Uniform Random Parameter Generation of Stable Minimum-Phase Real ARMA (p, q) Processes

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***Abstract*—An algorithm to randomly generate the parameters of stable invertible autoregressive moving average processes of order (p, q)—ARMA (p, q)—is presented. The AR and MA portions are independent of each other, and their respective parameters have jointly uniform distributions with support defined by stability and invertibility considerations. The uniform density insures that each possible model is equally likely. The algorithm uses the Levinson–Durbin recursion to guarantee the poles and zeros are inside the unit circle, thus avoiding coefficient resampling typical of "generate and test" methods. To initialize the Levinson–Durbin recursion for each model order, the reflection coefficients are generated using a rejection sampling technique.**

***Index Terms*—Bayesian parameter estimation, Gibbs sampling, Levinson–Durbin algorithm, system identification.**

I. Introduction

A PROBLEM common in many engineering fields is the estimation of parameters of an unknown system excited by white noise given an observed output data sequence. For stochastic linear time-invariant systems, the most often employed methods use the power spectral density, from which a stable minimum phase transfer function can be derived by spectral factorization. This type of analysis occurs in the areas of system identification [3] and adaptive signal processing [2]. Other uses of an input-output relations are for building models of random processes, such as in parametric time series modeling [4], [7]. For linear processes, the models are built by assuming that the observed data sequence was generated by exciting a linear system with white noise. Among the random process descriptions, the ARMA (p, q) type has been widely used in analytic studies in signal processing. Within this class, generally speaking, the stable minimum-phase ARMA models are preferred. Nonminimum-phase signal models can necessitate the inclusion of inherently unstable filters into the processing structure in order to implement the overall processing function.

In the above applications, during the course of the analysis, it is sometimes required to generate stable minimum phase system models completely at random. A specific example is Bayesian parametric estimation of a time series where the process coefficients are to be estimated using a technique such as the Gibbs sampler [5]. In situations where the process parameters appear as location parameters in the likelihood function, the most appropriate probability density function (pdf) for the process coefficients, the Jeffreys' prior, is uniform [1]. A uniform joint pdf on the AR and MA parameters respectively implies that all the possible stable invertible models are equally likely.

A straightforward but inefficient way to randomly generate a stable invertible ARMA (p, q) system with uniform pdf on the coefficients are the generate and test methods. If either the AR or MA polynomial, or both, possess zeros not in the interior of the unit circle, then that polynomial must be completely regenerated and retested. For high order systems, the repeating of the generation and test is computationally inefficient process. In this letter, for generation of the ARMA parameters, we propose an approach that does not require testing. The method, which is based on the Levinson–Durbin recursion algorithm, guarantees that the parameters are drawn from a uniform pdf over the stability and invertibility regions of the process.

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II. Algorithm Development

The algorithm presented is intended for use with the MA and AR portions separately, as it guarantees the synthesis of an arbitrary degree polynomial with zeros inside the unit circle and uniform pdf on the coefficients. Thus, a stable minimum-phase ARMA system can be realized by cascading a stable AR and minimum-phase MA system, each realized with the method. For the moment, we discuss the synthesis of only the AR(p) portion of an ARMA (p, q) system. Synthesis of the MA (q) portion follows similarly. The polynomial corresponding to the normalized AR(p) process is given by

$$A(z) = \sum_{i=0}^{p} a_{i,p} z^{-i} \tag{1}$$

where $a_{0,p} = 1$. The parameters of the AR(p) process, $a_{1,p}, \cdots, a_{p,p}$, can be generated by the well-known Levinson–Durbin recursion relations [6]

$$a_{i,k} = a_{i,k-1} + a_{k,k} a_{k-i,k-1} \tag{2}$$

where $i = 1, 2, \cdots, k-1$, $k = 2, 3, \cdots, p$, and $|a_{1,1}|, |a_{k,k}| < 1$. The Levinson–Durbin algorithm automatically enforces the stability constraint for the zero locations for polynomials of arbitrary degree. However, the uniform joint pdf requirement on the coefficients still needs attention. Straightforward construction of a pdf in

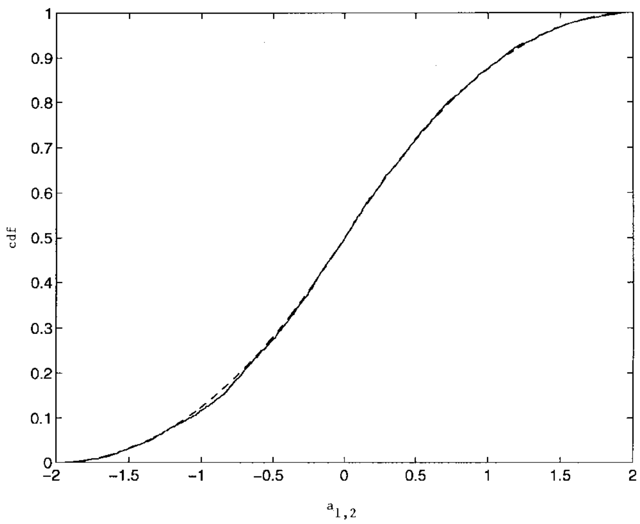

Fig. 1. Theoretical (solid line) and experimental (dashed line) cumulative distribution functions of the $a_{1,2}$ coefficient.

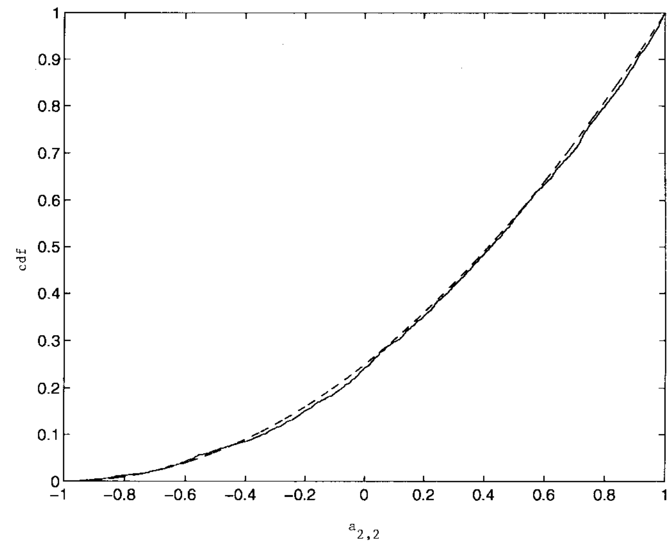

Fig. 2. Theoretical (solid line) and experimental (dashed line) cumulative distribution functions of the $a_{2,2}$ coefficient.

$\Re^p$ with support defined by the zero placement criteria is nontrivial. The following lemma reduces the burden of this calculation to a simple closed-form result useful for the AR and MA portions separately. The result relies on the Jacobian of each recursion in (2), assuming $a_{k,k}$ is given, to transform the pdf $f(a_{1,k-1}, \cdots, a_{k-1,k-1})$ to the pdf $f(a_{1,k}, \cdots, a_{k-1,k}|a_{k,k})$.

Lemma 1: If $a_{1,1}$ is drawn from a uniform pdf whose support is (−1, 1), $U(-1, 1)$, and $\{a_{k,k}\}_{k=2}^{p}$ is drawn from a pdf proportional to

$$|J_k(a_{k,k})| = |[a_{k,k} + (-1)^k]J_{k-1}(a_{k,k})| \tag{3}$$

where J_k is the Jacobian of the $(k-1)$th recursion and $J_1 \equiv 1$, then the Levinson–Durbin recursion (2) synthesizes p coefficients, $a_{1,p}, \cdots, a_{p,p}$, such that their joint pdf is uniform over the region of stability and zero elsewhere.

Proof: The case for $k = 1$ is trivial as no recursion is required. So consider $k = 2, \cdots, p$, and define the set $\mathcal{A}^k \subset \Re^k$ as the region of support for $f(a_{1,k}, \cdots, a_{k,k})$ such that $\forall (a_{1,k}, \cdots, a_{k,k}) \in \mathcal{A}^k$

$$\begin{aligned} f(a_{1,k}, \cdots, a_{k,k}) &= f(a_{k,k})f(a_{1,k}, \cdots, a_{k-1,k}|a_{k,k}) \\ &\propto \text{const} \end{aligned} \tag{4}$$

and all the zeros of $A(z) = \sum_{i=0}^{p} a_{i,p}z^{-i}$ are on the interior of the unit circle. For any $k \geq 2$ the transformations in (2) are 1:1 in $\Re^k$, because $|a_{k,k}| < 1$. Then assuming each $f(a_{1,k-1}, \cdots, a_{k-1,k-1}) \propto$ const, the recursion yields $f(a_{1,k}, \cdots, a_{k-1,k}|a_{k,k}) \propto |J_k(a_{k,k})|^{-1}$, where $J_k(a_{k,k})$ is the Jacobian of the $(k-1)$th recursion of (2), considering $a_{k,k}$ given. The Jacobians are expressed as

$$J_k = \begin{vmatrix} \dfrac{\partial a_{k-1,k}}{\partial a_{1,k-1}} & \cdots & \dfrac{\partial a_{k-1,k}}{\partial a_{k-1,k-1}} \\ \vdots & & \vdots \\ \dfrac{\partial a_{1,k}}{\partial a_{1,k-1}} & \cdots & \dfrac{\partial a_{1,k}}{\partial a_{k-1,k-1}} \end{vmatrix}. \tag{5}$$

So (4) is satisfied $\forall k \geq 2$ when $f(a_{k,k}) \propto |J_k(a_{k,k})|$ for $a_{k,k} \in (-1, 1)$ and $a_{1,1} \sim U(-1, 1)$. Also, with $a_{1,1} \sim U(-1, 1)$ and the $\{a_{k,k}\}_{k=2}^{p}$ determined by Levinson–Durbin's recursion (2), all the zeros of $A(z) = \sum_{i=0}^{k} a_{i,k}z^{-i}$, with $a_{0,k} = 1$, are guaranteed to be on the interior of the unit circle. Therefore, $(a_{1,k}, \cdots, a_{k,k}) \in \mathcal{A}^k$ with a uniform joint pdf.

The recursive form of the Jacobians, $J_k = \det(M_k)$, is derived from (5) with x replacing the appropriate $a_{k,k}$. For $k \geq 3$, each matrix $M_k \in \Re^{k-1,k-1}$ has the form

$$M_k = \begin{bmatrix} x & 0 & \cdots & 1 \\ 0 & & & 0 \\ \vdots & & M_{k-2} & \vdots \\ & & & \\ 1 & 0 & \cdots & x \end{bmatrix}. \tag{6}$$

Then $\det(M_k)$ can be found using co-factor expansions, and noting the sign pattern, we can write

$$\begin{aligned} \det(M_k) &= (x^2 - 1) \det(M_{k-2}) \\ &= [x + (-1)^k][x + (-1)^{k-1}] \det(M_{k-2}). \end{aligned} \tag{7}$$

By direct calculation, with $\det(M_1) \equiv 1$, we have

$$\det(M_2) = x + 1 \tag{8}$$

$$\det(M_3) = (x - 1) \det(M_2) \tag{9}$$

$$\det(M_4) = (x + 1) \det(M_3). \tag{10}$$

Using (7) and (8)–(10), we obtain by induction

$$\det(M_k) = [x + (-1)^k] \det(M_{k-1}). \tag{11}$$

Since $J_k \equiv \det(M_k)$, (11) yields the recursion

$$J_k(x) = [x + (-1)^k]J_{k-1}(x) \qquad \text{for } k \geq 2. \tag{12}$$

If we replace x with $a_{k,k}$, and use the definition $J_1 \equiv 1$, the recursive relationship (3) follows.

Q.E.D.

Further, note that the polynomial $J_k(a_{k,k})$ is a continuous function on the closed interval [−1, 1], and that $(a_{k,k})$ is bounded on (−1, 1). Therefore, $|J_k(a_{k,k})|$ satisfies the conditions for using rejection sampling [9] to generate the values of each $a_{k,k}$. Rejection sampling is well suited for this particular case, however, other methods for implementing random draws could also be used [8]. Combining the rejection sampling technique with the lemma, polynomials of appropriate degree can be generated to produce the AR(p), MA(q), and ARMA (p, q) systems with guaranteed stability and minimum-phase properties. In addition, the model generated has been chosen from all the possible stable and invertible systems completely at random because of the uniform joint pdf on the coefficients. The procedure is explicitly outlined below.

III. Synthesis Procedure

The procedure given here synthesizes an ARMA (p, q) system with each reflection coefficient $a_{k,k} \in (-1, 1)$.

Step 1)
Set $a_{0,p} = 1$.
Draw $a_{1,1}$ from the pdf $U(-1, 1)$.

Step 2)
Let $J_2(a_{2,2}) = [a_{2,2} + (-1)^2]J_1(a_{2,2}) = (a_{2,2} + 1)$.
Draw $a_{2,2}$ from a pdf proportional to $|J_2|$.
Use (2) with $k = 2$ to determine $a_{1,2}$.

Step 3)
Let $J_3(a_{3,3}) = [a_{3,3} + (-1)^3]J_2(a_{3,3}) = (a_{3,3}^2 - 1)$.
Draw $a_{3,3}$ from a pdf proportional to $|J_3|$.
Use (2) with $k = 3$ to determine $a_{1,3}$, and $a_{2,3}$.

$\vdots$

Step p)
Let $J_p(a_{p,p}) = [a_{p,p} + (-1)^p]J_{p-1}(a_{p,p})$.
Draw $a_{p,p}$ from a pdf proportional to $|J_p|$.
Use (2) with $k = p$ to determine $a_{1,p}, \cdots, a_{p-1,p}$.

Finally, use the so obtained coefficients to form the AR polynomial. Repeat the procedure (with q replacing p) to generate the MA coefficients.

IV. Example

To illustrate the results of the synthesis procedure, 2500 independent sets of coefficients of an AR(2) process were generated. Figs. 1 and 2 show the theoretical and empirical cumulative distribution functions of the process coefficients $a_{1,2}$ and $a_{2,2}$. Fig. 3 displays the scatter diagram of the individual outcomes. The experiment clearly shows agreement between the obtained and desired results. The algorithm generates the coefficients with a pdf that is uniform over the stability (invertibility) region.

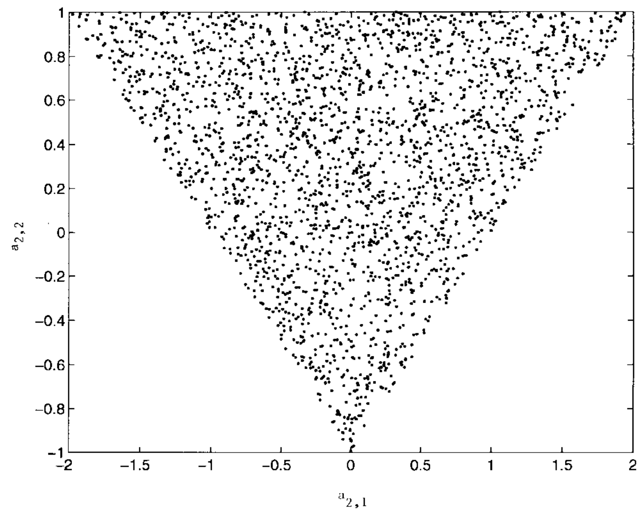

Fig. 3. Scatter diagram of 2500 independent draws of $a_{1,2}$ and $a_{2,2}$.

V. Conclusions

We have proposed a synthesis method that generates the coefficients of a stable and invertible ARMA (p, q) process such that the generated coefficients are drawn from a uniform pdf. The method uses the Levinson–Durbin's recursion algorithm to place the poles and zeros within the unit circle, and thereby guarantees a stable and invertible system. Lastly, the approach presented here is a single-pass synthesis algorithm, which is suitable for a digital computer program, since it avoids the redundant looping of the "generate and test" methods.

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