# BAYESIAN ESTIMATION OF CHIRPLET SIGNALS BY MCMC SAMPLING

*Chung-Chieh Lin and Petar M. Djurić*

Department of Electrical and Computer Engineering,
State University of New York at Stony Brook
Stony Brook, NY 11794, USA
email: clin@ece.sunysb.edu, djuric@ece.sunysb.edu

## ABSTRACT

We address the problem of parameter estimation of chirplets, which are chirp signals with Gaussian shaped envelopes. The procedure we propose is an extension of our previous work on estimation of chirp signals [5], and it is based on MCMC sampling. For fast convergence of the MCMC sampling based method, a critical step is the initialization of the method. Since the chirplets have finite durations and may or may not overlap in time, we propose initialization procedures for each of these cases. We have tested the method by extensive simulations and compared it with Cramer-Rao bounds. The obtained results have been excellent.

## 1. INTRODUCTION

Chirp signals appear in many phenomena of interest in science and engineering including physics, radar and sonar. For instance, in radar applications the chirp signals contain information about distance and velocity of moving objects. Furthermore, human voice and sounds of animals and insects can be decomposed into a series of chirp signals, and based on the decomposition, they can readily be classified. The main characteristic of the chirp signals is the linear change of their instantaneous frequencies, and therefore they have often been used in representing signals with time varying spectra. Parameter estimation of chirp signals has been of great interest in the past, and a wide variety of estimation procedures have been proposed and studied in great detail [2, 5].

Recently in the literature, increased attention has been given to chirplet signals, which are chirp signals that have Gaussian-shaped envelopes [1, 3, 6, 7]. They encompass important signal transforms, such as the short-time Fourier transform and the wavelet transform, and also, they represent the most general signal for which the Wigner distribution is positive throughout the time-frequency plane [2]. Clearly, the chirplets have more variables than the standard chirp signals, and therefore the parameter estimation of chirplets is more complicated.

Important work in developing estimation methods for chirplets is presented in [1] and [6], where the parameters of chirplet signals are limited to discrete values. They are easier to implement, but do not always fit the underlying signals well. The principles of maximum likelihood estimation were used in the sub-optimal method proposed in [7]. In [3] Markov Chain Monte Carlo (MCMC) sampling [8] was exploited for the estimation of narrowband chirplet signals. An adaptive chirplet based signal approximation was proposed in [9].

This work was supported by the National Science Foundation under Award No. CCR-9903120.

Here we extend some of our previous work on estimating chirp signals by the use of MCMC sampling [5]. MCMC based methods produce samples that are drawn from a target distribution, which is usually the posterior distribution of the parameters of interest. These samples can then be used to compute estimates based on various types of cost functions, or in general, to approximate the posterior itself. An important part of the procedure is its initialization. Initialization is particularly important for the problem of parameter estimation of chirp and chirplet signals because the posterior densities of the parameters of these signals are multimodal with many narrow peaks. In this paper we describe a scheme for initialization of the MCMC sampling that prevents trapping of the chain around local maxima of the posterior. Simulation results that illustrate the performance of the method are also provided.

## 2. BAYESIAN PARAMETER ESTIMATION OF CHIRPLET SIGNALS

The observed discrete-time sequence that is composed of $K$ chirplets embedded in noise is given by

$$y[n] = \sum_{k=1}^{K} a_k s_k[n] + w[n] \tag{1}$$

where

$$s_k[n] = e^{-\frac{\beta_k(n-\tau_k)^2}{2} + j\frac{\alpha_k(n-\tau_k)^2}{2} + j\omega_k(n-\tau_k)} \tag{2}$$

and $n = n_0, ...., n_0 + N - 1$ and $0 < \alpha_k, \omega_k \le 2\pi$. The parameters $\tau_k$, $\omega_k$ and $\alpha_k$ are the time center, frequency and frequency rate of the $k$-th chirplet, respectively, and $\beta_k$ controls the duration of the $k$-th chirplet. Each chirplet can, thus, be described by the vector of parameters $\theta_k = [a_k, \beta_k, \tau_k, \alpha_k, \omega_k]$. The amplitude of the $k$-th chirplet, $a_k$ is a complex number, and $w[n]$ is a complex white Gaussian noise (CWGN) with zero mean and variance $\sigma_w^2$, whose real and imaginary components are identically distributed. We see that the chirp signal is a special case of the chirplet, which is obtained by setting $\beta = 0$.

Our approach to estimating $\theta$ is based on the Bayesian methodology, and so we are interested in the posterior distribution of $\theta$. Its relation to the prior and the likelihood is given by

$$p(\theta|\mathbf{y}) \propto p(\mathbf{y}|\theta)p(\theta) \tag{3}$$

where $\theta = [\theta_1, \theta_2, ..., \theta_K, \sigma_w^2]$. We use uniform priors for $\{\beta_k, \tau_k, \alpha_k, \omega_k\}$, and the improper noninformative priors for $a_k$ and $\sigma_w^2$, i.e., $p(a_k) \sim$ const. and $p(\sigma_w^2) \sim 1/\sigma_w^2$.



0-7803-7041-4/01/$10.00 ©2001 IEEE

We apply the Bayesian approach by MCMC sampling. A Markov chain is generated whose target distribution is the multivariate posterior distribution, and it is implemented by sampling from lower-dimensional distributions. Once we obtain the samples from the posterior distribution, we can easily construct various point estimators.

The posterior distribution, $p(\theta|\mathbf{y})$, is a highly complicated function. We sample $\mathbf{a}$, $\sigma_w^2$ by Gibbs sampling and $\{\beta_k, \tau_k, \omega_k, \alpha_k\}_{k=1,\ldots,K}$ by the Metropolis-Hasting method. If $\tilde{\theta}$ denotes the most recent values of the parameters $\theta$ in the Markov chain, and $\tilde{\theta}_{-.}$ represents all the parameters $\tilde{\theta}$ except the ones next to the minus sign, the procedure can be described as follows: set $i = 0$ and perform the iterations along the following steps:

1. sample $\beta_k^{(i+1)} \sim p(\beta_k|\mathbf{y}, \tilde{\theta}_{-\beta_k})$
2. sample $\alpha_k^{(i+1)} \sim p(\alpha_k|\mathbf{y}, \tilde{\theta}_{-\alpha_k})$
3. sample $\omega_k^{(i+1)} \sim p(\omega_k|\mathbf{y}, \tilde{\theta}_{-\omega_k})$
4. sample $\tau_k^{(i+1)} \sim p(\tau_k|\mathbf{y}, \tilde{\theta}_{-\tau_k})$
5. jointly sample $\omega_k^{(i+1)}, \tau_k^{(i+1)} \sim p(\omega_k, \tau_k|\ \mathbf{y}, \tilde{\theta}_{-\omega_k, -\tau_k})$ along the line given by (9) and discussed in the next section, and set $k = k + 1$
6. repeat steps 1 to 5, until $k = K$
7. sample $\mathbf{a}^{(i+1)} \sim p(\mathbf{a}|\mathbf{y}, \tilde{\theta}_{-\mathbf{a}})$
8. sample $\sigma_w^{2(i+1)} \sim p(\sigma_w^2|\mathbf{y}, \tilde{\theta}_{-\sigma_w^2})$
9. repeat steps 1-8 $M$ times.

We sample $\alpha_k, \beta_k, \omega_k$ and $\tau_k$ in the first five steps, using the random walk Metropolis-Hasting algorithm with Taylor expansion around the current parameter values. Since the posterior $p(\mathbf{a}|\mathbf{y}, \theta_{-\mathbf{a}})$ is a Gaussian distribution, we can sample $\mathbf{a}$ easily. As for $\sigma_w^2$, its posterior is an inverted Gamma distribution, and the sampling of $\sigma_w^2$ is also easy.

## 3. CRITICAL ISSUES

There are two issues of great importance to our approach to estimating the parameters of the chirplet signals. One is the effect of unknown timings of the chirplets on the convergence of the MCMC, and the other is the presence of highly peaked local maxima in the chirpograms of the chirplets [5]. The latter can lead to incorrect initialization of the proposed MCMC method, and thereby to very slow convergence of the Markov chain.

### 3.1. Effect of Unknown Timings of the Chirplets

The initialization of MCMC methods when they are applied to frequency estimation of sinusoids is usually done by the periodogram. We are, thus, tempted to initialize the MCMC sampling for chirplets by using chirpograms. It turns out that this is not as straightforward, and as will be shown below, their use without careful consideration will provide only good initialization of the frequency rate but not the frequency. In brief, without knowing the timing of the chirplet, the estimated frequency will be inaccurate, and therefore for good initialization we cannot use the original definition of the chirpogram, but rather a modified version of it.

For simplicity, we only present the case of data that represent a single chirplet without noise, where the true parameters of the chirplet are $a_1, \beta_1, \tau_1, \omega_1, \alpha_1$. The chirpogram is then given by

$$P(\omega, \alpha) = \frac{1}{N}\left|\sum_{n=0}^{N-1} y[n]e^{-j(\omega n + \frac{\alpha}{2}n^2)}\right|^2$$

$$= \frac{1}{N}\left|\sum_{n=0}^{N-1} a_1 e^{-\frac{\beta_1(n-\tau_1)^2}{2}} e^{j(\frac{1}{2}(\alpha_1-\alpha)n^2 + (\omega_1 - \alpha_1\tau_1 - \omega)n)}\right|^2.$$

Clearly, without knowing $\tau_1$, the values of $\omega$ and $\alpha$ that maximize $P(\omega,\alpha)$ are moved from $(\omega_1, \alpha_1)$ to a different value given by $((\omega_1 - \alpha_1\tau_1)_{\bmod 2\pi}, \alpha_1)$. Thus, the initialization of the frequency from the chirpogram depends on $\tau_1$. So, we modify the definition of the chirpogram to

$$P(\omega, \alpha, \tau) = \frac{1}{N}\left|\sum_{n=0}^{N-1} y[n]e^{-j(\omega(n-\tau) + \frac{\alpha}{2}(n-\tau)^2)}\right|^2 \quad (4)$$

where $0 \leq \omega, \alpha < 2\pi$, $0 \leq \tau < N-1$. The discretized version of it takes the form

$$P(l, k, \tau) = \frac{1}{N}\left|\sum_{n=0}^{N-1} y[n]e^{-j2\pi[(\frac{l}{M_f})(n-\tau) + (\frac{k}{M_\alpha})(n-\tau)^2]}\right|^2 \quad (5)$$

where $0 < l \leq M_f$, $0 < k \leq M_\alpha$, and $M_f$ and $M_\alpha$ are positive integers that are not necessarily equal to $N$.

Furthermore, assume that the frequency rate $\alpha_1$ of the chirplet is known, and let $\widehat{\tau}_1 = \tau_1 + \Delta\tau$ be the estimated time center of the chirplet. Then, from (4) and under the noiseless assumption, we get

$$P(\omega|\widehat{\tau}_1, \alpha_1) = \frac{1}{N}\left|\sum_{n=0}^{N-1} y[n]e^{-j(\omega(n-\widehat{\tau}_1) + \frac{\alpha_1}{2}(n-\widehat{\tau}_1)^2)}\right|^2$$

$$= \frac{1}{N}\left|\sum_{n=0}^{N-1} a e^{-\frac{\beta_1(n-\tau_1)^2}{2}} e^{j(\omega_1 - \omega + \alpha_1\Delta\tau)(n-\tau_1)}\right|^2 \quad (6)$$

and the estimate for $\omega_1$ is

$$\widehat{\omega}_1 = \arg\max_{\omega} P(\omega|\widehat{\tau}_1, \alpha_1) = \omega_1 + \alpha_1\Delta\tau. \quad (7)$$

From the above relations, we see that an estimation error of the time center propagates in the estimate of the frequency. Given $\alpha_1$ and $\widehat{\tau}_1$, the pair $(\omega_1, \tau_1)$ is initialized by the sets of pairs

$$(\widehat{\omega}_1, \widehat{\tau}_1) = (\omega_1 + \alpha_1\Delta\tau, \tau_1 + \Delta\tau) \quad (8)$$

where

$$\widehat{\omega}_1 - \alpha_1\widehat{\tau}_1 = \omega_1 - \alpha_1\tau_1. \quad (9)$$

This relationship is shown in Figure 1.

### 3.2. Mirror Points of the Chirplets

We now briefly discuss the mirror points of a single chirplet. As expected, given $\tau_1$, the chirplet signal does have mirror points in its chirpogram. At the true location of the chirplet in the frequency-frequency rate plane, $(l_1, k_1)$, the magnitude of the modified chirpogram is equal to $P(l_1, k_1|\tau_1)$. The magnitudes at the mirror points located at $(l_1, k_1 + 0.5M_\alpha)$ and $(l_1 + 0.5M_f, k_1 + 0.5M_\alpha)$ are less than half of $P(l_1, k_1|\tau_1)$. This will be a problem when we work with multiple chirplets. For more discussion on this issue, see [5].

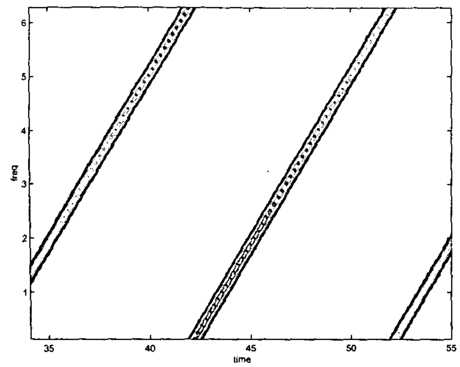


Figure 1: A contour of $P(\omega, \tau|\alpha_1)$ of a chirplet signal with parameters $\{\beta_1, \tau_1, \alpha_1, \omega_1\} = \{0.01, 44, 0.6283, 1.2566\}$, in the vicinity of $\omega_1, \tau_1$.

## 4. INITIALIZATION PROCEDURE

Our purpose here is to propose a simple procedure for finding good initial values of all the parameters sampled by the MCMC method. According to the degree of signal overlap in time domain, we divide the problem into two categories: overlapped chirplets and non-overlapped chirplets. For instance, if the chirplet signals can be distinguished directly from the signal energy in the time domain, we say that we have the non-overlapped case (even if the chirplets are partially overlapped). Otherwise the chirplets are overlapped.

### 4.1. Non-overlapped Chirplets

The following procedures are for single or multiple chirplets, which may be slightly overlapped in the time domain. We use the fact that these chirplets can be distinguished directly by the energy detector, and the initialization proceeds as follows:

**Step 1.** Initialization of $\beta$, $\tau$, and $a$

The energy of the observations during the presence of a chirplet at time $n$ is expressed by

$$z_1[n] = |y[n]|^2 = |a|^2 \exp(-\beta(n-\tau)^2) \qquad (10)$$

We can find the approximate values $(\hat{a}, \hat{\tau})$ from the maximum value of $z_1[n]$. The parameter $\beta$ can be estimated easily by first taking the logarithm of $z_1[n]$,

$$\log z_1[n] = 2\log|a| - \beta(n-\tau)^2 \qquad (11)$$

and then solving for $\beta$, where we substitute for $a$ and $\tau$ their estimated values.

**Step 2.** Initialization of $\alpha$ and $\omega$

The first order phase difference equation of the observations can be expressed as follows.

$$\begin{aligned} z_2[n] &= y[n]\,y[n-1]^H \qquad (12) \\ &= b[n]b^H[n-1]e^{j(\alpha(n-\tau)+w-\frac{\alpha}{2})} + v[n] \end{aligned}$$

where $b[n] = ae^{-\frac{\beta(n-\tau)^2}{2}}$, $\phi_n = \frac{\alpha(n-\tau)^2}{2} + \omega(n-\tau)$ and $v[n]$ is a summation of noise product terms with the nonoverlapping signal. We can find $\widehat{\alpha}$ by applying the periodogram to $z_2[n]$. The frequency $\omega$ can then be obtained easily.

### 4.2. Overlapped Chirplets

If the chirplet signals are overlapped, the above initialization procedure will not be suitable. Although, overlapped chirplets cannot be separated in the time-domain, they are distinguishable by its chirpogram. The initialization procedure in this case is implemented as follows:

**Step 1.** Initialization of $\omega$ and $\alpha$

We assume that the time center is zero, and we find $(\omega, \alpha)$ by applying (4). From the previous discussions, we know that the maximum point is located at $(\widehat{\omega}', \widehat{\alpha})$ and $\widehat{\omega}' = (\omega - \widehat{\alpha}\tau)_{\text{ mod } 2\pi}$.

**Step 2.** Initialization of $\tau$

Once $(\widehat{\omega}', \widehat{\alpha})$ are found, we de-chirp the chirplet signals and find the maximum point of energy in time domain as our estimation for $\widehat{\tau}$. This is done according to

$$\widehat{\tau} = \arg\max_n \left|y[n]\,s_1^H[n]\right|^2 \qquad (13)$$

where

$$s_1[n] = \exp\left(j(\widehat{\omega}'n + \frac{\widehat{\alpha}}{2}n^2)\right).$$

**Step 3.** Update $\omega$

When $\widehat{\tau}$ is estimated, we can update $\widehat{\omega}$ from $\widehat{\omega}' = (\omega - \widehat{\alpha\tau})_{\text{ mod } 2\pi}$. That means $\widehat{\omega} = \widehat{\omega}' \pm 2k\pi + \widehat{\alpha}\widehat{\tau}$, and $0 \le \widehat{\omega} < 2\pi$. Here, we simply assume that $\widehat{\beta} = 0$.

### 4.3. General Procedure for Initialization

When there are multiple chirplet signals in the observed data, we initialize the individual chirplets sequentially by applying one of the two procedures discussed in the previous two subsections. We start with initializing the strongest chirplet. Once it is initialized, that is, its set of parameters are estimated, that chirplet $a_1\mathbf{s}_1$, is reconstructed and removed from the data $\mathbf{y}$ according to

$$\mathbf{y}_1 = \mathbf{y} - \widehat{\mathbf{s}}_1\widehat{a}_1. \qquad (14)$$

The next chirplet is initialized from the data $\mathbf{y}_1$ along the same lines as the first chirplet. It is then reconstructed and removed from $\mathbf{y}_1$, and a new set of data $\mathbf{y}_2$ is obtained. The procedure continues until all the chirplets are initialized.

## 5. MCMC SIMULATION RESULTS

In this section we provide simulation results based on the proposed initialization procedures. In the simulations, all the parameters $\{a, \beta, \omega, \alpha, \sigma_w^2\}$ are unknown and need to be estimated, whereas the number of chirplets is known. We present results for all the parameters except for $a$.

The results for a single chirplet embedded in CWGN are shown in Figure 2. The parameters of the signal were $(\beta_1, \tau_1, \alpha_1, \omega_1) = (0.01, 44, 0.6283\ , 1.2566)$, $N = 100$. The figure shows 1000 iterations of the Markov chains. The values from the last 500 iterations are used to find the mimum mean square error (MMSE) estimates. The estimation results with 100 trials under different SNRs are displayed in Figure 3. They are compared with the respective Cramer-Rao Lower Bounds (CRLBs). The performance for a single chirplet follows the CRLB for SNRs down to $-3$ dB.

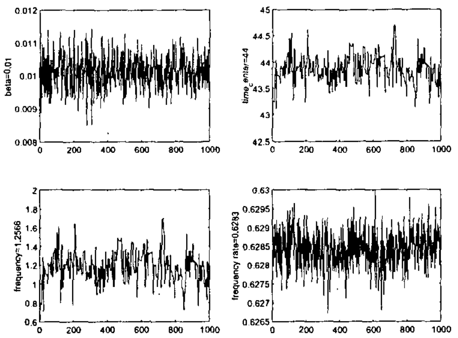

Figure 2: A MCMC simulation result of single chirplet signal with $(\beta_1, \tau_1, \alpha_1, \omega_1) = (0.01, 44, 0.6283, 1.2566)$ . The left top, right top, left bottom and right bottom are for beta, time center, frequency and frequency rate in order.

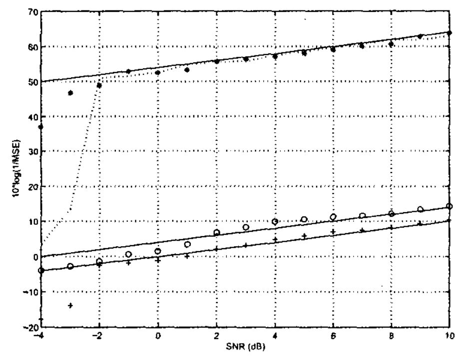


Figure 3: Estimation performance for a single chirplet under different SNRs and compared with CRLBs (solid line), respectively. The solid lines from top represent the CRLBs for frequency rate (and $\beta$), frequency, and time center, respectively. The symbols $\cdot$, $*$, $\circ$, and $+$ represent the results for frequency rate, $\beta$, frequency and time center, respectively.

Next, we made experiments with two chirplets overlapped in the time domain. To the single chirplet signal in the first experiment, we added another chirplet whose parameters were $(\beta_2, \tau_2, \alpha_2, \omega_2) = (0.01, 46, 1.0053, 1.8850)$. The simulation were carried out under different SNRs and the results are shown in Figure 4. Again, they are very close to the CRLB, this time for SNR $\geq 3dB$.

## 6. CONCLUSION

We have proposed an MCMC sampling based method for parameter estimation of chirplet signals. A critical part of the method is its initialization. We have described an initialization that avoids lengthy trappings of the chain for long periods around local optima of the posterior distribution and that leads the chain to quick convergence. The generated samples by the chain are used to find various point estimates or to approximate the posterior distribution. Further work on this approach includes estimation of the chirplet parameters when the number of chirplets is assumed unknown.

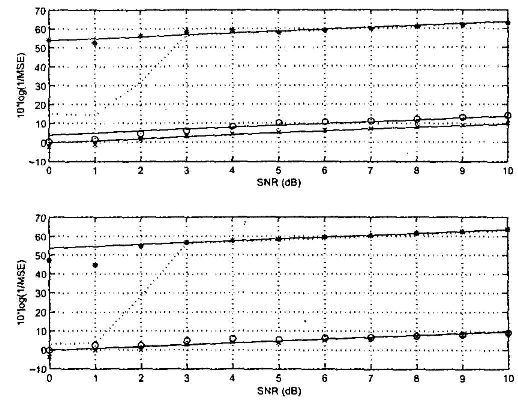


Figure 4: Estimation performance for two chirplet signals under different SNRs and compared with CRLBs (solid line). The symbols $\cdot$, $*$, $\circ$, and $x$ represent the results for frequency rate, $\beta$, frequency and time center, respectively.

## 7. REFERENCES


[1] R. G. Baraniuk and D. L. Jones, "Shear madness: New orthonormal bases and frames using chirp functions," *IEEE Trans. Signal Processing*, 41(12), pp. 3543-3549, 1993.

[2] L. Cohen, ,"Generalized phase-space distribution functions," *Math. Phys.*, Vol. 7, pp. 781-786, 1996.

[3] P. M. Djurić and S. Godsill, "Parametric modeling and estimation of time-varying spectra," *Proceedings of the Asilomar Conference on Signals, Systems, and Computers*, vol 1., pp. 292-296, Asilomar, CA 1998.

[4] M. Z. Ikram, K. Abed-Meraim and Y. Hua, "Fast quadratic phase transform for estimating the parameters of multicomponent Chirp Signals." *Digital Signal Processing*, 7, pp. 127-135, 1997.

[5] C. C. Lin and P. M. Djurić, "Estimation of Chirp Signal by MCMC," In *Proc. of the IEEE Int. Conf. on Acoust., Speech, and Signal Processing*, pp. 265-268, 2000.

[6] S. Mann and S. Haykin., "The chirplet transform: physical considerations," *IEEE Trans. Signal Processing*, 43(11), pp. 2745-2761, 1995.

[7] J. O'Neill. and P. Flandrin, "Chirp hunting," *Proceedings of the IEEE Int. Symp. on Time-Frequency and Time-Scale Analysis*, 1998.

[8] J. J. K. O'Ruanaidh.and W. J. Fitzgerald, *Numerical Bayesian Methods Applied to Signal Processing*, Springer-Verlag, 1996.

[9] S. Qian and D. Chen,"Adaptive chirplet based signal approximation," *Proc. of the IEEE Int. Conf. on Acoust., Speech, and Signal Processing*, 1781-1784, 1998.