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THE GEOMETRY OF LATTICES

FOR

MARKOVIAN QUEUEING NETWORKS

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ABSTRACT

The existence of product form solutions for the equilibrium probabilities of Markovian queueing networks is shown to be a consequence of the geometric and algebraic structure of the state transition diagram. The methodology calls for the decomposition of the global balance equations into a consistent set of partial balance equations. Necessary and sufficient conditions for consistency are given. The approach introduced allows for the construction of a class of Markovian state diagrams out of basic geometric shapes. This makes feasible the design of a class of protocols of seemingly arbitrary topologies which possess product form solution. Significantly, the class of queueing networks with blocking and state dependent routing that have product form solution is obtained. An algorithm for evaluating the equilibrium probabilities and some applications to deadlock prevention in computer communication networks are also discussed.

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1. Introduction

Markovian queueing networks are commonly used to model computer communication networks and computer systems. The mathematical foundation of these networks is based on the theory of Markov population processes [13]. These processes describe the statistical movement of packets (items) among sets of integral number of nodes (colonies).

Many problems involving Markovian structures entail the determination of the equilibrium probabilities. Typical examples are the determination of the average throughput and average time delay for optimal flow control of computer communication networks [16], [30]. The determination of the equilibrium probabilities involves the solution of a set of linear equations known as the global balance equations [14], [32].

Analytical methods to find the equilibrium probabilities abound in the literature [22]. Good discussions of their relative merits are given in [31] and [26]. In order to find closed form solutions certain simplifying assumptions are usually made. Jackson [7] showed that if the Markovian queueing network is assumed to have an infinite number of buffers at each node, the joint probability distribution of the number of packets in the system has the form of a product of marginal distributions. Jackson [8] and Gordon and Newell [5] showed that for closed Markovian networks similar product form solutions can be obtained. (If product form solutions are possible, a normalizing constant must be calculated in a numerically accurate manner, as in Buzen's algorithm [3].) A question naturally arises regarding the significance of the product form. What is the common structure which closed queueing networks share with open networks that leads to product form solutions?

Product form solutions have, in the past, been attributed to a quantitative behavior of the state transitions. The methodology introduced by Whittle [37], [38], proved to be very useful in finding the equilibrium probabilities for many classes of Markovian queueing networks [2], [11], [29]. Basket et al. [2] have noted, however, that Whittle's partial balance method does not always lead to a solution. For a given queueing model one has no assurance *a priori* that the set of partial balance equations are consistent. Note that if product form solutions do not exist, finding the solution to the global balance equations becomes practically unfeasible as the number of the states increases.

One area where the absence of product form solutions has produced difficulties is the study of tandem queueing systems with finite buffers. The inherent blocking introduces an interdependence between queues which complicates analysis. It has been observed, however, that for certain state transition diagrams (protocols) the product form solution applies even for the blocking case. Such solutions have mostly been derived for a two tandem queueing system with protocols that have applications in computer systems modeling. Variations of this system have been extensively analyzed in the literature (Neuts [26]). An excellent discussion of the frustration and

difficulties encountered in dealing with this "simple" network of two queues in tandem appears in [27]. Limited examples of queueing networks with blocking having product form equilibrium probabilities were also treated by Lam [15], Pittel [28], Kelly [11], and Hordijk and Van Dijk [6].

Again the question arises over the existence and nature of the common structure that characterizes these networks and the networks previously studied by Jackson, Gordon and Newell and others. Why do some queueing networks with blocking have a product form solution while others do not? What is the basic underlying structure that leads to product form solution? Can the class of Markovian queueing networks with product form solutions be completely characterized? These are the type of questions examined in this paper.

The issues raised above have been studied in the past by many investigators. Lam [15] reported results that extended the classical BCMP networks by using the method of local balance. Chandy et al. [4] attempted a complete characterization of queueing networks with product form solutions and related this property to the notion of "station balance". The fruitful study of reversibility [11], a quantitative property of the state transition diagram first analyzed by Kolmogorov, seemed to point to an algebraic solution. Kelly [12] has considered the $M \Rightarrow M$ property previously introduced by Muntz [24] and the resulting product form solution in a more general context of an interconnected network of quasireversible nodes. Walrand [36] presented a detailed analysis and proposed probabilistic arguments to explain the product form, the output theorems, the distribution at the jumps, and the Poisson character of the flows in order to provide a more intuitive justification of those properties.

The approach adopted in this paper is primarily of a geometric nature. The study of simple queueing systems consisting of one or two queues reveals that the topological structure of the state transition diagram is intimately connected with the product form solution of the equilibrium probabilities (see section II). Consequently, a geometric interpretation of the state transition diagram and its associated global balance equations in higher dimensions is pursued. The state transition diagram of an arbitrary Markovian queueing network can be decomposed along certain basic geometric shapes (building blocks). Taking these blocks in isolation a set of simple global balance equations can be written. These equations represent, for the original system, a possible set of partial balance equations. Geometrically, the original state transition diagram can be reconstructed by pasting the building blocks together. Algebraically, this procedure corresponds to a summation of the partial balance equations resulting in the set of global balance equations. There is no guaranty, however, that by following the approach delineated above the partial balance equations are consistent, *i.e.*, have a solution. The necessary and sufficient condition for consistency is found. Using this condition in some particular cases of interest, the topology of the state transition diagram can be qualitatively and quantitatively modified so that product form solutions result. Quantitative behavior on the boundary of the state transition diagram can, for example, lead to product form.

This paper is organized as follows. In section II, starting with simple examples of queueing systems consisting of one or two queues, it is shown that the product form is a result of the geometry of certain lattice structures within the state transition diagram. Replication of the same basic cell structure leads to the product form solution of the equilibrium probabilities. If the cell structure is repeated while assigning different transitions to the arcs, the consistency of the resulting system of partial balance equations is no longer assured. The necessary and sufficient consistency condition for this is given in section III (Theorem 1). For simplicity, this condition is defined using a consistency graph. In section IV a constructive process of *geometric replication* is introduced. This topological construction permits the design of a very general class of protocols (see section IV). A systematic methodology to find state transition diagrams

with product form equilibrium probabilities is given in Theorem 2. Using this framework it is shown how to find the invariant equilibrium probabilities for a large class of networks used in practice. In section V the explicit evaluation algorithm for the equilibrium probabilities is discussed. It is also shown how state transition diagrams can be changed so as to give rise to protocols with product form equilibrium probabilities of the type investigated in section IV. Finally, a deadlock prevention mechanism is also presented.

II. The Geometric Structure of the State Transition Diagram

In what follows, the geometric structure of the state transition diagram for Markovian queueing networks with an arbitrary number of nodes and packets will be studied. Our analysis begins with a queueing network with only one node. Using an inductive geometric approach the equilibrium probabilities for the system with N packets is found. Subsequently the analysis is extended to a queueing network with 2 nodes. The general results for networks with M nodes and an arbitrary number of packets are formulated in sections III and IV.

Consider a queueing system consisting of one queue with an exponential server and non-homogeneous Poisson arrivals as shown in Fig. 1a. The above queueing system is completely described by a Markov process $Q=(Q_t), t \in R_+$, with arrival and departure rates $\lambda=(\lambda_{k_1})$ and $\mu=(\mu_{k_1})$, where k_1 takes values in a state space E, with $E=\{1,2,\cdots,N\}$, N being the maximum buffer size. The corresponding state transition diagram for this system containing 1, 2 and N packets is given in Fig. 1b, 1c and 1e, respectively.

If only one packet is permitted into the network the product form solution naturally applies. For, if $N=1$ the balance equation at node {1} (see Fig. 1b) can be written:

$$\mu_1 p_1 = \lambda_0 p_0 \tag{1}$$

and therefore, using the conservation law of probabilities, $p_0 + p_1 = 1$, the equilibrium probabilities p_1 and p_0 are easily obtained.

Let us add a new state {2} to the state transition diagram as in Fig. 1c. This corresponds to the case where the buffer is of size 2 ($N=2$). By abuse of notation, the balance equation at node {2} is given by

$$\mu_2 p_2 = \lambda_1 p_1 \quad . \tag{2}$$

Therefore, the global balance equation at node {1} (see Fig. 1c)

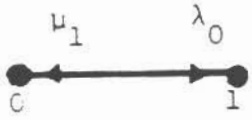
$$\lambda_0 p_0 + \mu_2 p_2 = (\lambda_1 + \mu_1) p_1 \quad , \tag{3}$$

can be decomposed into two partial balance equations

$$\lambda_0 p_0 = \mu_1 p_1 \tag{4}$$

and (2). From equations (2) and (4) the equilibrium probabilities p_1 and p_2 can be easily found as a function of p_0. Finally, the normalization factor p_0 can be derived from the conservation law of probabilities, $\sum_{k=0}^{2} p_k = 1$. Note that equation (1) representing the *global* balance equation at node {1} for the system with only one buffer is equivalent to the *partial* balance equation (4) for the system with two buffers at the same node. Therefore, when compared with the system with buffer size $N=1$, the

(a)



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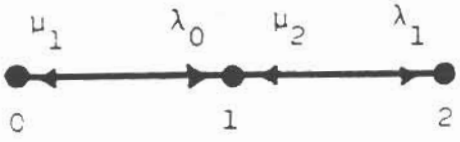
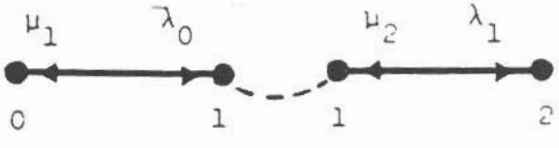


(c)



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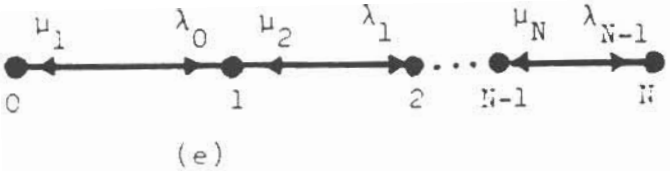
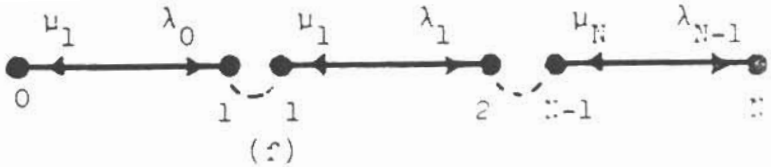


(e)



(f)

Fig. 1 A Markovian queueing system and its transition diagram.

overall effect of introducing the additional state {2} (Fig. 1c) on p_1 is merely the change of the scaling factor p_0.

The partial balance equations (2) and (4) can be also obtained by geometrically partitioning the transition diagram across the network state {2}. The procedure is demonstrated in Fig. 1d. The resulting building blocks of the state transition diagram are, then, simple arcs. In what follows these arcs will be referred to as one-dimensional cells (1-cells). By writing the *global* balance equations for this geometrical shapes in isolation, one obtains the equations (1) and (2), *i.e.*, the *partial* balance equations (2) and (4) for the system of Fig. 1c. A summation of the latter leads to the global balance equation of the original system (3).

In view of the above analysis, the state transition diagram of the queueing system with N packets (Fig. 1e) can be decomposed into one-dimensional cells (see Fig. 1f). The global balance equations for these cells are:

$$\lambda_{k-1}p_{k-1} = \mu_k p_k \quad , \tag{5}$$

for all k, $1 \le k \le N$. The set of equations (5) also represents a set of partial balance equations of the original state transition diagram (Fig. 1e). The intuitive meaning of these equations is that the flow into a state due to an arrival is equal to the flow out of that state due to a departure.

When pasting these cells together one obtains the original state transition diagram. Algebraically, this corresponds to a summation of the partial balance equations (5) resulting in the global balance equations of the original system. For example by adding the corresponding partial balance equations at an interior node l one has:

$$\lambda_{l-1}p_{l-1} + \mu_{l+1}p_{l+1} = (\lambda_l + \mu_l)p_l \quad , \tag{6}$$

for all l, $1 \le l \le N-1$. Although we have shown that by summing the partial balance equations we obtain the global balance equations that by no means guarantees that the system (5) is consistent, that is, it has a solution.

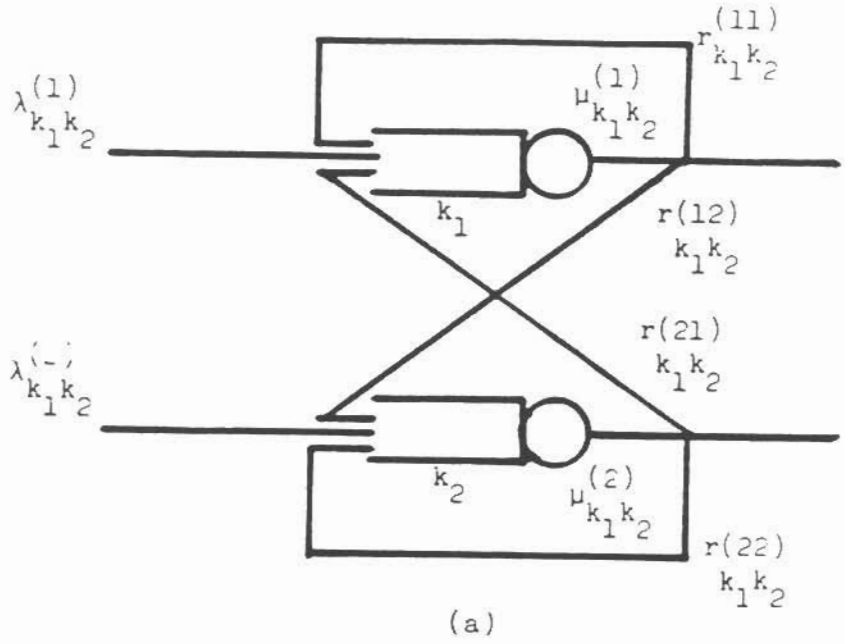
In the simple case studied above, the consistency of the system (5) of linear equations can be directly verified. The partial balance equations are in product form and:

$$p_k = \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} p_0 \quad , \tag{7}$$

for all k, $1 \le k \le N$. Finally, the normalization constant p_0 can be obtained from the conservation law of probabilities, *i.e.*, $\sum_{k=0}^{N} p_k = 1$.

Remark. The geometric method introduced above represents a reinterpretation of some well known results for the M/M/1 queue. A discussion of the above classical results and its connection to reversibility can be found in Kelly [11]. The key to the analysis above is the decomposition of the state transition diagram into its building blocks. The algebraic generalization of the previous proof for general queueing networks, although straightforward in its outline, is more difficult to grasp without the geometric interpretation.

We are now ready to generalize these observations to a more generic network of two queues. Our analysis will again be very detailed. For simplicity the queueing network is represented in Fig. 2a below. The Markovian process $Q=(Q_t^{(1)}, Q_t^{(2)})$, $t \in R_+$, representing the queueing system in Fig. 2a has arrival and departure rates $\lambda=(\lambda_{k_1k_2}^{(1)}, \lambda_{k_1k_2}^{(2)})$ and $\mu=(\mu_{k_1k_2}^{(1)}, \mu_{k_1k_2}^{(2)})$ and state dependent routing probabilities $r=(r_{k_1k_2}^{(ij)})$, $1 \le i,j \le 2$, $(k_1,k_2) \in E_1 \times E_2$, with $E_i=\{1,2,\cdots,N_i\}$, where N_i is the maximum



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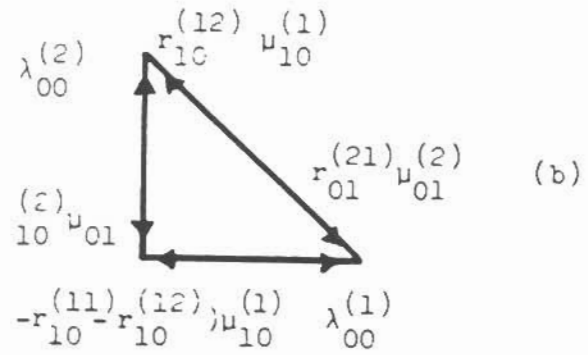

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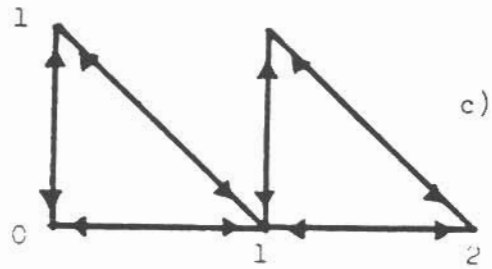
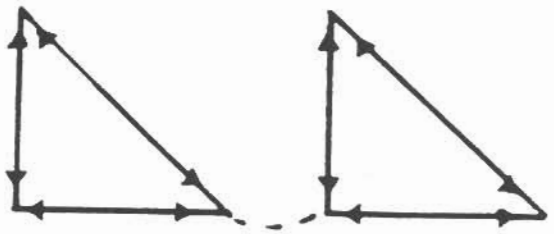


c)



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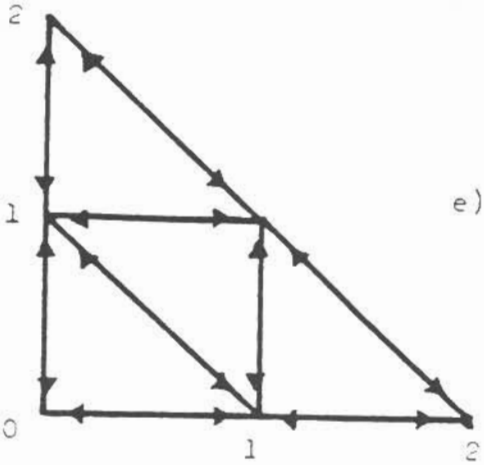

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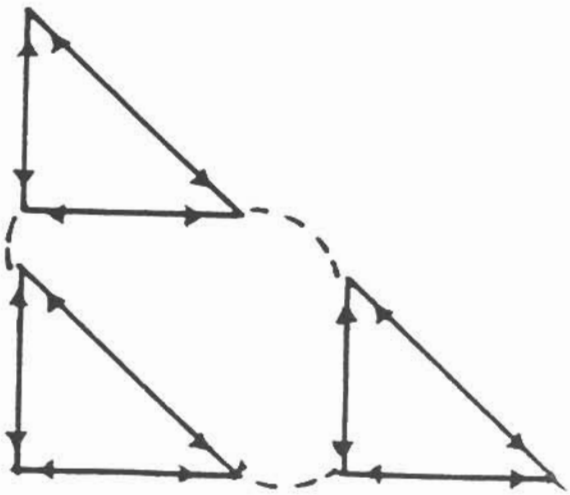


f)

Fig. 2 A Markovian queueing network consisting of two ...

buffer size at queue i, $i=1,2$. In what follows our attention will be focussed on the essential cell (or building block) that leads to product form for two dimensional state transition diagrams.

Consider first a state transition diagram consisting of a triangle in isolation with the nodes at (0,0), (1,0) and (0,1). This geometric shape will be referred to as a two-dimensional cell or simply a 2-cell (see Fig. 2b). Note that the choice of the geometry of the 2-cell is very natural since it comes with a simple queueing theoretical interpretation. It corresponds to a protocol accepting at most one packet into the network. The expressions for the equilibrium probabilities at (1,0) and (0,1) are straightforward to derive. These are:

$$r_{01}^{(21)}\mu_{01}^{(2)}p_{01} + \lambda_{00}^{(1)}p_{00} = (1-r_{10}^{(11)})\mu_{10}^{(1)}p_{10} \quad . \tag{8}$$

and

$$r_{10}^{(12)}\mu_{10}^{(1)}p_{10} + \lambda_{00}^{(2)}p_{00} = (1-r_{01}^{(22)})\mu_{01}^{(2)}p_{01} \quad . \tag{9}$$

and finally at node (0,0) we have

$$(1-r_{01}^{(21)}-r_{01}^{(22)})\mu_{01}^{(2)}p_{01} + (1-r_{10}^{(11)}-r_{10}^{(12)})\mu_{10}^{(1)}p_{10} = (\lambda_{00}^{(1)} + \lambda_{00}^{(2)})p_{00} \quad . \tag{10}$$

where $r=(r_{k_1k_2}^{(ij)})$, $1\le i,j\le 2$, is the set of state dependent routing probabilities. This leads to

$$p_{10} = \frac{\vartheta_{00}^{(1)}}{\mu_{10}^{(1)}}p_{00} \quad . \tag{11}$$

$$p_{01} = \frac{\vartheta_{00}^{(2)}}{\mu_{01}^{(2)}}p_{00} \quad . \tag{12}$$

where

$$\vartheta_{00}^{(1)} = \lambda_{00}^{(1)} + r_{10}^{(11)}\vartheta_{00}^{(1)} + r_{01}^{(21)}\vartheta_{00}^{(2)} \tag{13}$$

and

$$\vartheta_{00}^{(2)} = \lambda_{00}^{(2)} + r_{10}^{(12)}\vartheta_{00}^{(1)} + r_{01}^{(22)}\vartheta_{00}^{(2)} \quad . \tag{14}$$

respectively. Equations (13) and (14) will be called in the sequel *generalized* traffic flow equations. For the case of the Jacksonian network, $\vartheta=(\vartheta_{k_1,k_2}^{(j)})$, $1\le j\le M$, $(k_1,k_2)\varepsilon E_1\times E_2$, represents the average available load on each of its queues [22]. The renormalization constant p_{00} can be computed using the equations (11), (12) and the law of conservation of probabilities.

Now consider the addition of a second 2-cell at (1,0), (2,0), (1,1) or at (0,1), (1,1), (0,2) to the previous 2-cell. The resulting state transition diagram when the (1,0), (2,0), (1,1) cell is added, is shown in Fig. 2c. Both cells can be obtained by a geometric translation of the cell located at (0,0), (1,0), (0,1).

When considered in isolation, the associated global balance equations with the cell (k_1,k_2), (k_1+1,k_2), (k_1,k_2+1) are given by:

$$r_{01}^{(21)}\mu_{01}^{(2)}p_{k_1,k_2+1} + \lambda_{00}^{(1)}p_{k_1,k_2} = (1-r_{10}^{(11)})\mu_{10}^{(1)}p_{k_1+1,k_2} \quad . \tag{15}$$

and

$$r_{10}^{(12)}\mu_{10}^{(1)}P_{k_1+1,k_2} + \lambda_{00}^{(2)}P_{k_1,k_2} = (1-r_{01}^{(22)})\mu_{01}^{(2)}P_{k_1,k_2+1} \quad . \tag{16}$$

and finally at node (k_1,k_2) we have

$$(1-r_{01}^{(21)}-r_{01}^{(22)})\mu_{01}^{(2)}P_{k_1,k_2+1} + (1-r_{10}^{(11)}-r_{10}^{(12)})\mu_{10}^{(1)}P_{k_1+1,k_2} = (\lambda_{00}^{(1)} + \lambda_{00}^{(2)})P_{k_1,k_2} \quad . \tag{17}$$

The solution to the set of global balance equations above is given by

$$P_{k_1+1,k_2} = \frac{\theta_{00}^{(1)}}{\mu_{10}^{(1)}}P_{k_1,k_2} \quad . \tag{18}$$

and

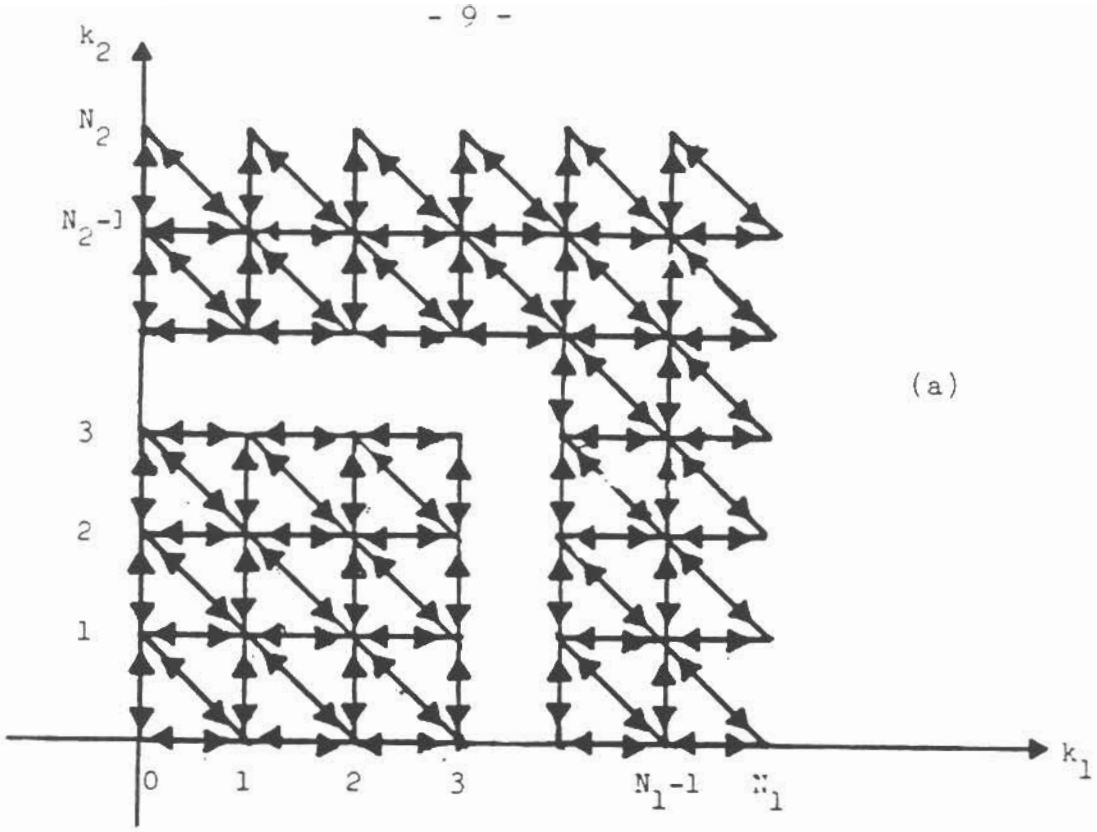
$$P_{k_1,k_2+1} = \frac{\theta_{00}^{(2)}}{\mu_{01}^{(2)}}P_{k_1,k_2} \quad . \tag{19}$$

By pasting to the cell located at (0,0), (1,0) and (0,1) a cell at (1,0), (2,0), (1,1) the equilibrium probabilities given by the equations (18) and (19) remain valid (for $(k_1,k_2)=(1,0)$). To see this, note that the global balance equations for the two cells at the nodes (2,0), (1,1) and (0,1), (0,0) are identical with the global balance equations of the original state transition diagram. By summing the global balance equations at the node (1,1) for the two cells taken in isolation, one obtains the global balance equations of the original state transition diagram. Therefore, the *global* balance equation of the two cells in isolation represent a set of *partial* balance equations of the original state transition diagram. Using equations (11), (12), (18) and (19) with $(k_1,k_2)=(1,0)$, the equilibrium probabilities of the appended 2-cell can be recursively calculated as a function of p_{00}. As before, p_{00} is computed using the equations (11), (12), (18), (19) and the conservation law of probabilities. This proves the consistency of the set of partial balance equations.

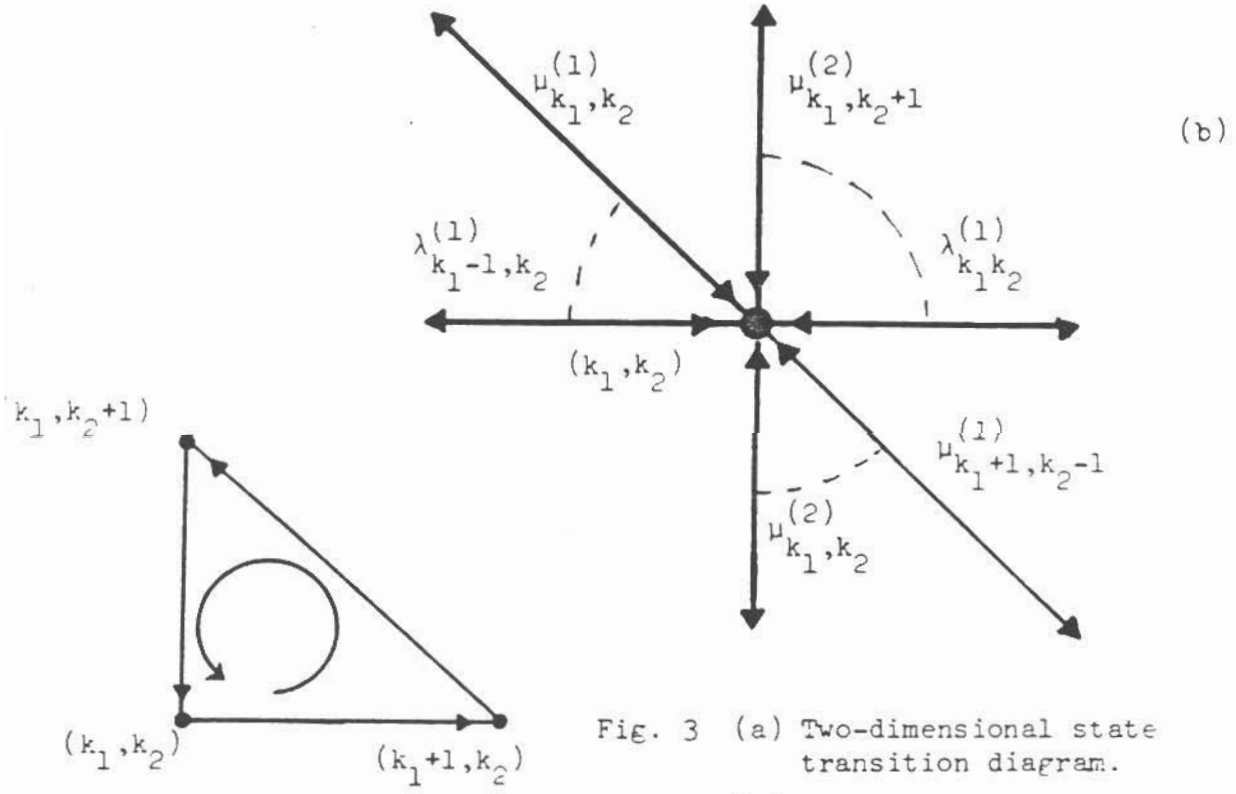
Let us add a third triangle to the previous state transition diagram at (0,1), (1,1), (0,2) as in Fig. 2e. In the case of a network consisting solely of one queue we did not have to consider the case with three cells. The analysis of this case is necessary since the state transition diagram is described by two states. By pasting the building blocks together, the global balance equations at the nodes (1,1) and (0,1) are obtained by summing the global balance equations of the cells taken in isolation (Fig. 2f). The equilibrium probabilities of the appended 2-cell can be found using the equations (18) and (19) with $(k_1,k_2)=(0,1)$.

It is not difficult to show that if, in general, the state transition diagram is obtained by translating the basic 2-cell (0,0), (1,0), (0,1), the global balance equations can be easily decomposed into a consistent set of partial balance equations. To derive this note first that $\lambda_{k_1,k_2}^{(1)}=\lambda_{00}^{(1)}$, $\mu_{k_1+1,k_2}^{(1)}=\mu_{10}^{(1)}$, and $\mu_{k_1,k_2+1}^{(2)}=\mu_{01}^{(2)}$, for all pairs (k_1,k_2), $(k_1,k_2)\varepsilon E_1\times E_2$. All other transitions are zero. Fig. 3a corresponds to the state transition diagram for the situation of Fig. 2 when the queue buffers are of arbitrary size (N_1 and N_2, respectively). It should be noted that this diagram consists exclusively of an aggregation of 2-cells. Such an aggregate is referred to as a complex [21]. This state transition diagram is a modified version of the one usually associated with two tandem queues with finite buffers. It differs in that upper and side transitions have been removed. This aspect will be further discussed in section V (Fig. 14).

At an interior node of Fig. 3a, the transitions can be paired in the manner shown in Fig. 3b. Equations (15), (16), and (17) apply for any 2-cell in isolation. That is, by



(a)



(b)

Fig. 3 (a) Two-dimensional state transition diagram.

(b) In and outgoing transitions at an interior node (k_1,k_2)

summing up at an interior node of the state transition diagram the global balance equations of the 2-cells adjacent to (k_1,k_2) in isolation, one obtains the global balance equation at node (k_1,k_2). Similar equations can be obtained at boundary points. This proves that the geometrical decomposition of the state transition diagram into its building blocks corresponds to an algebraic decomposition of the global balance equations into a set of partial balance equations.

By direct computation one can easily find that the solution to the set of partial balance equations (18), (19) is:

$$p_{k_1,k_2} = \prod_{i_1=0}^{k_1} \left(\frac{\vartheta_{00}^{(1)}}{\mu_{10}^{(1)}}\right)^{i_1} \cdot \prod_{i_2=0}^{k_2} \left(\frac{\vartheta_{00}^{(2)}}{\mu_{01}^{(2)}}\right)^{i_2} p_{00} \quad , \tag{20}$$

for all $(k_1,k_2)\varepsilon E_1\times E_2$. The renormalization constant p_{00} is computed from the equations (20) and the conservation law of probabilities.

Remark. To illustrate the generality of our results obtained thus far, consider now two queues in tandem as depicted in Fig. 4a. The Markovian process $Q=(Q_t^{(1)}, Q_t^{(2)})$, $t\varepsilon R_+$, representing the queueing system above has arrival and departure rates $\lambda=(\lambda_{k_1,k_2}^{(1)}=\lambda_{00}^{(1)}, \lambda_{k_1,k_2}^{(2)}=0)$, $\mu=(\mu_{k_1+1,k_2}^{(1)}=\mu_{10}^{(1)}, \mu_{k_1,k_2+1}^{(2)}=\mu_{01}^{(2)})$, and routing matrix $r=(r_{k_1,k_2}^{(i)}{}_{j}=1$ for $i=1$, $j=2$, and $r_{k_1,k_2}=0$ for $i=2$, $j=1)$, $(k_1,k_2)\varepsilon E_1\times E_2$, with $E_i=\{1,2,\cdots,N_i\}$, where N_i is the maximum buffer size at queue i, $i=1,2$. All other transitions are zero. Thus the queueing system is not reversible, since the "arrows" of the state transition diagram point only in a single direction.

The intuitive meaning of the equations (8), (9) and (10) is illustrated in Figs. 4b, 4c and 4d. For instance, equation (8)

$$\mu_{10}^{(1)} p_{k_1+1,k_2} = \lambda_{00}^{(1)} p_{k_1,k_2} \quad , \tag{21}$$

corresponds to Fig. 4b. In an interior state, the flow into the state due to an arrival at the first queue equals the flow out of the state due to a departure from the same queue. Similarly, the second equation (9)

$$\mu_{10}^{(1)} p_{10} = \mu_{01}^{(2)} p_{01} \quad , \tag{22}$$

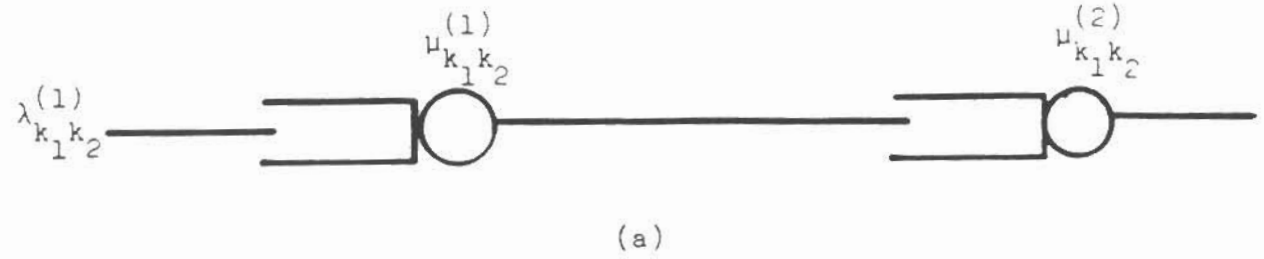
describes the equality of the flow into an interior state due to an arrival at the second queue and the flow out of that state due to a second queue departure (see Fig. 4c). Finally, equation (10) (see Fig. 4d)

$$\mu_{01}^{(2)} p_{01} = \lambda_{00}^{(1)} p_{00} \quad , \tag{23}$$

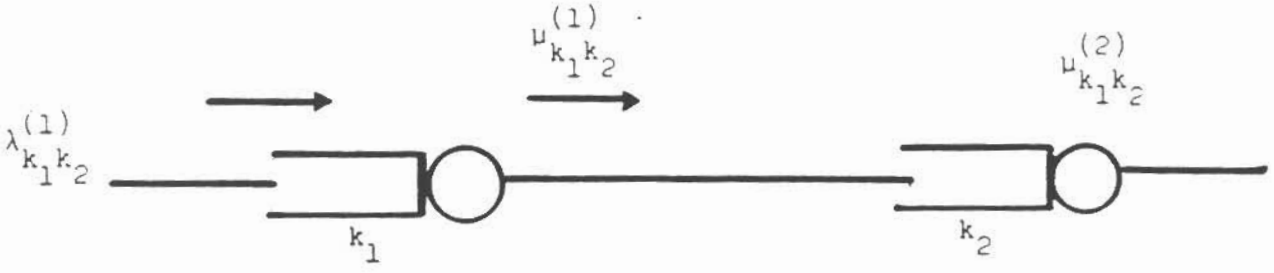
describes the equality of the flow into an interior state due to a second queue departure and the flow out of the state due to a first queue arrival. This last situation can be visualized in terms of a third unseen queue whose input consists of second queue departures (sink) and whose output serves as the first queues input (source). Note that the equilibrium probabilities have a product form although the equations (21), (22) and (23) are *not* detailed or local balance equations (as discussed in [11] or [31]). The latter require double transitions (or arrows) on each 1-dimensional cell.

In the previous analysis we have shown that if the *same* basic 2-cell is replicated the equilibrium probabilities have a product form. What if the 2-cell structure is replicated but the associated flows are changed? Will the product form solution again apply? The answer is simply *no* as the following simple counterexample shows.

Example 1. Consider a state transition diagram consisting of three 2-cells as shown in Fig. 2e. Assume that the transitions of the two lower cells are the same. The

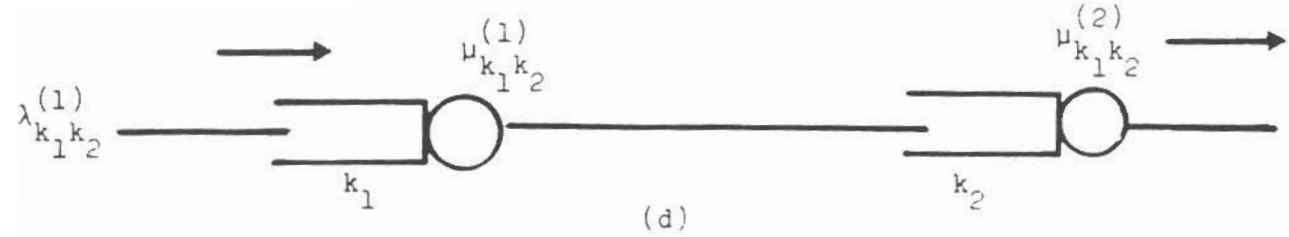


(a)



(b)

(c)



(d)

Fig. 4 Queueing theoretical interpretation of the two-dimensional state transition diagram.

transitions of the cell located at (0,2), (0,1) and (1,1) are taken to be different from the previous two. It is not difficult to see that the partial balance equations (15), (16), and (17) (with $(k_1,k_2)=(0,0), (1,0), (1,1)$) are not necessarily consistent. Note, for example, that the probability at node (1,1) can be computed as a function of the reference probability p_{00} either using the path (1,1), (1,0), (0,0) or (1,1), (0,1), (0,0). Since the arcs (1,1), (0,1) and (1,1), (1,0) belong to two different cells, different results for the same probability can be obtained! A numerical example to this effect appears in [30]. The question of consistency, therefore, previously mentioned in [2], has to be stated and solved in a precise mathematical setting.

III. The Consistency Graph

Recall that the approach adopted in the previous section calls for decomposing the original state transition diagram into its building blocks and for solving the associated partial balance equations. In the two dimensional case, when pasting the cells together we have noted the existence of distinct paths connecting some arbitrary nodes with the reference node. This property was not observed in the one dimensional case since the state transition diagram is a tree and hence a unique path links any node to the reference node.

Thus, the problem of consistency of the set of partial balance equations arises as a result of the existence of distinct paths from an arbitrary node to the reference node. If, by taking alternate paths, the "equilibrium probabilities" for the same node are different, the set of partial balance equations is not consistent.

Consider a Markovian network consisting of M queues. In the general case the queueing process $Q=(Q_t^{(1)}, Q_t^{(2)}, \cdots, Q_t^{(M)})$, $t\varepsilon R_+$, has the arrival and departure rates $\lambda=(\lambda^{(1)}_{k_1k_2\cdots k_M}, \lambda^{(2)}_{k_1k_2\cdots k_M}, \cdots, \lambda^{(M)}_{k_1k_2\cdots k_M})$ and $\mu=(\mu^{(1)}_{k_1k_2\cdots k_M}, \mu^{(2)}_{k_1k_2\cdots k_M}, \cdots, \mu^{(M)}_{k_1k_2\cdots k_M})$, $(k_1,k_2,\cdots,k_M)\varepsilon E_1\times E_2\times\cdots\times E_M$ with $E_i=\{1,2,\cdots,N_i\}$, where N_i is the maximum buffer size for all i, $1\le i\le M$.

The equilibrium probabilities are the solution of the set of global balance equations:

$$\sum_{j=1}^{M}\lambda^{(j)}_{k_1k_2\cdots k_M}P_{k_1k_2\cdots k_M}+\sum_{j=1}^{M}(1-\eta^{(j)}_{k_1k_2\cdots k_M})\mu^{(j)}_{k_1k_2\cdots k_M}P_{k_1k_2\cdots k_M}=$$

$$\sum_{j=1}^{M}\lambda^{(j)}_{k_1k_2\cdots k_j-1\cdots k_M}P_{k_1k_2\cdots k_j-1\cdots k_M}+$$

$$\sum_{i=1}^{M}\sum_{\substack{j=1\\ j\neq i}}^{M}\eta^{(i)}_{k_1k_2\cdots k_i+1\cdots k_j-1\cdots k_M}\mu^{(j)}_{k_1k_2\cdots k_i+1\cdots k_j-1\cdots k_M}P_{k_1k_2\cdots k_i+1\cdots k_j-1\cdots k_M}+$$

$$\sum_{j=1}^{M}(1-\sum_{i=1}^{M}\eta^{(i)}_{k_1k_2\cdots k_j+1\cdots k_M})\mu^{(j)}_{k_1k_2\cdots k_j+1\cdots k_M}P_{k_1k_2\cdots k_j+1\cdots k_M} \qquad (24)$$

for all $(k_1,k_2,\cdots,k_M)\varepsilon E_1\times E_2\times\cdots\times E_M$.

As in the lower dimensional case, the set of global balance equations (24) will be decomposed into a set of partial balance equations that can be solved in a straightforward manner. Such a decomposition is, however, not unique. In addition, it is not clear whether the resulting set of partial balance equations is consistent (see Example

1, section II). If the partial balance equations are consistent, however, they are equivalent with the global balance equations. This is because the latter has a unique solution.

To derive the necessary and sufficient conditions for the consistency of a set of partial balance equations, we define the following *consistency graph* [18].

Definition 1. A consistency graph is an oriented graph topologically equivalent with the original state transition diagram. In addition, it has the property that the probability at each node is equal to the probability of any adjacent node multiplied with the value of the associated edge connecting the two nodes.

The class of consistency graphs of interest to us very often derives the algebraic values associated with the arcs directly from the set of partial balance equations. Thus, the consistency graph can be seen as an "easy to read" graphical representation of the *proposed solution* for the partial balance equations (see for example equation (32), section IV).

Example 2. Consider the graph given in Fig. 5a. It corresponds to a Markovian queueing system with a processor sharing discipline having a server with a two stage Erlangian distribution. The example is taken from [31] pp. 67-70. The states are defined by the pair (k_1,k_2), where there are k_1 packets in the first stage and k_2 packets in the second stage. A set of partial balance equations can be computed for each two dimensional cell using equations (17), (18), and (19). These equations can be extracted from the consistency graph shown in Fig. 5b. For example, the relationship between the probabilities attached to nodes (1,1) and (2,1) is given by:

$$p_{21} = \frac{3\lambda}{2\mu^{(1)}} p_{11} \quad . \tag{25}$$

Choose any path (open or closed) on the consistency graph. Associated with it is an algebraic value called the *product* of the arcs or simply the product. For example, with the path (1,2), (1,1), (0,1) and (0,0) is associated the product:

$$\frac{2\mu^{(2)}}{3r^{(12)}\lambda} \frac{\mu^{(1)}}{2\lambda} \frac{\mu^{(2)}}{r^{(12)}\lambda} \quad . \tag{26}$$

Note that reversing the orientation of an arc of the consistency graph with value a results in an arc with value a^{-1}. We can now state the following consistency theorem [18].

Theorem 1. *A system of partial balance equations is consistent iff any closed path of the consistency graph has the product equal to one.*

Proof: Assume that there is a closed path in the consistency graph such that its product is not equal to one. Therefore, for at least one node belonging to this path, there are two distinct arcs leading to the reference node which do not have the same product. Thus two different values for the probability associated with such a node can be obtained. Hence, the partial balance equations are not consistent. The sufficiency part of the theorem can be shown by direct computation. The product of an arbitrary path connecting node $(k_1,k_2,\cdots,k_M)$ with the reference node $(0,0,\cdots,0)$ is an algebraic invariant. This is because, by assumption the product of any closed path is equal to one and reversing the orientation of any arc with value a results in an arc with value a^{-1}. The probability at node $(k_1,k_2,\cdots,k_M)$ is, therefore, equal to the product of the path connecting this node with the reference node times the probability of the reference node. •

Hence, to determine the consistency of the set of partial balance equations, the product of any closed path of the consistency graph has to be computed and compared

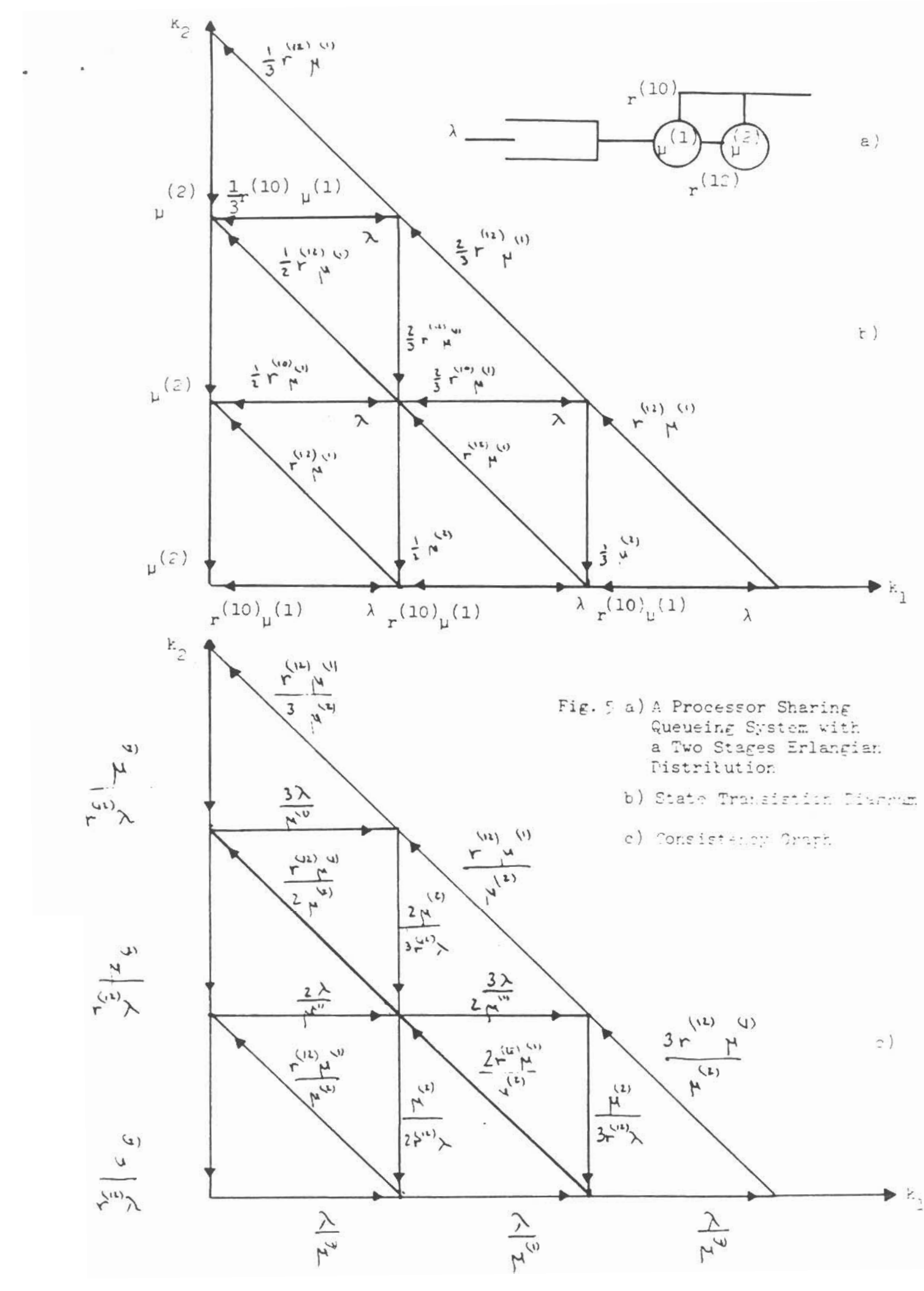


Fig. 5 a) A Processor Sharing Queueing System with a Two Stages Erlangian Distribution

b) State Transition Diagram

c) Consistency Graph