$\dot{\underline{x}} = A\underline{x}$

THERE IS A P SUCH THAT $PAP^{-1} = J$ IS IN JCF.

LET $\bar{\underline{x}} = P\underline{x}$. THEN $\dot{\bar{\underline{x}}} = J\bar{\underline{x}}$

& $\|\bar{\underline{x}}\| \leq \|P\| \, \|\underline{x}\|$ & $\|\underline{x}\| \leq \|P^{-1}\| \, \|\bar{\underline{x}}\|$

SO $\bar{\underline{x}}$ IS BOUNDED OR $\to \underline{0}$ IF AND ONLY IF $\underline{x}$ DOES.

SO FOR EACH JORDAN BLOCK, J_k,

$$\dot{\underline{x}}_k = J_k \underline{x}_k$$

$$\Rightarrow \underline{x}_k(t) = e^{J_k t} \underline{x}_k(0)$$

$$= \underbrace{\begin{pmatrix} e^{\lambda_k t} & te^{\lambda_k t} & \cdots & \\ & \ddots & \ddots & \\ 0 & & \ddots & te^{\lambda_k t} \\ & & & e^{\lambda_k t} \end{pmatrix}}_{m_k} \underline{x}_k(0)$$

SO IF $Re(\lambda_k) < 0$, $\|\underline{x}_k(t)\| \to 0$.

IF $Re(\lambda_k) > 0$, $\|\underline{x}_k(t)\| \to \infty$

IF $Re(\lambda_k) = 0$, THEN

$\|\underline{x}_k(t)\| = \|\underline{x}_k(0)\|$ IF $m_k = 1$

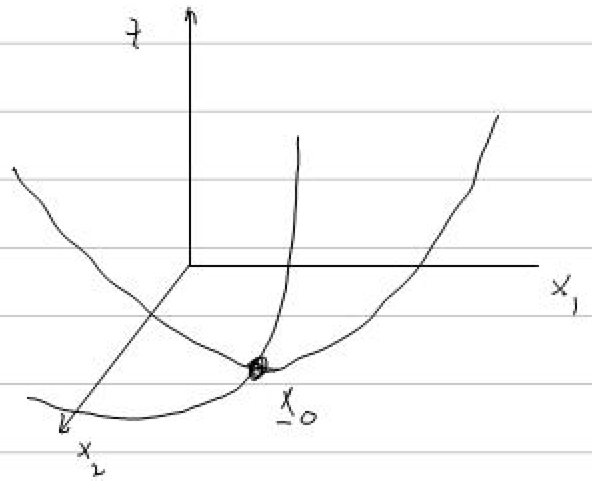
& $\|\underline{x}_k(t)\| \to \infty$ IF $m_k > 1$

$z = g(\underline{x})$

with $g(\underline{x}_0) = 0$

& $g(\underline{x}) > 0$ for $\underline{x} \neq \underline{x}_0$

$\dot{\underline{x}} = \underline{f}(\underline{x})$



IF WE KNOW THAT $\dot{\underline{x}}$ ALWAYS POINTS DOWNHILL ON THE SURFACE $z = g(\underline{x})$, THEN THE SYSTEM IS STABLE.

SO WE NEED $\nabla g(\underline{x}) \cdot \dot{\underline{x}} < 0$ FOR ALL $\underline{x}$

$$\Longleftrightarrow \quad \nabla g(\underline{x}) \cdot \underline{f}(\underline{x}) < 0 \quad " \quad " \quad \underline{x}$$

LINEAR CASE: $z = \underline{x}^T M \underline{x}$ WITH M POSITIVE DEFINITE.

FIND:

$$\begin{aligned}
\nabla z(\underline{x}) \cdot \dot{\underline{x}} &= \nabla z(\underline{x}) \cdot A\underline{x} \\
&= 2\underline{x}^T M \cdot A\underline{x} \\
&= \underline{x}^T (2MA)\underline{x} \\
&= \underline{x}^T (MA + A^T M^T)\underline{x} \\
&= \underline{x}^T (A^T M + MA)\underline{x}
\end{aligned}$$

SO IF $\nabla z(\underline{x}) \cdot \dot{\underline{x}} < 0$ FOR $\underline{x} \neq 0$, THEN

$$A^T M + MA = -N \quad \text{WITH } N \text{ POS. DEF.}$$

$$A^T M + MA = \int_0^\infty \left(A^T e^{A^T t} N e^{At} + e^{A^T t} N e^{At} A\right) dt$$

$$= \int_0^\infty \frac{d}{dt}\left(e^{A^T t} N e^{At}\right) dt$$

$$= 0 - e^{A^T 0} N e^{A 0}$$

$$= -N$$

$$= \int_0^\infty e^{A^T t} N e^{At}\, dt$$

$$\underline{x}^T \int_0^\infty e^{A^T t} N_0^T N_0 e^{At}\, dt\, \underline{x} = \int_0^\infty \underline{x}^T e^{A^T t} N_0^T N_0 e^{At} \underline{x}\, dt$$

$$= \int_0^\infty \left(N_0 e^{At} \underline{x}\right)^T \left(N_0 e^{At} \underline{x}\right) dt$$

$$= \int_0^\infty \underline{y}^T \underline{y}\, dt = \int_0^\infty \|\underline{y}\|^2 dt$$

WHERE $\underline{y} = N_0 e^{At} \underline{x}$

Ex. 5.5 $$\dot{\underline{x}} = \begin{pmatrix} -1 & e^{2t} \\ 0 & -1 \end{pmatrix} \underline{x} = A(t)\underline{x}$$

EIGENVALUES OF $A(t)$: $\lambda = -1$, MULT. 2.

BUT THE SOLUTION:

$$\dot{x}_1 = -x_1 + e^{2t} x_2$$

$$\& \quad \dot{x}_2 = -x_2 \implies x_2(t) = e^{-t} x_2(0)$$

$$\& \quad \dot{x}_1 = -x_1(t) + e^{t} x_2(0)$$

$$\implies x_1(t) = Ae^{-t} + Be^{t} \quad \text{— UNBOUNDED.}$$

SO SYSTEM IS UNSTABLE.

$$\underline{u}(t) = -B^T e^{A^T(t_1-t)} W_c^{-1}(t_1)\left[e^{A^T t_1}\underline{x}_0 - \underline{x}_1\right] \qquad e^{A^T t}$$

$$\underline{x}(t) = e^{At}\underline{x}_0 + \int_0^t e^{A(t-\tau)} B\underline{u}(\tau)\,d\tau$$

$$\Rightarrow \underline{x}(t) = e^{At}\underline{x}_0 - \int_0^t e^{A(t-\tau)} B B^T e^{A^T(t_1-\tau)} W_c^{-1}(t_1)\left(e^{A^T t_1}\underline{x}_0 - \underline{x}_1\right) d\tau$$

$$= e^{At}\underline{x}_0 - \int_0^t e^{A(t-\tau)} B B^T e^{A^T(t_1-\tau)}\,d\tau \;\; W_c^{-1}(t_1)\left(e^{A^T t_1}\underline{x}_0 - \underline{x}_1\right)$$

$$\Rightarrow \quad \underline{x}(t_1) = e^{At_1}\underline{x}_0 - W_c(t_1) W_c^{-1}(t_1)\left(e^{A^T t_1}\underline{x}_0 - \underline{x}_1\right)$$

$$= \underline{x}_1$$

$$\underline{x}' e^{At} B = 0 \quad \text{FOR ALL } t \in [0, t_1] \text{ WITH } t_1 > 0$$

$$\Longleftrightarrow \quad \underline{x}' A^k B = 0 \quad \text{FOR ALL } k,\ 0 \le k \le n-1$$

OR

$$\underline{x}'\left(\sum_{l=0}^{\infty} \frac{A^l}{l!} t^l\right) B = 0$$

$$\Longleftrightarrow \quad \sum_{l=0}^{\infty} \underline{x}' A^l B \cdot \frac{t^l}{l!} = 0 \quad \text{FOR ALL } t \in [0, t_1]$$

$$\Longleftrightarrow \quad \underline{x}' A^l B = 0 \quad \text{FOR ALL } l \ge 0$$

$$\Longleftrightarrow \quad \underline{x}' A^l B = 0 \quad \text{FOR ALL } l \text{ WITH } 0 \le l \le n-1$$

(BY CAYLEY-HAMILTON)