Bayesian Spectrum Estimation of Harmonic Signals

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Abstract—**A Bayesian spectrum estimator of harmonic signals in Gaussian noise is derived. It is based on the expected value of the theoretical signal spectrum over the joint posterior density function of the signal and noise parameters. Simulation results are provided that show its performance and comparison with MUSIC.**

I. Introduction

SPECTRUM estimation is an important area of research in signal processing. It is applied in many scientific and engineering disciplines, such as communications, radar, sonar, astronomy, geophysics, and biomedical engineering. Its main objective is the estimation of average power of signals as a function of frequency.

Recently the long list of techniques for spectrum estimation has been expanded by a Bayesian one [1], [2]. The Bayesian spectrum estimate is defined as the expected value of the theoretical signal spectrum over the joint posterior density function of the signal and noise parameters. The so-defined estimator provides estimates of the signal spectrum *only*. These estimates should be contrasted to the ones of the known approaches as the latter are spectrum estimates of the observed data rather than the signals. In this letter, starting with the basic definition, we derive a Bayesian spectrum estimator of multiple sinusoids in Gaussian noise. The advantage of the derived estimator over existing approaches is its excellent performance. The disadvantage is the increased computational intensity needed for its evaluation. The simulation results show spectrum estimates of sinusoidal signals whose frequencies are separated by a much smaller value than the Rayleigh resolution.

II. Problem Formulation

Let $y[n]$, $n \in Z_N = \{0, 1, 2, \cdots, N-1\}$ be a set of observed samples composed of superimposed sinusoids in additive noise, that is

$$y[n] = \sum_{k=1}^{m} A_k \cos(\omega_k n + \phi_k) + e[n] \quad (1)$$

where

$$\omega_k \neq \omega_i \text{ for } k \neq i, \text{ and } \omega_k \neq 0.$$

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A_k, ω_k, and ϕ_k are the parameters of the kth sinusoid that are unknown, $e[n]$ is a sample of a zero-mean, white, Gaussian noise, that is

$$e[n] \sim \mathcal{N}(0, \sigma^2) \quad (2)$$

where the variance of the noise σ^2 is also unknown, and finally, m is the number of superimposed sinusoids that is assumed known. Based on the data $y[n]$, $n \in Z_N$, the objective is to find the spectrum estimate of the superimposed sinusoids. Note that we want the estimate of the signal spectrum only, and not the spectrum of the observed data $y[n]$.

III. Signal Spectrum Estimation

Let us define the estimate of the signal spectrum by

$$S(\omega|\mathbf{y}) = \int_{\boldsymbol{\Theta},\boldsymbol{\Psi}} S_0(\omega, \boldsymbol{\theta}) f(\boldsymbol{\theta}, \boldsymbol{\psi}|\mathbf{y}) d\boldsymbol{\theta} d\boldsymbol{\psi}. \quad (3)$$

$S_0(\omega, \boldsymbol{\theta})$ is the theoretical spectrum of the signal, $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$ the parameters of the signal and noise, respectively, $f(\boldsymbol{\theta}, \boldsymbol{\psi}|\mathbf{y})$ the *a posteriori* density of the unknown parameters $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$, and $\mathbf{y}$ an $N \times 1$ vector that represents $y[n], n \in Z_N$. $S_0(\omega, \boldsymbol{\theta})$ is clearly a function of the radial frequency ω and the signal parameters $\boldsymbol{\theta}$. If $\boldsymbol{\theta}$ were known, it would have been the true spectrum of the signal. Since the parameters are not known we estimate $S_0(\omega, \boldsymbol{\theta})$ by (3), which is the expected value of the theoretical signal spectrum over the joint posterior density function of the signal and noise parameters. The *a posteriori* density $f(\boldsymbol{\theta}, \boldsymbol{\psi}|\mathbf{y})$ is obtained by applying Bayes' theorem, i.e.,

$$f(\boldsymbol{\theta}, \boldsymbol{\psi}|\mathbf{y}) = \frac{f(\mathbf{y}|\boldsymbol{\theta}, \boldsymbol{\psi}) f(\boldsymbol{\theta}, \boldsymbol{\psi})}{f(\mathbf{y})} \quad (4)$$

where $f(\mathbf{y}|\boldsymbol{\theta}, \boldsymbol{\psi})$ is the probability density function of the data given the parameters $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$, $f(\boldsymbol{\theta}, \boldsymbol{\psi})$ is the prior density of $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$, and $f(\mathbf{y})$ is the marginal density of the data that can be found from

$$f(\mathbf{y}) = \int_{\boldsymbol{\Theta},\boldsymbol{\Psi}} f(\mathbf{y}|\boldsymbol{\theta}, \boldsymbol{\psi}) f(\boldsymbol{\theta}, \boldsymbol{\psi}) d\boldsymbol{\theta} d\boldsymbol{\psi} \quad (5)$$

where $\boldsymbol{\Theta}$ and $\boldsymbol{\Psi}$ are the parameter spaces of $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$, respectively. If there is some prior knowledge about the signal and noise parameters, it should be quantified by the prior $f(\boldsymbol{\theta}, \boldsymbol{\psi})$.

Now we want to apply (3) when the signals are defined as in (1). In order to facilitate the derivation, we reparameterize the sinusoids according to

$$s[n] = \sum_{k=1}^{m} a_{ck} \cos(\omega_k n) + a_{sk} \sin(\omega_k n). \quad (6)$$

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The vector of observations $\mathbf{y}$ can now be represented succinctly in a vector-matrix form as

$$\mathbf{y} = \mathbf{D}(\boldsymbol{\omega})\mathbf{a} + \mathbf{e} \tag{7}$$

where $\boldsymbol{\omega}$ and $\mathbf{a}$ are $m \times 1$ and $2m \times 1$ vectors of the signal parameters $\boldsymbol{\theta}^T = [\boldsymbol{\omega}^T \ \mathbf{a}^T]$ given by

$$\boldsymbol{\omega}^T = [\omega_1 \ \omega_2 \cdots \omega_m]$$

$$\mathbf{a}^T = [a_{c1} \ a_{s1} \ a_{c2} \ a_{s2} \cdots a_{cm} \ a_{sm}]$$

and $\mathbf{D}(\boldsymbol{\omega})$ is an $N \times 2m$ matrix defined by

$$\mathbf{D}(\boldsymbol{\omega}) = [\mathbf{d}_{c1} \ \mathbf{d}_{s1} \ \mathbf{d}_{c2} \ldots \mathbf{d}_{sm}] \tag{8}$$

where

$$\mathbf{d}_{ck}^T = [1 \ \cos(\omega_k) \ \cdots \ \cos(\omega_k(N-1))]$$

$$\mathbf{d}_{sk}^T = [0 \ \sin(\omega_k) \ \cdots \ \sin(\omega_k(N-1))].$$

From the assumption (2), the only unknown noise parameter is σ.

To begin the derivation, we identify the functions that are used in (3). First, the theoretical spectrum of the sinusoids is

$$S_0(\omega, \boldsymbol{\theta}) = \sum_{k=1}^{m} \frac{1}{4}(a_{ck}^2 + a_{sk}^2)\delta(\omega - \omega_k) \tag{9}$$

where

$$\int_{\boldsymbol{\Omega}} \delta(\omega - \omega_k) d\omega = 1.$$

Note that this is the power spectrum density of the superimposed sinusoids if we knew their parameters.

Second, the posterior density $f(\boldsymbol{\theta}, \boldsymbol{\psi}|\mathbf{y})$ is found from (4) where using (2)

$$\begin{aligned} f(\mathbf{y}|\boldsymbol{\theta}, \boldsymbol{\psi}) &= f(\mathbf{y}|\boldsymbol{\omega}, \mathbf{a}, \sigma) \\ &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} e^{-\frac{1}{2\sigma^2}(\mathbf{y}-\mathbf{D}(\boldsymbol{\omega})\mathbf{a})^T(\mathbf{y}-\mathbf{D}(\boldsymbol{\omega})\mathbf{a})}. \end{aligned} \tag{10}$$

The other factor in the numerator of (4) is the prior of the parameters for which we write

$$f(\boldsymbol{\theta}, \boldsymbol{\psi}) = f(\mathbf{a}, \boldsymbol{\omega}, \sigma) = f(\mathbf{a}, \boldsymbol{\omega}) f(\sigma). \tag{11}$$

For $f(\mathbf{a}, \boldsymbol{\omega})$, we assume

$$f(\mathbf{a}, \boldsymbol{\omega}) \propto \left|\mathbf{D}^T(\boldsymbol{\omega})\mathbf{D}(\boldsymbol{\omega})\right|^{\frac{1}{2}} \tag{12}$$

and for $f(\sigma)$

$$f(\sigma) \propto \frac{1}{\sigma} \tag{13}$$

where $\propto$ denotes proportionality. Thus, the priors for the signal parameters are known up to an unknown proportionality constant, and they are almost constant over $\boldsymbol{\Theta}$ except in regions where any two frequencies get close to each other. If two frequencies become identical or their value is equal to zero, the prior for these sets of values in $\boldsymbol{\Theta}$ becomes also equal to zero which reflects the conditions set for (1). In Fig. 1, we display the prior $f(\mathbf{a}, \boldsymbol{\omega})$ from (12) when there are two sinusoids. Finally, the marginal density is obtained by applying (10), (12), and (13) to (5).

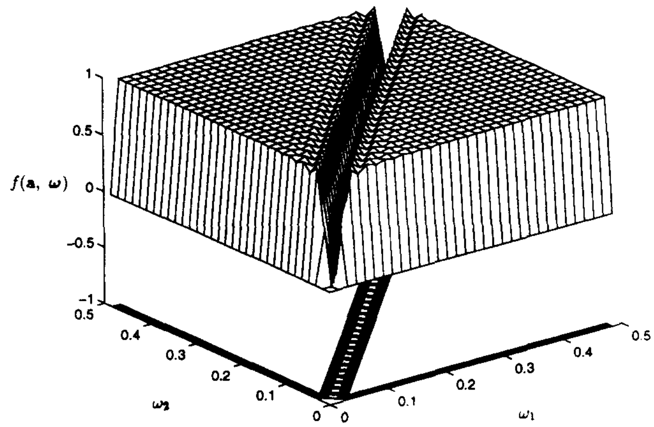


Fig. 1. Prior density for the parameters of two sinusoids as a function of their frequencies..

IV. Final Results

The rest is a straightforward, albeit tedious algebra. The signal spectrum estimate results in

$$S(\omega|\mathbf{y}) = m \sum_{i=1}^{4} I_i(\omega|\mathbf{y}) \tag{14}$$

where for $m > 1$

$$I_1(\omega|\mathbf{y}) = \frac{1}{4J} \int_{\tilde{\boldsymbol{\Omega}}} \frac{\hat{a}_c^2}{(\mathbf{y}^T\mathbf{P}^\perp\mathbf{y})^{\frac{N-2m}{2}}} d\tilde{\boldsymbol{\omega}} \tag{15}$$

$$I_2(\omega|\mathbf{y}) = \frac{1}{4(N-2m-2)\ J\ h_c(\omega)} \times \int_{\tilde{\boldsymbol{\Omega}}} \frac{1}{(\mathbf{y}^T\mathbf{P}^\perp\mathbf{y})^{\frac{N-2m-2}{2}}} d\tilde{\boldsymbol{\omega}} \tag{16}$$

$$I_3(\omega|\mathbf{y}) = \frac{1}{4J} \int_{\tilde{\boldsymbol{\Omega}}} \frac{\hat{a}_s^2}{(\mathbf{y}^T\mathbf{P}^\perp\mathbf{y})^{\frac{N-2m}{2}}} d\tilde{\boldsymbol{\omega}} \tag{17}$$

$$I_4(\omega|\mathbf{y}) = \frac{1}{4(N-2m-2)\ J\ h_s(\omega)} \times \int_{\tilde{\boldsymbol{\Omega}}} \frac{1}{(\mathbf{y}^T\mathbf{P}^\perp\mathbf{y})^{\frac{N-2m-2}{2}}} d\tilde{\boldsymbol{\omega}} \tag{18}$$

where $\tilde{\boldsymbol{\omega}} = [\tilde{\omega}_1, \ \tilde{\omega}_2 \ \cdots, \tilde{\omega}_{m-1}]^T$ and $\tilde{\boldsymbol{\Omega}}$ is the $m-1$-dimensional space of $\tilde{\boldsymbol{\omega}}$. Some of the variables in (15)–(18) have the following meaning:

$$J = \int_{\boldsymbol{\Omega}} \frac{1}{(\mathbf{y}^T\mathbf{P}^\perp\mathbf{y})^{\frac{N-2m}{2}}} d\boldsymbol{\omega} \tag{19}$$

$$h_c(\omega) = \sum_{n=0}^{N-1} \cos^2(\omega n) \tag{20}$$

$$h_s(\omega) = \sum_{n=0}^{N-1} \sin^2(\omega n) \tag{21}$$

The matrix $\mathbf{P}^\perp$ in the integrals in (15)–(18) is a projection matrix and is a function of ω and $\tilde{\boldsymbol{\omega}}$. It is defined by

$$\begin{aligned} \mathbf{P}^\perp(\omega, \tilde{\boldsymbol{\omega}}) &= \mathbf{I} \\ &\quad - \mathbf{D}(\omega, \tilde{\boldsymbol{\omega}})\left(\mathbf{D}^T(\omega, \tilde{\boldsymbol{\omega}})\mathbf{D}(\omega, \tilde{\boldsymbol{\omega}})\right)^{-1}\mathbf{D}(\omega, \tilde{\boldsymbol{\omega}})^T \end{aligned} \tag{22}$$

where $\mathbf{D}(\omega, \tilde{\boldsymbol{\omega}}) = [\mathbf{d}_c(\omega)\ \mathbf{d}_s(\omega)\ \mathbf{d}_c(\tilde{\omega}_1)\ \cdots\ \mathbf{d}_s(\tilde{\omega}_{m-1})]^T$. Finally, the amplitude estimates used in (15) and (17) are the first two elements of $\hat{\mathbf{a}}(\omega, \tilde{\boldsymbol{\omega}})$ obtained according to

$$\hat{\mathbf{a}}(\omega, \tilde{\boldsymbol{\omega}}) = \left(\mathbf{D}^T(\omega, \tilde{\boldsymbol{\omega}})\mathbf{D}(\omega, \tilde{\boldsymbol{\omega}})\right)^{-1}\mathbf{D}^T(\omega, \tilde{\boldsymbol{\omega}})\mathbf{y}. \quad (23)$$

For $m = 1$, the expressions (15)–(18) are slightly different and they take the forms

$$I_1(\omega|\mathbf{y}) = \frac{1}{4J}\frac{\hat{a}_c^2}{(\mathbf{y}^T\mathbf{P}^{\perp}\mathbf{y})^{\frac{N-2}{2}}} \quad (24)$$

$$I_2(\omega|\mathbf{y}) = \frac{1}{4(N-4)\ J\ h_c(\omega)\ (\mathbf{y}^T\mathbf{P}^{\perp}\mathbf{y})^{\frac{N-4}{2}}} \quad (25)$$

$$I_3(\omega|\mathbf{y}) = \frac{1}{4J}\frac{\hat{a}_s^2}{(\mathbf{y}^T\mathbf{P}^{\perp}\mathbf{y})^{\frac{N-2}{2}}} \quad (26)$$

and

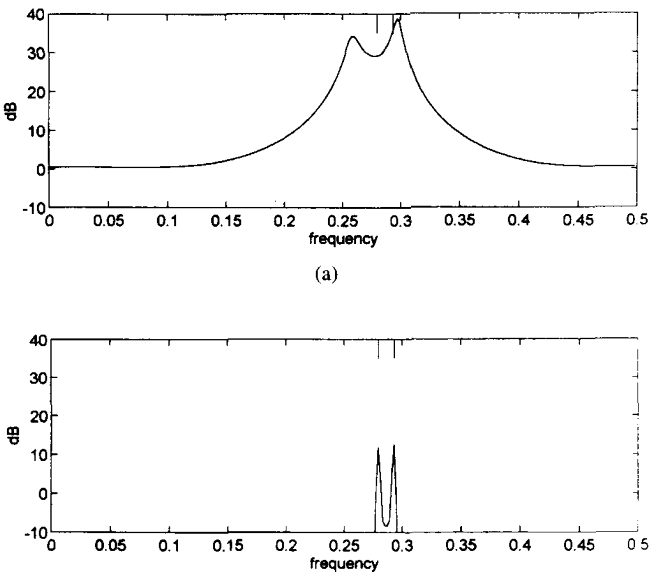
$$I_4(\omega|\mathbf{y}) = \frac{1}{4(N-4)\ J\ h_s(\omega)\ (\mathbf{y}^T\mathbf{P}^{\perp}\mathbf{y})^{\frac{N-4}{2}}}. \quad (27)$$

where $\hat{a}_c$ and $\hat{a}_s$ are the amplitude estimates of the sinusoid [as defined in (23)], provided its frequency is ω.

Clearly, this estimator requires multidimensional integrations that may be considered as its disadvantage. With the advance of multiple integration techniques however, this issue is losing its importance [3].

V. Computer Simulation Examples

To verify the performance of our spectrum estimator, we generated a data vector according to (1) with two sinusoids ($m = 2$). The length of $\mathbf{y}$ was $N = 25$, and the frequencies and phases of the sinusoids were $\omega_1 = 2\pi 0.28$, $\omega_2 = 2\pi\ (0.28 + \frac{1}{3N})$, $\phi_1 = 0.5$ rad, and $\phi_2 = 0$ rad, respectively. The sinusoids had identical amplitudes, and the signal-to-noise ratio for each sinusoid defined by SNR $= 10log\frac{A^2}{2\sigma^2}$, where A is the amplitude of the sinusoid, was 25 dB. Note that the frequency separation of the sinusoids is three times less the Rayleigh resolution. We compared the Bayesian spectrum estimator with MUSIC, which is considered to have excellent resolution.



(a)

(b)

Fig. 2. Power spectrum estimates of the (a) MUSIC and (b) Bayesian spectrum estimators.

The results are shown in Fig. 2. The estimated spectrum by MUSIC is displayed in Fig. 2(a), and the estimated spectrum by the Bayesian method in Fig. 2(b). The markers in the upper portion of each figure show the true location of the spectrum peaks. Repeated trials show very similar results. Clearly, the Bayesian estimator provides a much more accurate estimate. Also, it should be noted that the area under the Bayesian spectrum is an estimate of the true power of the signal.

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