1) \tag{5.8}$$

$$<\alpha_{m_1 n_1} | \alpha_{m_2 n_2}> = (-)^{m_1 n_2 - m_2 n_1}\; e^{-\frac{\pi}{\sqrt{3}}\left[M^2 + N^2 + MN\right]} \tag{5.9}$$

General lattice

$$\alpha_{mn} = m\,\omega^{(1)} + n\,\omega^{(2)}\,, \qquad m\,,\, n = 0, \pm 1, \pm 2, \cdots \tag{5.10}$$

where $\omega^{(1)}$ and $\omega^{(2)}$ are arbitrary complex numbers, satisfying $\mathrm{Im}\,[\,\omega^{(1)*}\omega^{(2)}\,] \equiv S_{\text{unit cell}} = \pi$:

$$|\,\alpha_{m_1 n_1} - \alpha_{m_2 n_2}\,|^2 = M^2\,|\omega^{(1)}|^2 + N^2\,|\omega^{(2)}|^2 + 2\,MN\,\mathrm{Re}\left[\omega^{(1)*}\omega^{(2)}\right] \tag{5.11}$$

$$\mathrm{Im}\,(\alpha^*_{m_1 n_1}\alpha_{m_2 n_2}) = S_{\text{unit cell}}\,(m_1 n_2 - m_2 n_1) \tag{5.12}$$

$$<\alpha_{m_1 n_1} | \alpha_{m_2 n_2}> = (-)^{m_1 n_2 - m_2 n_1}\;|\,\alpha_{m_1 n_1} - \alpha_{m_2 n_2}\,| \tag{5.13}$$

Coordinate representation:

First let us establish a convenient one-one correspondence between the *xy* plane and the complex plane *z* and put down an explicit expression for the operators of interest.

$$\vec{r} = (x\,,\,y) \qquad z = \frac{x + \mathrm{i}\,y}{\sqrt{2}\,l} \tag{6.1}$$

$$\vec{\nabla} = (\partial_x\,,\,\partial_y) \qquad \frac{\partial}{\partial z^*} = \frac{l\,(\partial_x + \mathrm{i}\,\partial_y)}{\sqrt{2}} \tag{6.2}$$

$$\left[\frac{\partial}{\partial z}\right]^\dagger = -\frac{\partial}{\partial z^*} \qquad \frac{\partial}{\partial z} = \frac{l\,(\partial_x - \mathrm{i}\,\partial_y)}{\sqrt{2}} \tag{6.3}$$

$$\vec{A} = (A_x\,,\,A_y) \qquad A = \frac{A_x + \mathrm{i}\,A_y}{\sqrt{2}\,Bl} \tag{6.4}$$

$$\vec{A} = \frac{B}{2}(-y\,,\,x) \qquad A = \frac{\mathrm{i}\,z}{2} \tag{6.5}$$

With these definitions we have

$$\frac{\partial A}{\partial z} = \frac{1}{2B}\left[\vec{\nabla}\cdot\vec{A} + \mathrm{i}\,\vec{\nabla}\times\vec{A}\right] \tag{6.6}$$

In the Coulombic gauge ($\vec{\nabla}\cdot\vec{A} = 0$) we have $\partial A/\partial z = \mathrm{i}/2$ whence

$$☛ \qquad A = \frac{\mathrm{i}\,z}{2} + A_0^* \tag{6.7}$$

where $A_0^* = [A_0(z)]^*$ is an arbitrary analytic function of z^*. This function expresses the remaining gauge freedom in the Coulombic gauge. The symmetric gauge, $\vec{A} = ½\,[\vec{B}\times\vec{r}]$ corresponds to $A_0 = 0$.

$$\vec{p} = -\mathrm{i}\hbar\vec{\nabla} \qquad p_x + \mathrm{i}\,p_y = -\frac{\mathrm{i}\,\hbar\sqrt{2}}{l}\frac{\partial}{\partial z^*} \tag{6.8}$$

$$\vec{p}^{\,\dagger} = \vec{p} \qquad p_x - \mathrm{i}\,p_y = -\frac{\mathrm{i}\,\hbar\sqrt{2}}{l}\frac{\partial}{\partial z} \tag{6.9}$$

$$\vec{\pi} = \vec{p} - \frac{e}{c}\vec{A} \qquad \frac{l\,(\hat{\pi}_x + \mathrm{i}\,\hat{\pi}_y)}{\hbar\sqrt{2}} = -\mathrm{i}\frac{\partial}{\partial z^*} - A \tag{6.10}$$

Gauge-independent expressions for the $\hat{a}$ and $\hat{b}$ operators are as follows:

$$\hat{a} = \mathrm{i}\,z^* - \hat{b}^\dagger \qquad \hat{b} = -\mathrm{i}\frac{\partial}{\partial z^*} - A \tag{6.11}$$

$$\hat{a}^\dagger = -\mathrm{i}\,z - \hat{b} \qquad \hat{b}^\dagger = -\mathrm{i}\frac{\partial}{\partial z} - A^* \tag{6.12}$$

Coordinate representation, continued:

In the symmetric gauge, $A = \mathrm{i}z/2$, we have

$$\hat{a} = \frac{\mathrm{i}z^*}{2} + \mathrm{i}\frac{\partial}{\partial z} \qquad \hat{b} = -\frac{\mathrm{i}z}{2} - \mathrm{i}\frac{\partial}{\partial z^*}$$

$$\hat{a}^\dagger = -\frac{\mathrm{i}z}{2} + \mathrm{i}\frac{\partial}{\partial z^*} \qquad \hat{b}^\dagger = \frac{\mathrm{i}z^*}{2} - \mathrm{i}\frac{\partial}{\partial z} \tag{7.1}$$

$$\hat{a}^\dagger\hat{a} = \frac{z\,z^*}{4} - \frac{\partial}{\partial z^*}\frac{\partial}{\partial z} - \frac{1}{2} - \frac{z^*}{2}\frac{\partial}{\partial z^*} + \frac{z}{2}\frac{\partial}{\partial z} \tag{7.2a}$$

$$\hat{b}^\dagger\hat{b} = \frac{z\,z^*}{4} - \frac{\partial}{\partial z^*}\frac{\partial}{\partial z} - \frac{1}{2} - \frac{z}{2}\frac{\partial}{\partial z} + \frac{z^*}{2}\frac{\partial}{\partial z^*} \tag{7.2b}$$

$$\frac{1}{\hbar}\hat{L}_z = (\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}) = z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*} \tag{7.3}$$

The vacuum state. Some Europeans might say that ket $|00\rangle$ represents the "loo state". Its coordinate representation $\langle\mathbf{r}|00\rangle \equiv \psi_{00}(\mathbf{r})$ is found from the system of two equations:

$$\langle z|\,\hat{a}\,|00\rangle = \left[\frac{\mathrm{i}z^*}{2} + \mathrm{i}\frac{\partial}{\partial z}\right]\psi_{00}(z) = 0 \tag{7.4a}$$

$$\psi_{00}(z) = \mathrm{e}^{-\frac{1}{2}|z|^2} \quad ✌$$

$$\langle z|\,\hat{b}\,|00\rangle = \left[-\frac{\mathrm{i}z}{2} - \mathrm{i}\frac{\partial}{\partial z^*}\right]\psi_{00}(z) = 0 \tag{7.4b}$$

Normalizing ψ_{00} under the scalar product, defined by

$$\langle 1|2\rangle \equiv \int \psi_1^*(\mathbf{r})\,\psi_2(\mathbf{r})\,\mathrm{d}^2\mathbf{r} = 2l^2\int\!\!\int \psi_1^*\,\psi_2\,\mathrm{d}(\mathrm{Re}\,z)\,\mathrm{d}(\mathrm{Im}\,z) \tag{7.5}$$

and using $\int \exp(-x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y = \pi$, we obtain

$$\psi_{00} = \frac{1}{l\sqrt{2\pi}}\exp\left[-\frac{|z|^2}{2}\right] = \frac{1}{l\sqrt{2\pi}}\exp\left[-\frac{x^2+y^2}{4l^2}\right] \tag{7.6}$$

Note the factor of 4 in the exponent (rather than 2 as it would be for the ground state of a 2D harmonic oscillator). This means that the uncertainty in x and y is $\sqrt{2}$ times larger than the cyclotron length l. States $\langle x,y|0\mu\rangle$ are the most localized states possible in the degenerate ground Landau level in a magnetic field B. The extra factor of 2 in the exponent of (7.6) can be interpreted as resulting from the zero-point motion in the Larmor radius (the $\hat{b}$'s) added to the zero-point motion of the Larmor orbit center (the $\hat{a}$'s).

Acting on (7.6) with an appropriate combination of displacement operators, we find representations of such kets as $|\nu\mu\rangle$. Since $\hat{a}^\dagger\psi_{00} = -\mathrm{i}z\,\psi_{00}$ and $\hat{b}^\dagger\psi_{00} = \mathrm{i}z^*\psi_{00}$, we have

$$\psi_{0\mu}(z) = \hat{D}_M(\mu)\,\psi_{00} = \mathrm{e}^{-\frac{1}{2}|\mu|^2}\,\mathrm{e}^{\mu\hat{a}^\dagger}\psi_{00} = (2\pi l^2)^{-1/2}\,\mathrm{e}^{-\frac{1}{2}|\mu|^2}\,\mathrm{e}^{-\frac{1}{2}|z|^2}\,\mathrm{e}^{-\mathrm{i}\mu z} \tag{7.7}$$

$$\psi_{\nu 0}(z) = \hat{D}_N(\nu)\,\psi_{00} = \mathrm{e}^{-\frac{1}{2}|\nu|^2}\,\mathrm{e}^{\nu\hat{b}^\dagger}\psi_{00} = (2\pi l^2)^{-1/2}\,\mathrm{e}^{-\frac{1}{2}|\nu|^2}\,\mathrm{e}^{-\frac{1}{2}|z|^2}\,\mathrm{e}^{\mathrm{i}\nu z^*} \tag{7.8}$$

$$\psi_{\nu\mu}(z) = \hat{D}_N(\nu)\,\hat{D}_M(\mu)\,\psi_{00} = (2\pi l^2)^{-1/2}\,\mathrm{e}^{-\frac{1}{2}(|\nu|^2+|\mu|^2)-\mu\nu}\,\mathrm{e}^{-\frac{1}{2}|z|^2}\,\mathrm{e}^{\mathrm{i}\nu z^* - \mathrm{i}\mu z} \tag{7.9}$$

Coordinate representation, continued:

Bargmann-like representation.

Operations such as those in Eqs. (7.7-9) can be made somewhat easier by redefining the operators so as to absorb into the definition the result of their action on $\exp(-½\,|z|^2)$:

$$\hat{O}\, e^{-½\,|z|^2}\, f = e^{-½\,|z|^2}\, \hat{O}_f\, f \tag{8.1}$$

In this way, we can represent an arbitrary operator $\hat{O}$ by the operator

$$\hat{O}_f = e^{½\,|z|^2}\, \hat{O}\, e^{-½\,|z|^2} \tag{8.2}$$

For example:

$$\left[\frac{\partial}{\partial z}\right]_f = \frac{\partial}{\partial z} - \frac{z^*}{2} \qquad \left[\frac{\partial}{\partial z^*}\right]_f = \frac{\partial}{\partial z^*} - \frac{z}{2} \tag{8.3}$$

Accordingly, an arbitrary ket $|\,\rangle$ is represented by the function

$$f(z) = \sqrt{2\pi\, l}\; e^{½\,|z|^2} \langle z\,|\,\rangle \tag{8.4}$$

Thus, $f_{00} = 1$. This is similar, but not equivalent, to the well-known Bargmann representation. The latter is usually defined for the Hilbert space isomorphic to either M or N and the function f is analytic in either z or z^*. The present case is different because in general the function $f(z)$ representing a state in $H = N \times M$ is neither analytic nor can it be factored into a product of analytic functions of z^* and z, respectively.

Explicit expressions for selected operators in the Bargmann space are:

$$\hat{a}_f = \mathrm{i}\frac{\partial}{\partial z} \qquad \hat{b}_f = -\mathrm{i}\frac{\partial}{\partial z^*} \tag{8.5a}$$

$$\hat{a}_f^\dagger = -\mathrm{i}\,z + \mathrm{i}\frac{\partial}{\partial z^*} \qquad \hat{b}_f^\dagger = \mathrm{i}\,z^* - \mathrm{i}\frac{\partial}{\partial z} \tag{8.5b}$$

$$\hat{D}_{M\,f}(\mu) = e^{-½\,|\mu|^2 - \mathrm{i}\mu z}\; e^{\mathrm{i}\mu\frac{\partial}{\partial z^*} - \mathrm{i}\mu^*\frac{\partial}{\partial z}} \tag{8.6 M}$$

$$\hat{D}_{N\,f}(\nu) = e^{-½\,|\nu|^2 + \mathrm{i}\nu z^*}\; e^{\mathrm{i}\nu^*\frac{\partial}{\partial z^*} - \mathrm{i}\nu\frac{\partial}{\partial z}} \tag{8.6 N}$$

Acting on any function of z and z^*, operator $\hat{D}_{M\,f}(\mu)$ displaces $z \to z - \mathrm{i}\mu^*$ and $z^* \to z^* + \mathrm{i}\mu$. Operation of $\hat{D}_{N\,f}(\nu)$ displaces $z \to z - \mathrm{i}\nu$ and $z^* \to z^* + \mathrm{i}\nu^*$.

Let us illustrate the newly acquired ease of operation by deriving Eq. (7.9):

$$f_{0\mu}(z) = \hat{D}_{M\,f}(\mu)\, f_{00} = e^{-½\,|\mu|^2 - \mathrm{i}\mu z} \tag{8.7 M}$$

$$f_{\nu 0}(z) = \hat{D}_{N\,f}(\nu)\, f_{00} = e^{-½\,|\nu|^2 + \mathrm{i}\nu z^*} \tag{8.7 N}$$

$$f_{\nu\mu}(z) = \hat{D}_{N\,f}(\nu)\, f_{0\mu} = e^{-½(|\mu|^2 + |\nu|^2)}\; e^{-\mu\nu}\; e^{\mathrm{i}\nu z^* - \mathrm{i}\mu z} \tag{8.8}$$

Of course, we could have used $\hat{D}_{M\,f}$ and $\hat{D}_{N\,f}$ in any order. Next, using Eq. (3.2), we find:

$$f_{n\mu} = \frac{(\mathrm{i}\,z^* - \mu)^n}{n!}\, f_{0\mu} \qquad f_{\nu m} = \frac{(-\mathrm{i}\,z - \nu)^m}{m!}\, f_{\nu 0} \tag{8.9}$$

Also, expanding the factor $\exp(\mathrm{i}\,\nu\, z^* - \mu\,\nu)$ in Eq. (8.8) and using Eq. (3.1), we have

$$f_{\nu\mu}(z) = e^{-½\,|\nu|^2} \sum_n \frac{\nu^n}{\sqrt{n!}} f_{n\mu} = e^{-½\,|\mu|^2} \sum_m \frac{\mu^m}{\sqrt{m!}} f_{\nu m} \tag{8.10}$$

Coordinate representation, continued:

Connection with the scalar product.

Comparing Eqs. (3.3) and (7.7), we see that the wave function $\psi_{0\mu}(z)$ is proportional to the scalar product of coherent states

$$\sqrt{2\pi l^2}\ \psi_{0\mu}(z) = \langle (\mathrm{i}z^*) | \mu \rangle = \langle (\mathrm{i}\mu^*) | z \rangle . \tag{9.1}$$

Similarly, comparing (3.3) and (7.8), we find

$$\sqrt{2\pi l^2}\ \psi_{\nu 0}(z) = \langle z | (\mathrm{i}\nu) \rangle = \langle \nu | (\mathrm{i}z) \rangle . \tag{9.2}$$

Let us emphasize that the left-hand side of Eqs. (9.1-2) represents the wave function *not* in the "coherent-state representation", which would obviously be roundabout, but in the *coordinate* representation. This gives a new twist to the Dirac notation!

There seems to be no analogous connection between Eqs. (3.4) and (7.9). Close but no cigar!

Close inspection. Let us examine Eq. (7.9), which gives ket $|\nu\mu\rangle$ in the coordinate representation, more closely:

$$\sqrt{2\pi l^2}\ \psi_{\nu\mu}(z) = e^{-\frac{1}{2}(|\nu|^2 + |\mu|^2) - \mu\nu}\ e^{-\frac{1}{2}|z|^2}\ e^{\mathrm{i}\nu z^* - \mathrm{i}\mu z}$$

$$\sqrt{2\pi l^2}\ \psi_{\nu_1\nu_2\mu_1\mu_2}(z) = e^{-\frac{1}{2}[z_1 - (\mu_2 - \nu_2)]^2}\ e^{-\frac{1}{2}[z_2 - (\mu_1 + \nu_1)]^2}\ e^{\mathrm{i}[z_1(\nu_1 - \mu_1) + z_2(\nu_2 + \mu_2) - (\mu_1\nu_2 + \nu_1\mu_2)]} \tag{9.3}$$

$$\sqrt{2\pi l^2}\ \psi_{x_\nu y_\nu x_\mu y_\mu}(x, y) = e^{-\frac{1}{4l^2}\left[[x - (x_\mu + x_\nu)]^2 + [y - (y_\mu + y_\nu)]^2\right]}\ e^{\frac{\mathrm{i}}{2l^2}\left[(x_\mu - x_\nu)y - (y_\mu - y_\nu)x + (x_\nu y_\mu - x_\mu y_\nu)\right]}$$

where $z \equiv z_1 + \mathrm{i}z_2 = \dfrac{x + \mathrm{i}y}{l\sqrt{2}}$ $\quad x \equiv l\sqrt{2}\,\mathrm{Re}(z)$ $\quad y \equiv l\sqrt{2}\,\mathrm{Im}(z)$; (9.4)

and

$$\mu \equiv \mu_1 + \mathrm{i}\mu_2 = \frac{y_\mu + \mathrm{i}x_\mu}{l\sqrt{2}} \qquad x_\mu \equiv l\sqrt{2}\,\mathrm{Im}(\mu) \qquad y_\mu \equiv l\sqrt{2}\,\mathrm{Re}(\mu)\ , \tag{9.5μ}$$

$$\nu \equiv \nu_1 + \mathrm{i}\nu_2 = \frac{y_\nu - \mathrm{i}x_\nu}{l\sqrt{2}} \qquad x_\nu \equiv -l\sqrt{2}\,\mathrm{Im}(\nu) \qquad y_\nu \equiv l\sqrt{2}\,\mathrm{Re}(\nu)\ . \tag{9.5ν}$$

We see that the wave packet is centered at the point with coordinates

$$(\bar{x}, \bar{y}) = (x_\mu + x_\nu\ ,\ y_\mu + y_\nu) \tag{9.6}$$

Of course, by letting in Eq. (9.3) either $\mu = 0$ or $\nu = 0$ we recover Eqs. (7.7-8) and (9.1-2).

Equation (9.3) shows that the conjugate wavefunction wave function $\psi^*_{\nu\mu}$ corresponds to the replacement

$$x_\nu\ ,\ x_\mu \ \longleftrightarrow\ x_\mu\ ,\ x_\nu\ . \tag{9.7}$$

Of course, this can be seen already from Eq. (7.9) or Eq. (8.8):

$$\psi^*_{\nu\mu} = \psi_{\mu^*\nu^*}\ . \tag{9.8}$$

This is worthy of note, because the first index in $\psi_{\nu\mu}$ has to do with the average electron kinetic energy $\langle\hat{n}\rangle = |\nu|^2$, whereas the second index does not affect $\langle\hat{n}\rangle$ (since $\hat{D}_M$ commutes with $\hat{n}$).

Motion of the coherent-state wave packets:

Coherent wave packet in a magnetic field. In a constant magnetic field B the Hamiltonian of a free electron is of the form

$$\hat{H}_0 = \frac{1}{2m}\left[\hat{\mathbf{p}} - \frac{e\hat{\mathbf{A}}}{c}\right]^2 = \hbar\omega(\hat{b}^\dagger\hat{b} + \tfrac{1}{2})\ . \tag{10.1}$$

Heisenberg's equation of motion give

$$\frac{\partial\hat{a}}{\partial t} = \frac{[\hat{a}, \hat{H}_0]}{\mathrm{i}\hbar} = 0\ , \tag{10.2a}$$

$$\frac{\partial\hat{b}}{\partial t} = \frac{[\hat{b}, \hat{H}_0]}{\mathrm{i}\hbar} = -\mathrm{i}\,\omega\,\hat{b}\ . \tag{10.2b}$$

Transforming to the Schrödinger representation, we have

$$\mu(t) = \mu_0 = \text{constant of the motion} \tag{10.3μ}$$

$$\nu(t) = \mathrm{e}^{-\mathrm{i}\omega t}\,\nu_0\ , \tag{10.3ν}$$

which means that the packet moves without dispersion in a quantum analog of the Larmor orbit: a circle of radius $|\nu|\,l\sqrt{2}$ about the center with coordinates (x_μ, y_μ).

Motion of a coherent wave packet in crossed electric and magnetic fields. In an electric field $\mathbf{F}$ parallel to the plane of the 2DEG, the hamiltonian is $\hat{H} = \hat{H}_0 + \hat{V}$ with the additional term $\hat{V}$ of the form:

$$\hat{V} = e\,\hat{\mathbf{r}}\cdot\mathbf{F} = -\frac{\mathrm{i}\,e\,l\,F}{\sqrt{2}}\left[\hat{a} + \hat{b}^\dagger\right] + \text{herm. conj.} \tag{10.4}$$

where $F \equiv F_x + \mathrm{i}F_y$. The Heisenberg equations now become:

$$\frac{\partial\hat{a}}{\partial t} = \frac{e\,l\,F^*}{\sqrt{2}\,\hbar}\ ; \qquad \frac{\partial\hat{b}}{\partial t} = -\frac{e\,l\,F}{\sqrt{2}\,\hbar} - \mathrm{i}\,\omega\,\hat{b}\ . \tag{10.5}$$

The solution corresponds to the center of the wave packet performing a cycloidal motion

$$\mu(t) = \mu_0 + \frac{e\,l\,F^*}{\sqrt{2}\,\hbar}\,t\ ; \tag{10.6μ}$$

$$\nu(t) = \mathrm{e}^{-\mathrm{i}\omega t}\nu_0 + \frac{e\,l\,F\,(\mathrm{e}^{-\mathrm{i}\omega t} - 1)}{\sqrt{2}\,\mathrm{i}\hbar\omega}\ . \tag{10.6ν}$$

In the coordinates (9.5) this is:

$$x_\mu(t) = x_{\mu_0} - \frac{e\,l^2 F_y}{\hbar}\,t\ , \qquad y_\mu(t) = y_{\mu_0} + \frac{e\,l^2 F_x}{\hbar}\,t \tag{10.7μ}$$

$$\begin{aligned} x_\nu(t) &= \cos(\omega t)\,x_{\nu_0} + \sin(\omega t)\,y_{\nu_0} + \frac{e\,l^2}{\hbar\omega}\left[F_x[\cos(\omega t) - 1] + F_y\sin(\omega t)\right]\ ; \\ y_\nu(t) &= \cos(\omega t)\,y_{\nu_0} - \sin(\omega t)\,x_{\nu_0} + \frac{e\,l^2}{\hbar\omega}\left[-F_x\sin(\omega t) + F_y[\cos(\omega t) - 1]\right]\ . \end{aligned} \tag{10.7ν}$$

The motion of wavepacket centers is identical to that of a classical particle in the same fields.

Completeness:

Coherent states $|n\,\mu>$ form a complete set in M_n and, similarly, states $|\nu\, m>$ form a complete set in N_m. Even though the member states are not orthogonal, these overcomplete sets allow resolutions of unity:

$$\pi^{-1} \int d^2\mu \ |n\,\mu><\mu\, n| = \hat{P}\,(M_n) \qquad (11.1M)$$

$$\pi^{-1} \int d^2\nu \ |m\,\nu><\nu\, m| = \hat{P}\,(N_m) \qquad (11.1N)$$

where the $\hat{P}$'s are projection operators; within the subspace they project on they represent the unit operator. The proof of Eqs. (1) has been given by Glauber [*Phys. Rev.* **131**, 2766 (1963)].

It is a trivial matter to generalize Glauber's argument and construct an analogous closure relation in H:

$$\pi^{-2} \int d^2\mu\, d^2\nu \ |\nu\,\mu><\mu\,\nu| = \hat{J}\ . \qquad (11.2H)$$

As seen from Eqs. (8.9), in the Bargmann representation states $|n\,\mu>$ (for a fixed μ) are binomials $\propto (z^*+i\mu)^n$ of power n [respectively, states $|\nu\, m>$ for a fixed ν are $\propto (z-i\nu)^m$]. These states are eigenstates of hermitean operators ($\hat{n}$ and $\hat{m}$, respectively, cf. p. 2) and are therefore complete in N_m and M_n, respectively. This means that an arbitrary state from these subspaces can be represented by an analytic function (respectively, of z^* and z). But analytic functions are determined by a set of their values on any convergent sequence of points $z_1^*, z_2^*, \cdots$ (respectively, $z_1, z_2, \cdots$). This proves that any subset of coherent states, that corresponds to a sequence of complex numbers with a limit point in the complex plane, is complete in its own subspace. It is, of course, overcomplete: removal of any finite number of elements does not affect the existence of a limit point.

Considering infinite sequences that *do not* have a limit point, Perelomov has shown [*Theor. Math. Phys.* **6**, 156 (1971)] that any regular lattice which is denser than the Neumann lattice (i.e., has a unit cell area $S<\pi$, cf. p. 5) is overcomplete, while that which is sparser ($S>\pi$) is incomplete. As to the Neumann Lattice itself, he proved a remarkable theorem:

Perelomov's Theorem: Let $\Gamma_A = \{\,|\alpha>\}$ be a set of coherent states forming a Neumann lattice in a subspace A. The set is complete in A and remains complete upon removal of any single state. It becomes incomplete on removal of any two states.

In our case, α stands for either $|n\,\mu>$ (for a fixed n) or $|\nu\, m>$ (for a fixed m) and, respectively, A for M_n or N_m.

It should be stressed that completeness *does not* entail the existence of a closure relation. Naively, one could anticipate – in analogy to (11.1) – a relation of the kind

$$\sum_{\alpha\in\Gamma_A} |\alpha>I_\alpha<\alpha| = \hat{P}\,(A) \qquad ☛ \ \textbf{Wrong!} \qquad (11.3)$$

It can be shown that no such measure I_α can be found that would validate Eq. (11.3). This follows from the existence of a conjugate set $\tilde{\Gamma}_A = \{\,|\tilde{\alpha}>\}$, bi-orthogonal to Γ_A (see p. 12). Indeed, assuming (11.3) we should be able to write

$$|\alpha> = \sum_{\mu\in\Gamma_A} |\mu>I_\mu<\mu|\alpha> \qquad (11.4)$$

$$☛ \quad \delta_{\alpha\beta} = <\tilde{\beta}|\alpha> = \sum_{\mu\in\Gamma_A} <\tilde{\beta}|\mu>I_\mu<\mu|\alpha> = I_\beta<\beta|\alpha> \neq \delta_{\alpha\beta} \qquad (11.5)$$

Thus, assumption that (11.3) exists leads to a contradiction. The conjugate set exists only if Γ_A is *minimal* complete set (so a similar argument would not be applicable to Eqs. 11.1). The proof (11.5) shows that the necessary condition for a minimal complete set to allow an expansion of unity is that *the set must consist of orthogonal states.* Of course, this condition is also sufficient.

Bi-orthogonal bases

Definition: A set Γ_A is complete in A if the projection on A of any state orthogonal to all members of Γ_A vanishes identically.

Definition: A complete set Γ_A is "minimal" if it becomes incomplete upon the removal of any member of the set. We shall refer to such sets as *bases.*

A Neumann lattice with one state removed is a basis.

Bari's Theorem (quoted by Perelomov, I have not seen the proof):

For any basis Γ_A there exists another basis $\tilde{\Gamma}_A$ such that if $|\alpha>, |\beta> \in \Gamma_A$ and $|\tilde{\alpha}>, |\tilde{\beta}> \in \tilde{\Gamma}_A$ then $<\tilde{\alpha}|\beta> = <\tilde{\beta}|\alpha> = \delta_{\alpha\beta}$.

Given Bari's theorem it is obvious that for any state $|\psi> \in A$ we can write $|\psi> = \sum |\alpha><\tilde{\alpha}|\psi>$. It then follows that bi-orthogonal bases *do allow* expansions of the unity:

$$\sum_{\alpha} |\alpha><\tilde{\alpha}| = \sum_{\alpha} |\tilde{\alpha}><\alpha| = \hat{J} \; . \qquad (12.1)$$

Matrices $<\tilde{\alpha}|\tilde{\beta}>$ and $<\alpha|\beta>$ are inverse to one another:

$$\sum_{\mu} <\tilde{\alpha}|\tilde{\mu}><\mu|\beta> = \delta_{\alpha\beta} \; . \qquad (12.2)$$

In order to do practical work in the coherent state basis, it is important to be able to construct the inverse to the scalar product matrix defined on the Neumann lattice (p. 5).

The explicit form of the Neumann-lattice basis depends on which of the NL elements has been thrown out. It seems that if we remove a state $|\alpha_0>$ far from the origin ($|\alpha_0| \gg 1$) then the actual shape of states $|\tilde{\alpha}>$ should not depend on α_0 – so long as $|\alpha| \ll |\alpha_0|$. Does this offer an opportunity to describe states in a finite 2D sample in a NL basis, corresponding to removed state far outside the sample boundaries?

My tentative answer is yes, but with great caution.

There seems to exist a deep connection between the Perelomov theorem and the well-known fact that it is impossible to construct a continuous set of atlases faithfully covering the surface of a sphere. (An atlas is a patch with a local coordinate system. In a continuous set, the atlases overlap and in the overlap region the coordinates correctly transform as a vector from one system to the other. The impossibility of covering a sphere in this way is a topological fact. I have heard C. N. Yang preaching on the profound importance of this circumstance in physics.) The connection to Perelomov's theorem arises in virtue of the exact correspondence of the motion of 2D electrons in a normal magnetic field B and the motion of 2D electrons confined to a sphere of radius R containing at the center a monopole of charge $g = BR^2$. The latter problem (Tamm) reduces to the motion on a sphere *without* a monopole – except the allowed values of the angular momentum start from $L = L_g \equiv eg/\hbar c$ rather than $L = 0$. It is well-known (Dirac) that the magnetic charge, if exists, is quantized so that L_g is integer or half-integer. Each of the shells $L = L_g,\ L_g + 1,\ \cdots$ is finite $(2L+1)$ dimensional but its degeneracy per unit area of the sphere coincides with the Landau level degeneracy $(2\pi l^2)^{-1}$. It seems that if we could construct a single coherent state lattice on a sphere (which would entail an atlas covering), we would then be able to map it on the Neumann lattice. A sphere with one point punched out *is* coverable and this seems to parallel the statement of Perelomov's theorem.