Degenerate electron gas

It appears obvious that most of the properties of a degenerate 2D electron gas would not be significantly different if the gas were confined to the surface of a sphere of radius R rather than a planar quantum well – provided that $R \gtrsim \lambda_{\rm in}$ where $\lambda_{\rm in}$ is the inelastic mean free path of electrons.

The kinematic behavior of a 2DEG of surface density n is determined by electrons in a narrow band δE near the Fermi level $E_{\rm F}$, where

$$E_{\rm F} = \frac{\pi \hbar^2 n}{m} \tag{1.1}$$

(two-fold spin degeneracy included). The density of kinematically active electrons is

$$\delta n = \frac{m\, \delta E}{\pi \hbar^2} \tag{1.2}$$

By the order of magnitude $\delta E \sim kT \sim \hbar/\tau_{\rm in}$, where

$$\tau_{\rm in} \equiv \frac{\lambda_{\rm in}}{v_{\rm F}} = \frac{\lambda_{\rm in}\, m}{\hbar \sqrt{2\pi n}} \tag{1.3}$$

We shall assume that $\delta E \ll E_{\rm F}$ and therefore $\delta n \ll n$.

Kinematically active electrons are forbidden to scatter into the subspace of the Hilbert space below $E_{\rm F}$. On a sphere, the forbidden subspace corresponds to a number of shells filled up to an angular momentum $\hbar(L_{\rm F}-1)$, defined by†

$$4\pi R^2 n = 2 \sum_{l=0}^{L_{\rm F}-1} (2l+1) = 2 L_{\rm F}^2 \tag{1.4}$$

The energy separation between the kinematically active (partially filled) $L_{\rm F}$-th shell and the highest completely filled $[(L_{\rm F}-1)$-st$]$ shell

$$\Delta_{\rm F} = E_{\rm F} - E_{L_{\rm F}-1} = \frac{\hbar^2 L_{\rm F}}{m R^2} = \frac{\hbar^2}{m R} \sqrt{2\pi n} \tag{1.5}$$

and the density of states in the $L_{\rm F}$-th shell is

$$\delta n_{\rm F} = \frac{2(2L_{\rm F}+1)}{4\pi R^2} = \frac{(2L_{\rm F}+1)\, n}{L_{\rm F}^2} \approx \frac{2n}{L_{\rm F}} \tag{1.6}$$

It is reasonable to set $\delta n \approx \frac{1}{2} \delta n_{\rm F}$ to ensure the consistency of definitions (with this identification the $L_{\rm F}$-th shell is approximately half filled). Thus, if we start from a given pair $(n\, , \delta n)$ [equivalently, $(E_{\rm F}, \delta E)$], then both the sphere radius R and the Fermi shell number $L_{\rm F}$ are fixed by Eqs. (4) and (6):

$$L_{\rm F} = \frac{n}{\delta n}\ ; \qquad 2\pi R^2 = \frac{n}{(\delta n)^2} \tag{1.7}$$

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† Alternatively, we can define $L_{\rm F}$ by requiring that the Fermi level resides within partially filled $L_{\rm F}$-th level:

$$\frac{\hbar^2 L_{\rm F}(L_{\rm F}+1)}{2m R^2} = E_{\rm F} \quad \text{which gives} \quad 4\pi R^2 n = 2 L_{\rm F}^2 \left[1 + \frac{1}{L_{\rm F}} \right] \tag{1.8}$$

The two definitions are equivalent to within terms of order

$$\frac{1}{L_{\rm F}} \sim \frac{1}{R\sqrt{2\pi n}} \lesssim (\lambda_{\rm in} k_{\rm F}) \sim \frac{1}{2} \frac{\hbar/\tau_{\rm in}}{E_{\rm F}} \ll 1 \tag{1.9}$$

Magnetic analogy. Suppression of $L_F - 1$ shells can be achieved by placing at the center of the sphere a magnetic monopole of charge g

$$\frac{eg}{\hbar c} = L_F \tag{2.1}$$

It is well known (Tamm) that the electronic motion on a sphere with a monopole (2.1) at the center is identical to that on a sphere without the monopole, except that the allowed values of the angular momentum start from L_F rather from 0. The magnetic charge, if exists, is quantized (Dirac) so that the combination (2.1) is integer or half integer.

Substituting into (2.1) the definition of magnetic charge $g \equiv B R^2$, where B is the normal magnetic field at the surface of the sphere, we find

$$L_F = \frac{R^2}{l^2} \ , \tag{2.2}$$

where $l^2 \equiv \frac{\hbar c}{eB}$. Using Eqs. (1.7), we can express l iin terms of the assumed δn :

$$l^2 = \frac{1}{2\pi\,\delta n} \tag{2.3}$$

The separation (1.5) between adjacent shells becomes identical to a cyclotron energy spacing:

$$\Delta_F = \frac{\hbar^2 L_F}{m R^2} = \frac{\hbar^2}{ml^2} \equiv \hbar\omega_c \tag{2.4}$$

Electron gas on a torus

It appears obvious that most of the properties of a degenerate 2D electron gas would not be significantly different if the gas were confined to the surface of a large enough torus – rather than a planar quantum well – provided that $L^{(i)} \gtrsim \lambda_{in}$ where λ_{in} is the inelastic mean free path of electrons and $L^{(i)}$ $(i = 1, 2)$ are the principal periods of the torus..

Magnetic analogy. In the large "straigt"torus limit ($L^{(1)} \gg L^{(2)} \gg l$), the eigenstates of a 2DEG in a magnetic field must coincide with those on a torus with a magnetically charge wire loop in the middle.

Periodic boundary conditions. Considering the general problem of finding a complete set of localized states on a "straight" torus, we can view the magnetic field B as an artificial object, that helps us define the operators $\hat{a}$, $\hat{a}^\dagger$, $\hat{b}$, $\hat{b}^\dagger$, etc., but does not enter into the Hamiltonian of the electronic system.

The wave functions defined on the torus *must* be periodic; otherwise quantum mechanics would make no sense.

$$\psi(x + L^{(1)}, y + L^{(2)}) = \psi(x , y) \ . \tag{A2.1}$$