Performance Comparison of Gaussian-Based Filters Using Information Measures

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***Abstract*—In many situations, solutions to nonlinear discrete-time filtering problems are available through approximations. Many of these solutions are based on approximating the posterior distributions of the states with Gaussian distributions. In this letter, we compare the performance of Gaussian-based filters including the extended Kalman filter, the unscented Kalman filter, and the Gaussian particle filter. To that end, we measure the distance between the posteriors obtained by these filters and the one estimated by a sequential Monte Carlo (particle filtering) method. As a distance metric, we apply the Kullback–Leibler and χ^2 information measures. Through computer simulations, we rank the performance of the three filters.**

***Index Terms*—Extended Kalman filter, filtering, information measures, sequential Monte Carlo, unscented Kalman filter.**

I. Introduction

THE discrete state–space (DSS) model is an indispensable mathematical model that describes the evolution and the observation function of the state [1]. A standard formulation of the model is as follows:

$$\mathbf{x}_t = g(\mathbf{x_{t-1}}) + \mathbf{u_t} \tag{1}$$

$$\mathbf{y}_t = h(\mathbf{x}_t) + \mathbf{v}_t \tag{2}$$

where $\mathbf{x}_t$ is the unknown state, $g(\cdot)$ is the state transition function, $\mathbf{y}_t$ are the measurements, $h(\cdot)$ is the measurement function, and $\mathbf{u}_t$ and $\mathbf{v}_t$ are noise processes.

The estimation of the filtering or posterior distribution $p(\mathbf{x}_t|\mathbf{y}_{1:t})$ or their statistics is a standard filtering problem. Closed-form analytical and optimal solutions to this filtering problem exist in a small number of cases, for example, when the functions $g(\cdot)$ and $h(\cdot)$ of the DSS model are linear, and the noise vectors $\mathbf{u}_t$ and $\mathbf{v}_t$ are zero-mean Gaussian with covariance matrices $\mathbf{C}_{u,t}$ and $\mathbf{C}_{v,t}$, respectively. However, in many real-world scenarios, closed-form solutions cannot be obtained [2].

Some approximate parametric solutions to these problems are obtained via the extended Kalman filter (EKF) and the unscented Kalman filter (UKF) [1], [3]. With the advent of more computing power, the last decade has seen a surge in Monte Carlo methods where the posterior distributions are approximated by a large weighted set of samples. These sequential Monte Carlo (SMC) methods, also known as particle filters (PFs), have been proposed in the last decade as more robust and close to optimal solutions in determining posterior distributions and their statistics [4]. The Gaussian particle filter (GPF) is an SMC method that approximates the posterior distributions with Gaussian distributions [5]. In this letter, we refer to the EKF, UKF, and GPF as Gaussian-based filters.

We reiterate that the underlying idea of these methods is that they approximate the posterior distribution with a Gaussian. However, the process by which this approximation is obtained is different, and therefore, the resulting distributions are also different. The EKF achieves the Gaussian approximation through a linearization of the DSS model, the UKF obtains it by means of an unscented transformation, while the GPF obtains the parameters of the approximations using SMC steps.

Popular metrics for assessing the performance of these approximations are through root mean square errors (RMSEs) of the point estimates of the states. An alternative is to estimate biases and variances of the estimates, thereby capturing only a limited picture of the filter's performance. In this letter, we provide a more complete performance comparison among the methods by measuring how close their posterior distributions are from the posterior distribution obtained by particle filtering. This has been suggested recently in [6]. The reason for choosing the posterior obtained by particle filtering is that the true posterior is not known and that particle filtering has the most ambitious aim while estimating unknown states by attempting to track the evolution of their posterior distributions. For measuring distance, we use information measures, more specifically the Kullback–Leibler (KL) and χ^2 information metrics. We develop and discuss the computation of these metrics in the context of the performance comparison of the filters. In the remainder of this letter, we provide a brief summary of the EKF, UKF, standard particle filter (SPF), and GPF and provide details of the comparison by information metrics.

II. Brief Summary of the Filters

Most recursive solutions to the filtering problem involve two key operations at each time instant: 1) propagation of the state estimate from the previous time instant to the current time instant and 2) updating of the state estimate using the current measurements. We now briefly summarize the various filters that provide approximate solutions to the filtering problem when the functions in the DSS model are nonlinear. In these filters, the following approximations are made: the predictive density of the state is approximated by a Gaussian, $p(\mathbf{x}_t|\mathbf{y}_{1:t-1}) \approx \mathcal{N}(\bar{\mathbf{x}}_{t|t-1}, \mathbf{P}_{t|t-1})$, where $\bar{\mathbf{x}}_{t|t-1}$ and $\mathbf{P}_{t|t-1}$ are the predictive mean and covariance matrix of $\mathbf{x}_t$ given $\mathbf{y}_{1:t-1}$, and the filtering density is approximated by another Gaussian, $p(\mathbf{x}_t|\mathbf{y}_{1:t}) \approx \mathcal{N}(\bar{\mathbf{x}}_{t|t}, \mathbf{P}_{t|t})$, where $\bar{\mathbf{x}}_{t|t}$ and $\mathbf{P}_{t|t}$ are the mean and covariance matrix of $\mathbf{x}_t$ given $\mathbf{y}_{1:t}$.

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A. EKF

In the EKF, through a Taylor series expansion, the DSS equations are linearized and transformed into a model for which the Kalman filter presents an optimal solution [1]. With initial mean, $\bar{\mathbf{x}}_0 = E(\mathbf{x}_0)$, and covariance matrix, $\mathbf{P}_{0|0} = \mathrm{Cov}(\mathbf{x}_0)$, and using the Gaussian approximation of the predictive and filtering distributions at time instant $t-1$, the two steps in the EKF at time instant t are as follows.

1) Time update step:

$$\begin{aligned}\bar{\mathbf{x}}_{t|t-1} &= g(\bar{\mathbf{x}}_{t-1|t-1})\\ \mathbf{P}_{t|t-1} &= \mathbf{C}_{u,t} + \mathbf{G}_{t-1}\mathbf{P}_{t-1|t-1}\mathbf{G}_{t-1}^{\top}.\end{aligned} \tag{3}$$

2) Measurement update:

$$\begin{aligned}\bar{\mathbf{x}}_{t|t} &= \bar{\mathbf{x}}_{t|t-1} + \mathbf{K}_t(\mathbf{y}_t - h(\bar{\mathbf{x}}_{t|t-1}))\\ \mathbf{P}_{t|t} &= \mathbf{P}_{t|t-1} - \mathbf{K}_t\mathbf{H}_t\mathbf{P}_{t|t-1}\end{aligned} \tag{4}$$

where $\mathbf{K}_t = \mathbf{P}_{t|t-1}\mathbf{H}_t^{\top}\mathbf{S}_t^{-1}, \mathbf{S}_t = \mathbf{H}_t\mathbf{P}_{t\,|\,t-1}\mathbf{H}_t^{\top} + \mathbf{C}_{v,t}, \mathbf{H}_t = (\partial h(\mathbf{x}))/(\partial \mathbf{x})_{\mathbf{x}=\bar{\mathbf{x}}_{t|t-1}}$, and $\mathbf{G}_t = (\partial g(\mathbf{x}))/(\mathbf{x})_{\mathbf{x}=\bar{\mathbf{x}}_{t|t}}$.

B. UKF

In the UKF, a set of deterministically chosen points known as sigma points $\boldsymbol{\Omega}_t^s$ are used in approximating the filtering and predictive distributions by a Gaussian distribution. In this filter, the state of the system $\mathbf{x}_t$ is represented by concatenating it with the process and measurement noise states, i.e., by forming $\mathbf{x}_t^s = [\mathbf{x}_t^{\top}, \mathbf{u}_t^{\top}, \mathbf{v}_t^{\top}\mathbf{P}^{\top}$. The main steps in the UKF are as follows.

1) Calculation of sigma points: $\boldsymbol{\Omega}_{t-1}^s = [\mathbf{x}_t^s, \mathbf{x}_t^s \pm \sqrt{(n_s+\lambda)\mathbf{P}_{t-1}^s)}]$, where n_s is the dimension of $\mathbf{x}_t^s$ and λ is a scaling parameter, and $\mathbf{P}_{t-1}^s$ is the covariance matrix of $\mathbf{x}_{t-1}^s$.
2) Time update step: Using (1), the sigma points $\boldsymbol{\Omega}_{t-1}^s$ are propagated to obtain $\boldsymbol{\Omega}_{t|t-1}^s$, which are then appropriately weighted. Using $\boldsymbol{\Omega}_{t|t-1}^s$ and the weights, the mean $\mathbf{x}_{t|t-1}$ and covariance matrix $\mathbf{P}_{t|t-1}$ of the predictive distribution are easily obtained. The predictive sigma points $\boldsymbol{\Omega}_{t|t-1}^s$ are transformed through the observation model (2) to obtain a new set of measurement points $\mathcal{Y}_{t-1}$ and their predictive measurement mean $\bar{\mathbf{y}}_{t|t-1}$.
3) Measurement update step: Using the points computed in the time update step, the covariance matrix $\mathbf{P}_{\mathbf{y}_t\mathbf{y}_t}$ of $\mathcal{Y}_{t-1}$ and the cross covariance matrix $\mathbf{P}_{\boldsymbol{\Omega}_t\mathbf{y}_t}$ between $\boldsymbol{\Omega}_{t\mathrm{mid}pt-1}^s$ and $\mathcal{Y}_{t-1}$ are computed. The parameters of the filtering density are obtained as follows:

$$\begin{aligned}\mathbf{K}_t &= \mathbf{P}_{\boldsymbol{\Omega}_t\mathbf{y}_t}\mathbf{P}_{\mathbf{y}_t\mathbf{y}_t}^{-1}\\ \bar{\mathbf{x}}_{t|t} &= \bar{\mathbf{x}}_{t|t-1} + \mathbf{K}_t(\mathbf{y}_t - \bar{\mathbf{y}}_{t|t-1})\\ \mathbf{P}_{t|t} &= \mathbf{P}_{t|t-1} - \mathbf{K}_t\mathbf{P}_{\mathbf{y}_t\mathbf{y}_t}\mathbf{K}_t^{\top}.\end{aligned} \tag{5}$$

More details about the expressions of the UKF can be found in [3] and [7].

C. SPF

In SMC methods, the filtering density is represented by a discrete random measure that is a set of weights and samples, $\boldsymbol{\Xi} = \{\mathbf{x}_t^{(m)}, w_t^{(m)}\}_{m=1}^M$ [4]. This is mathematically written as $p(\mathbf{x}_t\,|\,\mathbf{y}_{1:t}) \approx \sum_{m=1}^M w_t^{(m)}\delta(\mathbf{x}_t - \mathbf{x}_t^{(m)})$, where $\mathbf{x}_t^{(m)}$ is a randomly drawn sample of the unknown state $\mathbf{x}_t$, with corresponding weight $w_t^{(m)}$, and $\delta(\cdot)$ denotes the Dirac delta function. The main steps of the SPF are as follows.

1) Generation of particles: This is the time update step with particles $\mathbf{x}_t^{(m)}$ drawn from a proposal distribution function $\pi(\mathbf{x}_t\,|\,\mathbf{x}_{t-1}^{(m)}, \mathbf{y}_t)$.
2) Calculation of weights: Using the measurement $\mathbf{y}_t$, the weights are updated by

$$w_t^{(m)} \propto w_{t-1}^{(m)} \frac{p\left(\mathbf{y}_t \mid \mathbf{x}_t^{(m)}, \mathbf{y}_{1:t-1}\right) p\left(\mathbf{x}_t^{(m)} \,|\, \mathbf{x}_{t-1}^{(m)}\right)}{\pi\left(\mathbf{x}_t^{(m)} \,|\, \mathbf{x}_{t-1}^{(m)}, \mathbf{y}_t\right)}$$

and normalized such that $\sum_{m=1}^M w_t^{(m)} = 1$.

A third step known as resampling is performed to replace particles that have negligible weights with particles with non-negligible weights.

D. GPF

The GPF is an SMC filtering algorithm that also approximates the predictive and posterior distributions with Gaussian distributions [5]. The main steps of this filter are as follows.

1) Generation of particles: Samples $\mathbf{x}_{t-1}^{(m)}$ are drawn from $\mathcal{N}(\bar{\mathbf{x}}_{t-1\,|\,t-1}, \mathbf{P}_{t-1\,|\,t-1})$ and the samples $\mathbf{x}_t^{(m)}$ are drawn from $p(\mathbf{x}_t\,|\,p\mathbf{x}_{t-1}^{(m)})$.
2) Computation of the mean and covariance of the predictive density: Using the samples of the previous step, the sample mean $\bar{\mathbf{x}}_{t\,|\,t-1}$ and covariance matrix $\mathbf{P}_{t\,|\,t-1}$ are obtained.
3) Computation of the weights: Using the measurement $\mathbf{y}_t$, the weights are calculated by

$$\tilde{w}_t^{(m)} = \frac{p\left(\mathbf{y}_t \,|\, \mathbf{x}_t^{(m)}\right) p\left(\mathbf{x}_t^{(m)} \,|\, \mathbf{y}_{1:t-1}\right)}{\pi\left(\mathbf{x}_t^{(m)} \,|\, \mathbf{x}_{t-1}^{(m)}, \mathbf{y}_t\right)}$$

where $p(\mathbf{x}_t\,|\,\mathbf{y}_{1:t-1})$ is a Gaussian with a mean $\bar{\mathbf{x}}_{t\,|\,t-1}$ and a covariance matrix $\mathbf{P}_{t\,|\,t-1}$. The weights are subsequently normalized.
4) Computation of the mean and covariance of the filtering density: The sample mean $\bar{\mathbf{x}}_{t\,|\,t}$ and covariance $\mathbf{P}_{t,|\,t}$ of the filtering density are obtained using the weighted set of samples.

III. Information Metrics

For measuring the distance between the Gaussian posteriors produced by the Gaussian-based filters and the posterior obtained by the SPF, we use the KL and χ^2 information metrics [8]. However, the proposed approach for measuring performance is general and can be applied to a broad class of information measures.

With a slight abuse of notation, the KL and the χ^2 information metrics are defined as follows.

- KL information:

$$\mathcal{KL}(p, q) = \int p(\mathbf{x}) \log\left(\frac{p(\mathbf{x})}{q(\mathbf{x})}\right)\, d\mathbf{x}.$$

- χ^2 information:

$$\chi^2(p,q) = \int \frac{(p(\mathbf{x}) - q(\mathbf{x}))^2}{q(\mathbf{x})}\, d\mathbf{x} = \int \frac{p^2(\mathbf{x})}{q(\mathbf{x})}\, d\mathbf{x} - 1.$$

In our problem, $p(\mathbf{x})$ is the posterior obtained by the SPF, and $q(\mathbf{x})$ is the posterior estimated by a Gaussian-based filter. A straightforward computation of these measures is not feasible because the random measures obtained by the SPF are discrete while the Gaussian posterior approximation is continuous. We avoid this by computing

$$\begin{aligned} p(\mathbf{x}_t \,|\, \mathbf{y}_{1:t}) &\propto p(\mathbf{y}_t \,|\, \mathbf{x}_t, \mathbf{y}_{1,t-1}) p(\mathbf{x}_t \,|\, \mathbf{y}_{1:t-1}) \\ &\propto p(\mathbf{y}_t \,|\, \mathbf{x}_t) \int p(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}) \times p(\mathbf{x}_{t-1} \,|\, \mathbf{y}_{1:t-1}) d\mathbf{x}_t. \end{aligned} \tag{6}$$

In the integral in (6), we express the posterior $p(\mathbf{x}_{t-1} \,|\, \mathbf{y}_{1:t-1})$ as

$$p(\mathbf{x}_{t-1} \,|\, \mathbf{y}_{1:t-1}) = \sum_{m=1}^{M} w_{t-1}^{(m)} \delta(\mathbf{x}_{t-1} - \mathbf{x}_{t-1}^{(m)}) \tag{7}$$

and for the posterior $p(\mathbf{x}_t \,|\, \mathbf{y}_{1:t})$, we obtain

$$p(\mathbf{x}_t \,|\, \mathbf{y}_{1:t}) \propto p(\mathbf{y}_t \,|\, \mathbf{x}_t) \sum_{m=1}^{M} w_{t-1}^{(m)} p(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}^{(m)}). \tag{8}$$

We will write $p(\mathbf{x}_t \,|\, \mathbf{y}_{1:t}) = c\tilde{p}(\mathbf{x}_t \,|\, \mathbf{y}_{1:t})$, where

$$\tilde{p}(\mathbf{x}_t \,|\, \mathbf{y}_{1:t}) = p(\mathbf{y}_t \,|\, \mathbf{x}_t) \sum_{m=1}^{M} w_{t-1}^{(m)} p\left(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}^{(m)}\right). \tag{9}$$

A. Computation of $\mathcal{KL}$ Information

The $\mathcal{KL}$ distance at time instant t is expressed as

$$\mathcal{KL}_t = \int p \log\left(\frac{p}{q}\right) d\mathbf{x}t = \int \frac{p}{q} \log\left(\frac{p}{q}\right) q d\mathbf{x}t. \tag{10}$$

The integral in (10) can by computed by Monte Carlo integration by drawing samples from $q(\mathbf{x}_t)$. It is easy to show that

$$\mathcal{KL}_t \simeq \frac{1}{N} \sum_{n=1}^{N} \rho_t^{(n)} \log \rho_t^{(n)} \tag{11}$$

where N is the number of drawn samples from $q(\mathbf{x}_t)$ and

$$\rho_t^{(n)} = \frac{\tilde{\rho}_t^{(n)}}{\frac{1}{N}\sum_{k=1}^{N} \tilde{\rho}_t^{(k)}} \tag{12}$$

and

$$\tilde{\rho}_t^{(n)} = \frac{\tilde{p}(\mathbf{x}_t^{(n)} \,|\, \mathbf{y}_{1:t})}{q(\mathbf{x}_t^{(n)})}.$$

B. Computation of χ^2 Information

The χ^2 at time instant t is given by

$$\begin{aligned} \chi_t^2 &= \int \frac{p^2}{q} d\mathbf{x}_t - 1 \\ &= \int \frac{p^2}{q^2}\, q\, d\mathbf{x}_t - 1. \end{aligned} \tag{13}$$

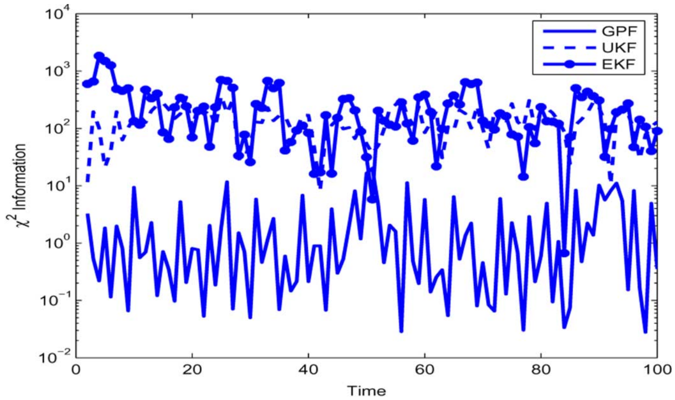

Fig. 1. The χ_t^2 information of the GPF, EKF, and UKF filters.

A Monte Carlo estimate of χ_t^2 is obtained in a similar way, and it is given by

$$\chi_t^2 \simeq \frac{1}{N} \sum_{n=1}^{N} \left(\rho_t^{(n)}\right)^2 - 1 \tag{14}$$

where the symbols have the same meaning as in (11).

IV. Simulations

We provide two examples where we compare the performance of the EKF, UKF, and GPF.

A. Univariate Nonlinear Model

Consider the following one-dimensional nonlinear time series:

$$\begin{aligned} x_t &= 0.5x_{t-1} + 25\frac{x_{t-1}}{1 + x_{t-1}^2} + 8\cos(1.2(t-1)) + u_t \\ y_t &= \frac{x_t^2}{20} + v_t. \end{aligned} \tag{15}$$

Here, u_t and v_t are both zero mean Gaussian noise processes with unit variance. The initial distribution $p(x_0) \sim \mathcal{N}(0,1)$ [5]. In the simulation of the Monte Carlo filters, $M = 10\,000$ particles were used. The number of samples N that were generated for computing the information measures was also 10 000. For obtaining the weights associated with each sigma point of the UKF, the parameters α, β, and κ were set to $\alpha = 0.95, \beta = 2, \kappa = 0$. We measured the performance of the algorithms by computing the information metric using $K = 100$ different trajectories. In Figs. 1 and 2, the averaged χ_t^2 information metric and the RMSEs are shown. From the plots, it can be seen that the Gaussian approximation to the posterior distribution with GPF is considerably better than the ones of the EKF and the UKF. However, the GPF is computationally more intensive.

B. Bearings Only Target Tracking

We considered a target tracking scenario using angle measurements, which is a four-dimensional DSS model, with partially observed states

$$\mathbf{x}_t = \mathbf{F}\mathbf{x}_{t-1} + \mathbf{u}_t, \quad t \in \mathbb{N} \tag{16}$$

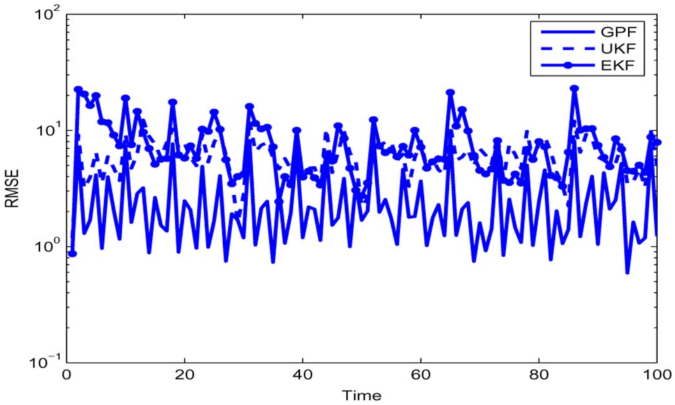

Fig. 2. RMSE with the GPF, EKF, and UKF filters.

where $\mathbf{x}(t) = [x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t)]^\top$ are the target's position and velocity in a two-dimensional plane, $\mathbf{F}$ is a 4×4 transition matrix, and $\mathbf{u}_t$ is a 4×1 Gaussian noise vector. The transition matrix $\mathbf{F}$ and the covariance matrix of the noise process are given by

$$\mathbf{F} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C}_{u,t} = \sigma_u^2 \begin{bmatrix} \frac{T^3}{3} & \frac{T^2}{2} & 0 & 0 \\ \frac{T^2}{2} & T & 0 & 0 \\ 0 & 0 & \frac{T^3}{3} & \frac{T^2}{2} \\ 0 & 0 & \frac{T^2}{2} & T \end{bmatrix}$$

where T is the sampling interval.

The measurement equation is given by

$$y_t = \arctan\left(\frac{x_2(t)}{x_1(t)}\right) + v_t \tag{17}$$

where v_t is a measurement noise considered zero mean and Gaussian.

The initial distribution of $p(\mathbf{x}_0)$ is modeled as a Gaussian distribution $\mathcal{N}(\mu_0, \mathbf{P}_0)$, where $\mu_0 = [-10.0,\ -0.5,\ -10.0,\ 0.5\mathbf{P}^\top$ and $\mathbf{P}_0$ is given as diag[1]$([0.25,\ 0.25,\ 0.25,\ 0.25)$. The values of the other parameters were $\sigma_u = 0.1$ and $\sigma_v = 0.001$. In the Monte Carlo SPF and GPF, we used $M = 10\,000$ particles. The initial particles were all drawn from $p(\mathbf{x}_0)$. Similarly, when implementing the EKF and UKF, the initial mean and covariance matrix of the state vector were those used in $p(\mathbf{x}_0)$. We compute the $\mathcal{KL}_t$ metric and the RMSE by averaging them over $K = 100$ different trajectories. In Figs. 3 and 4, we present the obtained results. Again, the performance of the GPF is better than the other two filters.

V. Conclusions

In this letter, we provided a novel method for computing the metrics that measure the distance between the posterior distributions obtained with the SPF and the Gaussian-based filters, EKF, UKF, and GPF. Through simulation studies, we note that the GPF provides a better approximation to the posterior distribution than the EKF and the UKF but at a higher computational cost.

[1]The diag$(\mathbf{x})$ operation refers to the formation of the diagonal matrix with vector $\mathbf{x}$ along the main diagonal.

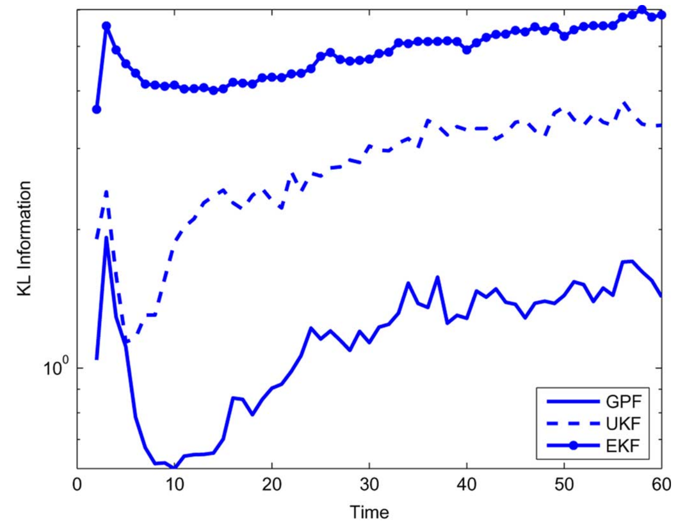

Fig. 3. The $\mathcal{KL}_t$-information for the GPF, EKF, and UKF filters.

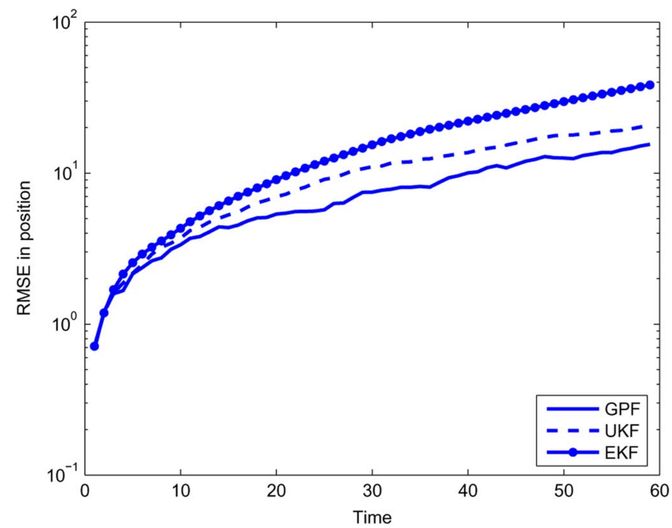

Fig. 4. RMSE in position with the GPF, EKF, and UKF filters.

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