Notes:

4 State-Space Solutions & Realizations (Chapter 4):

4.1 Solution of Linear, Time-Invariant Equations:

If

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

then

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau$$

Proof: Substitute.

Note: Second term is a convolution, and using the convolution theorem for the Laplace transform gives:

$$\mathbf{X}(s) = (sI - A)^{-1}B\mathbf{U}(s) + (sI - A)^{-1}\mathbf{x}(0)$$

— already known, so gives alternative proof.

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Notes:

Computing Matrix Exponential:

Need to compute exponential: four possible ways

1. Find eigenvalues of A , and find polynomial which is equal on spectrum — often easiest.
2. Use Jordan form — usually complicated.
3. Power series — usually impractical
4. Laplace transform — need to compute $(sI - A)^{-1}$
   (a) Direct inverse — symbolic
   (b) Equality on spectrum — calculate exponential directly?
   (c) Jordan form — usually complicated.
   (d) Geometric series — usually impractical
   (e) Leverrier algorithm (problem 3.26)

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Notes:

Matrix Exponential Examples:

Example 4.1: Compute $(sI - A)^{-1}$ using a and b.

Example 4.2: Solve

$$d\mathbf{x}/dt = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Solution:

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix} \mathbf{x}(0) \\ &+ \begin{bmatrix} -\int_0^t (t-\tau)e^{-(t-\tau)}u(\tau)d\tau \\ \int_0^t [1-(t-\tau)]e^{-(t-\tau)}u(\tau)d\tau \end{bmatrix} \end{aligned}$$

Note that each entry of e^{At} is analytic.

Read Section 4.2.1

Slide 3

Notes:

Discrete-Time Solution:

The discrete-time state equations:

$$\begin{aligned}\mathbf{x}_{n+1} &= A\mathbf{x}_n + B\mathbf{u}_n \\ \mathbf{y}_n &= C\mathbf{x}_n + D\mathbf{u}_n\end{aligned}$$

have the solutions

$$\begin{aligned}\mathbf{x}_k &= A^k\mathbf{x}_0 + \Sigma_{m=0}^{k-1} A^{k-1-m} B\mathbf{u}_m \\ \mathbf{y}_k &= CA^k x(0) + \Sigma_{m=0}^{k-1} CA^{k-1-m} B\mathbf{u}_m + D\mathbf{u}_k\end{aligned}$$

Again agrees with transform solution:

$$\mathbf{X}(z) = (zI - A)^{-1} B\mathbf{U}(z) + z(zI - A)^{-1}\mathbf{x}(\mathbf{0})$$

Slide 4

Notes:

Algebraic (State) Equivalence (Sec. 4.3):

What happens when we choose a different basis for the state space?

Let P be the matrix which gives the change of basis. Then $\bar{\mathbf{x}} = P\mathbf{x}$, and so

$$\begin{aligned} d\bar{\mathbf{x}}/dt &= d(P\mathbf{x})/dt \\ &= Pd\mathbf{x}/dt \\ &= PA\mathbf{x}(t) + PB\mathbf{u}(t) \\ &= PAP^{-1}\bar{\mathbf{x}}(t) + PB\mathbf{u}(t) \end{aligned}$$

and

$$\begin{aligned} \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t) \\ &= CP^{-1}\bar{\mathbf{x}}(t) + D\mathbf{u}(t) \end{aligned}$$

So $\bar{A} = PAP^{-1}$; $\bar{B} = PB$; $\bar{C} = CP^{-1}$; $\bar{D} = D$.

– Algebraic equivalence.

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Notes:

Algebraic Equivalence continued:

Algebraically equivalent systems have the same eigenvalues and same transfer function:

$$\begin{aligned}(\lambda I - \bar{A})^{-1} &= (\lambda P P^{-1} - P A P^{-1})^{-1} \\ &= (P(\lambda I - A)P^{-1})^{-1} \\ &= P(\lambda I - A)^{-1} P^{-1}\end{aligned}$$

and so $\bar{A}$ and A have the same eigenvalues, and

$$\begin{aligned}\bar{C}(sI - \bar{A})^{-1}\bar{B} &= C P^{-1} P (sI - A)^{-1} P^{-1} P B \\ &= C(sI - A)^{-1} B\end{aligned}$$

which shows that the two systems have the same transfer function (matrix).

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Notes:

Transfer Function (Zero-State) Equivalence:

Zero-State equivalence: Systems have the same transfer function.

Equivalently:

$$\begin{aligned} \bar{C}(sI-\bar{A})^{-1}\bar{B} &= C(sI-A)^{-1}B \\ \Longleftrightarrow \frac{1}{s}\bar{C}(I-\frac{1}{s}\bar{A})^{-1}\bar{B} &= \frac{1}{s}C(I-\frac{1}{s}A)^{-1}B \\ \Longleftrightarrow \sum_{n=0}^{\infty}\frac{1}{s^{n+1}}\bar{C}\bar{A}^n\bar{B} &= \sum_{n=0}^{\infty}\frac{1}{s^{n+1}}CA^nB \end{aligned}$$

(expanding both sides in a geometric series) and so transfer function equivalence is equivalent to

$$\begin{aligned} \bar{D} &= D \\ \bar{C}\bar{A}^m\bar{B} &= CA^mB \end{aligned}$$

for all m.

–Weaker.

Example 4.4

Read sec. 4.3.1(complex to real) & sec. 4.3.2 (op-amp scaling).

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Notes:

4.2 Realizations (Section 4.4):

Given $G(s)$, find (A, B, C, D) with
$G(s) = C(sI - A)^{-1}B + D$.

Theorem 4.2: $G(s)$ has a realization if, and only if, $G(s)$ is proper.

First, suppose there is a realization, and let
$G_{sp}(s) = C(sI - A)^{-1}B$. Then $G_{sp}(s)$ has a denominator with degree n and each term in the numerator has degree at most $n - 1$. Therefore, since
$G(s) = C(sI - A)^{-1}B + D$, $G(s)$ is proper.

Conversely, given $G(s)$, let $G(s) = G_{sp}(s) + G(\infty)$. In any realization, $G(\infty) = D$, so focus on $G_{sp}(s)$. Let

$$d(s) = s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r$$

be the least common denominator of all entries of $G_{sp}(s)$.
Then

$$G_{sp}(s) = \frac{1}{d(s)}\left[N_1 s^{r-1} + N_2 s^{r-2} \cdots + N_{r-1}s + N_r\right]$$

where N_i are $q \times p$ constant matrices.

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Notes:

Realization (continued):

Then (A, B, C) is a realization (controllable canonical form) of $G_{sp}(s)$, where:

$$A = \begin{bmatrix} -\alpha_1 I_p & -\alpha_2 I_p & \cdots & -\alpha_{r-1} I_p & -\alpha_r I_p \\ I_p & 0 & \cdots & 0 & 0 \\ 0 & I_p & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_p & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} I_p \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} N_1 & N_2 & \ldots & N_{r-1} & N_r \end{bmatrix}$$

A is in block companion form, and is $rp \times rp$. The state dimension is rp .

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Notes:

Realization (continued):

Proof: Let

$$(sI - A)^{-1}B = Z(s) = \begin{bmatrix} Z_1(s) \\ Z_2(s) \\ \vdots \\ Z_r(s) \end{bmatrix}$$

where each $Z_i(s)$ is $p \times p$.

Then

$$C(sI - A)^{-1}B = N_1Z_1 + N_2Z_2 + \cdots + N_rZ_r$$

and

$$(sI - A)Z = B$$

or

$$sZ = AZ + B$$

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Notes:

Realization (continued):

This gives

$$sZ_2 = Z_1, sZ_3 = Z_2, \ldots, sZ_r = Z_{r-1}$$

and so

$$Z_2 = Z_1/s, Z_3 = Z_1/s^2, \ldots, Z_r = Z_1/s^{r-1}$$

Substitute in the first block of $Z(s)$ to get:

$$\begin{aligned} sZ_1 &= -\alpha_1 Z_1 - \alpha_2 Z_2 \cdots - \alpha_r Z_r + I_p \\ &= -\left(\alpha_1 + \alpha_2/s \cdots + \alpha_r/s^{r-1}\right) Z_1 + I_p \end{aligned}$$

and so

$$I_p = \left(s + \alpha_1 + \alpha_2/s \cdots + \alpha_r/s^{r-1}\right) Z_1 = \frac{d(s)}{s^{r-1}} Z_1$$

Therefore,

$$Z_1 = \frac{s^{r-1}}{d(s)} I_p, Z_2 = \frac{s^{r-2}}{d(s)} I_p, \ldots, Z_r = \frac{1}{d(s)} I_p$$

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Notes:

Realization (continued):

Substitute in

$$C(sI-A)^{-1}B = N_1 Z_1 + N_2 Z_2 \cdots + N_r Z_r$$

to get

$$C(sI-A)^{-1}B = \frac{1}{d(s)}\left[N_1 s^{r-1} + N_2 s^{r-2} \cdots + N_r\right]$$

as required.

Example 4.6, p. 103.

Note: There are numerous realizations, and this is not normally the best (or even good): there are also the observable canonical form; combinations of SIMO; combinations of MISO, etc., none of which is usually minimal.

Example 4.7, p. 105; note different dimensions, so *zero-state equivalent*, but not *algebraically equivalent*

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Notes:

4.3 Solution of Linear Time-Varying Equations (Sec. 4.5):

Given a *time-varying* state-variable system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) \\ \mathbf{y}(t) &= C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t) \end{aligned}$$

what can be said about the solution?

Note:

1. For constant A, the solution of $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$ is $e^{At}\mathbf{x}(0)$.
2. For scalar $a(t)$, the solution of $\dot{x}(t) = a(t)x(t)$ is $e^{\int_0^t a(\tau)d\tau}x(0)$.

BUT!!

The corresponding expression DOES NOT work for matrix $A(t)$; There is NO CLOSED FORM!!!

From the theory of ordinary differential equations, for each initial vector $\mathbf{x}_i(t_0)$, there is a unique solution $\mathbf{x}_i(t)$.

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Fundamental and State Transition Matrices:

If $X = [\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)]$, where each $\mathbf{x}_i(t)$ is a solution of the equation, then $\dot{X}(t) = A(t)X(t)$.

If the initial states $\{\mathbf{x}_1(t_0), \mathbf{x}_2(t_0), \ldots, \mathbf{x}_n(t_0)\}$ are linearly independent, then $\{\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)\}$ are linearly independent for all t; this follows from the uniqueness of the solution.

Then $X(t)$ is called a *fundamental matrix* for the equation, and the linear independence implies that $X(t)$ is invertible for all t.

Definition: $\Phi(t, t_0) = X(t)X^{-1}(t_0)$ is called the state transition matrix of the equation $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$.

Alternative Definition: $\Phi(t, t_0)$ is the solution of

$$\frac{d}{dt}\Phi(t, t_0) = A(t)\Phi(t, t_0)$$

with initial condition $\Phi(t_0, t_0) = I$.

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Notes:

Notes:

State Transition Matrix Properties:

Properties:

1. $\Phi(t_0, t_0) = I$
2. $\Phi^{-1}(t, t_0) = \Phi(t_0, t)$
3. $\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)$

The solution of

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$

is given by:

$$\begin{aligned}\mathbf{x}(t) &= \Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)\mathbf{u}(\tau)d\tau \\ &= \Phi(t, t_0)\left[\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t_0, \tau)B(\tau)\mathbf{u}(\tau)d\tau\right]\end{aligned}$$

and the impulse response for state equations is:

$$\begin{aligned}G(t, \tau) &= C(t)\Phi(t, \tau)B(\tau) + D(t)\delta(t - \tau) \\ &= C(t)X(t)X^{-1}(\tau)B(\tau) + D(t)\delta(t - \tau) \\ &= M(t)N(\tau) + D(t)\delta(t - \tau)\end{aligned}$$

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Notes:

Time-Varying Equivalence (Sec. 4.6):

For the equation

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) \\ \mathbf{y}(t) &= C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t) \end{aligned}$$

Assume a time-varying change of basis: $\bar{\mathbf{x}} = P(t)\mathbf{x}(t)$, where $P(t)$ is $n \times n$, $P(t)$ is invertible, and assume that $\dot{P}(t)$ is continuous. Then

$$\begin{aligned} \frac{d}{dt}\bar{\mathbf{x}}(t) &= P(t)\dot{\mathbf{x}}(t) + \dot{P}(t)\mathbf{x}(t) \\ &= P(t)A(t)\mathbf{x}(t) + P(t)B(t)\mathbf{u}(t) + \dot{P}(t)\mathbf{x}(t) \\ &= \bar{A}(t)\bar{\mathbf{x}}(t) + \bar{B}(t)\mathbf{u}(t) \end{aligned}$$

and

$$\begin{aligned} \mathbf{y}(t) &= C(t)P(t)^{-1}\bar{\mathbf{x}}(t) + D(t)\mathbf{u}(t) \\ &= \bar{C}(t)\bar{\mathbf{x}}(t) + D(t)\mathbf{u}(t) \end{aligned}$$

where

$$\begin{aligned} \bar{A}(t) &= P(t)A(t)P(t)^{-1} + \dot{P}(t)P(t)^{-1} \\ \bar{B}(t) &= P(t)B(t), \quad \text{and} \quad \bar{C}(t) = C(t)P(t)^{-1} \end{aligned}$$

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Notes:

Time-Varying Equivalence (continued):

If $X(t)$ is a fundamental matrix for the original system, then $P(t)X(t)$ is a fundamental matrix for the new system.

Proof:

$$\begin{aligned}\frac{d}{dt}P(t)X(t) &= P(t)\frac{d}{dt}X(t) + \dot{P}(t)X(t)\\ &= P(t)A(t)X(t) + \dot{P}(t)X(t)\\ &= (P(t)A(t) + \dot{P}(t))P^{-1}(t)P(t)X(t)\\ &= \bar{A}(t)P(t)X(t)\end{aligned}$$

Theorem 4.3:

If A_0 is any constant matrix, there is a $P(t)$ which transforms the system into a system with $\bar{A}(t) = A_0$.

Proof:

Let $X(t)$ be a fundamental matrix of $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$. Then

$$\begin{aligned}0 &= \dot{I}\\ &= \frac{d}{dt}(X^{-1}(t)X(t))\\ &= \frac{d}{dt}(X^{-1}(t))X(t) + X^{-1}(t)\dot{X}(t)\end{aligned}$$

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Notes:

Time-Varying Equivalence (continued):

Therefore

$$\begin{aligned}\frac{d}{dt}(X^{-1}(t)) &= -X^{-1}(t)\dot{X}(t)X^{-1}(t)\\ &= -X^{-1}(t)A(t)X(t)X^{-1}(t)\\ &= -X^{-1}A(t)\end{aligned}$$

Now let $P(t) = \bar{X}(t)X^{-1}(t) = e^{A_0 t}X^{-1}(t)$. Then

$$\begin{aligned}\bar{A}(t) &= P(t)A(t)P^{-1}(t) + \dot{P}(t)P(t)^{-1}\\ &= e^{A_0 t}X^{-1}A(t)Xe^{-A_0 t}\\ &\quad + \left(A_0 e^{A_0 t}X^{-1} + e^{A_0 t}\frac{d}{dt}(X^{-1})\right)Xe^{-A_0 t}\\ &= e^{A_0 t}X^{-1}A(t)Xe^{-A_0 t}\\ &\quad + \left(A_0 e^{A_0 t}X^{-1} - e^{A_0 t}X^{-1}A(t)\right)Xe^{-A_0 t}\\ &= A_0 e^{A_0 t}X^{-1}Xe^{-A_0 t}\\ &= A_0\end{aligned}$$

Special Case: If $A_0 = 0$, then $P = X^{-1}(t)$, and $\bar{A} = 0$, $\bar{B} = X^{-1}B$, and $\bar{C} = CX$ — block diagrams.

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Notes:

Time-Varying Equivalence (continued):

Notes:

1. The impulse response $C(t)X(t)X^{-1}(\tau)B(\tau) + D(t)\delta(t-\tau)$ is invariant under a time-varying state transformation, since $\bar{X}(t) = P(t)X(t)$.
2. The A matrix properties (especially stability) are *NOT* preserved under a time-varying state transformation (obvious).
3. For this reason, general time-varying state transformations are not much use; the unbounded gains of an unstable system can be masked by unbounded gains in $P(t)$

Lyapunov Equivalence:

A more useful concept is given by

Definition 4.3: $P(t)$ is a Lyapunov transformation if, in addition to the previous conditions, $P(t)$ and $P^{-1}(t)$ are bounded for all t.

Then

1. Two systems are Lyapunov equivalent if they are related by a time-varying state transformation $P(t)$ where $P(t)$ is a Lyapunov transformation.
2. Stability, and asymptotic stability, are preserved by Lyapunov transformations.

The most important example of Lyapunov equivalence is the following: if a system is periodic, then it is Lyapunov equivalent to a time-invariant system.

Proof (outline): Let $X(t)$ be a fundamental matrix for the system, and let $Q = X^{-1}(0)X(T) = e^{\bar{A}T}$; then $X(t+T) = X(t)Q$ (uniqueness).

Define $P(t) = e^{\bar{A}t}X^{-1}(t)$. Then check that $P(t)$ is periodic, and so $P(t)$ and $P^{-1}(t)$ are bounded.

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Notes: To prove $X(t+T) = X(t)Q$:

$$\dot{X}(t+T) = A(t+T)X(t+T) = A(t)X(t+T)$$

and

$$(X(\dot{t})Q) = A(t)\,(X(t)Q)$$

so that $X_1(t) = X(t+T)$ and $X_2(t) = X(t)Q$ satisfy the same differential equation.

Also, $X_1(0) = X(T)$ and $X_2(0) = X(0)Q = X(T)$, and so $X_1(t)$ and $X_2(t)$ have the same initial conditions.

Therefore, $X_1(t) = X_2(t)$, or $X(t+T) = X(t)Q$.

Notes:

4.4 Time-Varying Realizations (Sec. 4.7):

A time-varying impulse response $G(t,\tau)$ has a time-varying realization if and only if there are time-varying matrix functions $M(t)$ ($q \times n$), $N(t)$ ($n \times p$), and $D(t)$ ($q \times p$) such that

$$G(t,\tau) = M(t)N(\tau) + D(t)\delta(t-\tau)$$

for all $t \geq \tau$.

Proof: One direction is obvious from

$$G(t,\tau) = C(t)X(t)X^{-1}(\tau)B(\tau) + D(t)\delta(t-\tau)$$

For the other direction, use $M(t) = C(t)X(t)$, $N(\tau) = X^{-1}(\tau)B(\tau)$, and $A = 0$ (and so $X(t) = I$).

Do example 4.10

(Note requirement to have solution functions available anyway.)

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Notes:

5 Stability (Sec. 5.2):

Definitions:

1. A function $f(t)$ is *bounded* if there is a constant K such that $|f(t)| \leq K$ for all t.
2. A system is *bounded-input, bounded-output (BIBO) stable* if every bounded input produces a bounded output.
3. A function $f(t)$ is *absolutely integrable* (or $f(t) \epsilon L_1$, or $f(t) \epsilon L_1(-\infty, \infty)$) if $\int_{-\infty}^{\infty} |f(t)| \, dt$ is finite.

Theorem 5.1: A linear, time-invariant system is BIBO stable if, and only if, its impulse response $h(t)$ is absolutely integrable, i.e., $h(t) \epsilon L_1$.

Notes:

Stability (continued):

Proof: Let $u(t)$ denote the input, and $y(t)$ the output. For one direction, assume that the input is bounded by K, and that $h(t)\epsilon L_1$. Then

$$\begin{aligned} |y(t)| &= \left|\int_{-\infty}^{\infty} u(t-\tau)h(\tau)\,d\tau\right| \\ &\leq \int_{-\infty}^{\infty} |u(t-\tau)h(\tau)|\ d\tau \\ &\leq K\int_{-\infty}^{\infty} |h(\tau)|\,d\tau \end{aligned}$$

which is finite, and so $y(t)$ is bounded.

Conversely, assume that $\int_{-\infty}^{\infty} |h(t)|\,dt$ is infinite. Then let $u(t) = sgn(h(-t))$. Then $u(t)$ is bounded by 1, and $y(0) = \int_{-\infty}^{\infty} |h(t)|\,dt$ is infinite. Therefore, a bounded input gives an unbounded output, and so the system is not BIBO stable.

Notes:

1. Extends easily to matrices.
2. $f(t)\epsilon L_1$ implies $\lim_{t\to\infty}\int_t^{\infty} |f(\tau)|\,d\tau = 0$, but *not* that $f(t)$ is bounded.

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Notes:

Stability (continued):

Theorem 5.2: If a system (LTI) is BIBO stable with impulse response $g(t)$, its response $y_a(t)$ to $e^{j\omega_0 t}U(t)$ satisfies

$$\lim_{t\to\infty} \left(y_a(t) - G(j\omega_0)e^{j\omega_0 t}\right) = 0$$

Proof:

$$\begin{aligned} & G(j\omega_0)e^{j\omega_0 t} - y_a(t) \\ &= e^{j\omega_0 t}\int_0^\infty g(\tau)e^{-j\omega_0\tau}\,d\tau - \int_0^t g(\tau)e^{j\omega_0(t-\tau)}\,d\tau \\ &= e^{j\omega_0 t}\int_t^\infty g(\tau)e^{-j\omega_0\tau}\,d\tau \end{aligned}$$

and so

$$\begin{aligned} & \lim_{t\to\infty}\left(G(j\omega_0)e^{j\omega_0 t} - y_a(t)\right) \\ &= \lim_{t\to\infty}\left(e^{j\omega_0 t}\int_t^\infty g(\tau)e^{-j\omega_0\tau}\,d\tau\right) \\ &= 0 \end{aligned}$$

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Notes:

Stability (continued):

In particular:

1. response $y_d(t)$ to a unit step $U(t)$:

$$\lim_{t\to\infty} y(t) = G(0)$$

2. response $y(t)$ to $\cos(\omega_0 t)U(t)$:

$$\lim_{t\to\infty} \left(y(t) - |G(j\omega_0)|\cos(\omega_0 t + \phi)\right) = 0$$

   where $\phi = \angle G(j\omega_0)$.

3. response $y(t)$ to $\sin(\omega_0 t)U(t)$:

$$\lim_{t\to\infty} \left(y(t) - |G(j\omega_0)|\sin(\omega_0 t + \phi)\right) = 0$$

   where $\phi = \angle G(j\omega_0)$.

In other words, if a linear, time-invariant system is BIBO stable, there is a steady-state sinusoidal response to a sinusoidal input, with *gain* $|G(j\omega_0)|$, and phase-shift $\phi = \angle G(j\omega_0)$.

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Notes:

Stability (continued):

Theorem 5.3: If a LTI system has a (proper) *rational* transfer function $G(s)$, then it is BIBO stable if, and only if, all of the poles of $G(s)$ are in the left half-plane, i.e., all poles have real part < 0.

Proof: By partial fraction expansion, the impulse response is a finite sum of terms of the form $t^k e^{s_p t}$. Let σ_p be the real part of s_p; then $|t^k e^{s_p t}| = t^k e^{\sigma_p t}$, and $\int_0^\infty t^k e^{\sigma_p t}\, dt$ is finite if, and only if, $\sigma_p < 0$.

These properties extend directly to multi-input, multi-output systems, by looking at each entry in the transfer function matrix.

Example 5.1: Note non-rational transfer function.

Note (relevant to next topic): If $G(s)$ is the transfer function of a state-variable system, each pole is an eigenvalue of the A matrix; so if all eigenvalues of A are in the left half-plane, the system is BIBO stable, but the converse may not be true.

Example (Discrete time): Cascade of two systems: the first with transfer function $z^{-1}/(1 - 2z^{-1})$, and the second with transfer function $1 - 2z^{-1}$.

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Notes:

5.1 Internal Stability (Sec. 5.3):

For the equation

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) \Longrightarrow \mathbf{x}(t) = e^{At}\mathbf{x}_0$$

1. **Marginal Stability** means: for every initial condition $\mathbf{x}_0$, there is a constant K such that $||\mathbf{x}(t)|| < K$ for all t.
2. **Asymptotic Stability** means: for every initial condition $\mathbf{x}_0$, $\lim_{t\to\infty} ||\mathbf{x}(t)|| = 0$.

Theorem 5.4:

1. Marginal stability $\Longleftrightarrow$ for each eigenvalue λ of A, $\Re(\lambda) \leq 0$, and if $\Re(\lambda) = 0$, λ is a simple root of the MINIMAL polynomial.
2. Asymptotic stability $\Longleftrightarrow$ for each eigenvalue λ of A, $\Re(\lambda) < 0$.

Proof: Transform to Jordan canonical form.

Example 5.4

Note that asymptotic stability of a state-variable system $\Longrightarrow$ the system is BIBO stable.

Read discrete-time case.

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Notes:

Lyapunov Theorem (Sec. 5.4):

Terminology: A is stable $\Longleftrightarrow \Re(\lambda) < 0$ for every eigenvalue λ of A.

Theorem 5.5: A stable if, and only if, for any given positive definite matrix N, the Lyapunov equation

$$A'M + MA = -N$$

has a unique symmetric positive definite solution M.

Corollary 5.5: A is stable if, and only if, for any given $m \times n$ matrix N_0 with $m < n$ and $rank(\mathbf{O}) = n$, where

$$\mathbf{O} = \begin{bmatrix} N_0 \\ N_0 A \\ \vdots \\ N_0 A^{n-1} \end{bmatrix}$$

the equation

$$A'M + MA = -N_0' N_0$$

has a unique symmetric positive definite solution M.

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Notes:

Lyapunov Theorem continued:

Proof: Necessity: Assume A is stable; by substitution, the solution is

$$M = \int_0^\infty e^{A't} N e^{At} dt$$

which is clearly symmetric and positive definite.

A and A' have the same eigenvalues, and so, if $\Re(\lambda(A)) < 0$ for every eigenvalue λ of A, $\lambda_i + \lambda_j \neq 0$. Therefore the solution M is unique for any N.

For the corollary, the solution is also unique, and is given by

$$M = \int_0^\infty e^{A't} N_0' N_0 e^{At} dt$$

If this is not positive definite, then there is a vector $\mathbf{x} \neq \mathbf{0}$ such that

$$\begin{aligned} \mathbf{0} &= \mathbf{x}' M \mathbf{x} \\ &= \mathbf{x}' \left(\int_0^\infty e^{A't} N_0' N_0 e^{At} dt \right) \mathbf{x} \\ &= \int_0^\infty ||N_0 e^{At} \mathbf{x}||^2 dt \end{aligned}$$

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Notes:

Lyapunov Theorem continued:

Then $N_0 e^{At}\mathbf{x} = 0$ for all t, and, by successive differentiation, $N_0 A^k \mathbf{x} = 0$ for all k. Therefore $\mathbf{O}\mathbf{x} = 0$, and so $\mathbf{O}$ does not have rank n.

Sufficiency: Assume that

$$A'M + MA = -N$$

with N and M positive definite. Let λ be an eigenvalue of A and $\mathbf{x}$ the corresponding eigenvector.

Note that $A\mathbf{x} = \lambda\mathbf{x} \Longrightarrow \mathbf{x}^* A' = \mathbf{x}^* A^* = \lambda^* \mathbf{x}^*$. Then

$$\begin{aligned} -\mathbf{x}^* N \mathbf{x} &= \mathbf{x}^* A' M \mathbf{x} + \mathbf{x}^* M A \mathbf{x} \\ &= (\lambda^* + \lambda)\mathbf{x}^* M \mathbf{x} \\ &= 2\Re(\lambda)\mathbf{x}^* M \mathbf{x} \end{aligned}$$

and so $\Re(\lambda) < 0$.

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Notes:

Lyapunov Theorem continued:

For the corollary, $\Re(\lambda) \leq 0$, and if $\Re(\lambda) = 0$, then $\mathbf{x}^* N_0' N_0 \mathbf{x} = 0$, and so $N_0 \mathbf{x} = \mathbf{0}$. Since $\mathbf{x}$ is an eigenvector of A,

$$\begin{bmatrix} N_0 \\ N_0 A \\ \vdots \\ N_0 A^{n-1} \end{bmatrix} \mathbf{x} = \begin{bmatrix} N_0 \mathbf{x} \\ \lambda N_0 \mathbf{x} \\ \vdots \\ \lambda^{n-1} N_0 \mathbf{x} \end{bmatrix} = \mathbf{0}$$

and so $\mathbf{O}$ is not rank n.

Note: can also prove uniqueness by using integral: if there are two solutions, denote the difference by M; then $A'M + MA = 0$. Then, for all t

$$\begin{aligned} 0 &= \int_0^\infty e^{A't}(A'M + MA)e^{At}\,dt \\ &= \int_0^\infty \frac{d}{dt}(e^{A't} M e^{At})\,dt \\ &= e^{A't} M e^{At}|_0^\infty \\ &= 0 - M \end{aligned}$$

and so $M = 0$, as required.

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Notes:

Discrete-time Lyapunov equation (Sec. 5.4.1):

Read discrete-time case.

Note that the Lyapunov equation is

$$M - A'MA = N$$

and that its solution is

$$M = \sum_{m=0}^{\infty} (A')^m N A^m$$

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Notes:

Stability of Linear Time-Varying Systems (Sec. 5.5):

1. BIBO stable $\Leftrightarrow$ there is a constant K such that $\int_{t_0}^{t} |g(t,\tau)| d\tau \leq M$ for all t and t_0.
2. Zero-Input response is marginally stable $\Leftrightarrow$ there is a constant K such that $||\Phi(t, t_0)|| \leq M$ for all t and t_0
3. Zero-Input response is asymptotically stable $\Leftrightarrow \lim_{t\to\infty} ||\Phi(t, t_0)|| \to 0$ for all t_0.

Do Example 5.5.

Theorem 5.7: Marginal and asymptotic stability are invariant under a Lyapunov transformation.

Proof: If $\Phi_1(t, \tau)$ and $\Phi_2(t, \tau)$ are the state transition matrices of the two systems, then $\Phi_1(t, \tau) = P(t)\Phi_2(t, \tau)P^{-1}(\tau)$, and the theorem follows from this.

Note: for LTV systems, asymptotically stable does *not* imply BIBO stable.

Notes:

6 Controllability & Observability (Chapter 6):

The equation

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

is *controllable*, or the pair (A, B) is controllable, if for any initial state $\mathbf{x}(0) = \mathbf{x}_0$, and any final state $\mathbf{x}_1$, there is an input $\mathbf{u}(t)$ and a time t_0 such that $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(t_0) = \mathbf{x}_1$.

In other words, controllability means that we can drive any state to any other state by using an appropriate input.

Look at figure 6.1 and examples 6.1 – 6.3 (figs. 6.2 – 6.3).

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Notes:

Controllability Characterizations:

Theorem 6.1: The following are equivalent:

1. (A, B) is controllable
2. The matrix

$$\begin{aligned} W_c(t) &= \int_0^t e^{A\tau} BB' e^{A'\tau}\, d\tau \\ &= \int_0^t e^{A(t-\tau)} BB' e^{A'(t-\tau)}\, d\tau \end{aligned}$$

   is nonsingular for any $t > 0$.
3. The controllability matrix $\mathbf{C} = [B, AB, A^2B \ldots A^{n-1}B]$ ($n \times np$) has rank n
4. The matrix $[A - \lambda I, B]$ ($n \times (n+p)$) has rank n at every eigenvalue λ of A (and so has rank n for all λ).
5. If, in addition, A is stable, then the unique solution of

$$AW_c + W_c A' = -BB'$$

   is positive definite, where $W_c = W_c(\infty)$ above (the controllability Gramian).

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Notes:

Controllability Characterizations (continued):

Proof:

(1$\Leftarrow$2) $W_c(t)$ nonsingular $\Rightarrow$ controllable: given $\mathbf{x}_0$ and $\mathbf{x}_1$, and the solution

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)\,d\tau$$

the input $\mathbf{u}(t) = -B'e^{A'(t_1-t)}W_c^{-1}(t_1)[e^{At_1}\mathbf{x}_0 - \mathbf{x}_1]$ will drive $\mathbf{x}_0$ to $\mathbf{x}_1$, by substitution.

(1$\Rightarrow$2) First $W_c(t)$ is singular if, and only if, there is a vector $\mathbf{x} \neq \mathbf{0}$ such that

$$\begin{aligned} 0 &= \int_0^{t_1} \mathbf{x}'e^{A(t_1-\tau)}BB'e^{A'(t_1-\tau)}\mathbf{x}\,d\tau \\ &= \int_0^{t_1} ||B'e^{A'(t_1-\tau)}\mathbf{x}||^2\,d\tau \end{aligned}$$

and so $W_c(t)$ is singular if, and only if, there is a vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{x}'e^{A(t_1-\tau)}B = \mathbf{0}$ for all τ.

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Notes:

(1⇐2 continued) Now controllability implies that there is an input $\mathbf{u}(t)$ which takes $e^{-At_1}\mathbf{x}$ at $t = 0$ to 0 at $t = t_1$, that is,

$$0 = \mathbf{x} + \int_0^{t_1} e^{A(t_1-\tau)} B\mathbf{u}(\tau)d\tau$$

and so

$$\begin{aligned} 0 &= \mathbf{x}'\mathbf{x} + \int_0^{t_1} \mathbf{x}' e^{A(t_1-\tau)} B\mathbf{u}(\tau)d\tau \\ &= ||\mathbf{x}||^2 + 0 \end{aligned}$$

and so $\mathbf{x} = \mathbf{0}$, a contradiction.

(2⟺3) From the previous part, W_c is singular if, and only if, there is a vector $\mathbf{x}' \neq \mathbf{0}$ with $\mathbf{x}' e^{At} B = 0$ for all t.

Now if $\rho(\mathbf{C}) < n$, there is a vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{x}'\mathbf{C} = \mathbf{0}$, and then, by the Cayley-Hamilton theorem, $\mathbf{x}' A^k B = 0$ for all k. Therefore $\mathbf{x}' e^{At} B = 0$ for all t.

Conversely, if $\mathbf{x}' e^{At} B = 0$ for all t, then $\mathbf{x}' A^k B = 0$ for all k by differentiation, and so $\mathbf{x}'\mathbf{C} = 0$.

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Notes:

(3⟺4) If $\rho([A-\lambda I, B]) < n$, there is a vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{x}'[A-\lambda I, B] = 0$. Then $\mathbf{x}'A = \lambda \mathbf{x}'$, and so $\mathbf{x}'A^k = \lambda^k \mathbf{x}'$. Also, $\mathbf{x}'B = \mathbf{0}$, and so $\mathbf{x}'\mathbf{C} = \mathbf{0}$. Therefore $\rho(\mathbf{C}) < n$.

Conversely, if $\rho(\mathbf{C}) = n - m$ it will be proved that there is a nonsingular $n \times n$ matrix P such that $\bar{A} = PAP^{-1}$ and $\bar{B} = PB$, and

$$\bar{A} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

where $A_{\bar{c}}$ is $m \times m$. Now let $\mathbf{x}_1'$ be a left eigenvector of $A_{\bar{c}}$ with eigenvalue λ_1, so that $\mathbf{x}_1' A_{\bar{c}} = \lambda_1 \mathbf{x}_1'$.

Then $\mathbf{x}' = [\mathbf{0}_{n-m}, \mathbf{x}_1']$, is an $n-$dimensional eigenvector of $\bar{A}$, and $\mathbf{x}'[\bar{A} - \lambda_1 I,\ \bar{B}] = \mathbf{0}$, so that $\rho([\bar{A} - \lambda_1 I,\ \bar{B}]) < n$.

(2⟺5) W_c nonsingular ⟺ Lyapunov condition: this follows from Lyapunov's theorem (Corollary 5.5).

Look at example 6.2, and especially 6.3 and 6.4

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Notes:

Controllability Indices:

For each column $\mathbf{b}_m$ of B (with $\rho(B) = p$), let $\mu_m =$ # of columns in the set $\{\mathbf{b}_m, A\mathbf{b}_m, A^2\mathbf{b}_m \ldots, A^{n-1}\mathbf{b}_m\}$ which are linearly independent of columns to their left in $\mathbf{C}$. This set of columns must be of the form $\{\mathbf{b}_m, A\mathbf{b}_m, A^2\mathbf{b}_m \ldots, A^{\mu_m-1}\mathbf{b}_m\}$.

The μ_m are called the controllability indices, and their max is called the controllability index, i.e., $\mu =$smallest integer such that $\rho(\mathbf{C}_\mu) = \rho([B, AB, \ldots, A^{\mu-1}B]) = n$.

Note: $n/p \leq \mu \leq \min(\bar{n}, n-p+1)$ where $\bar{n}$ is the degree of the minimal polynomial and $\rho(B) = p$.

Corollary 6.1: (A, B) is controllable $\Leftrightarrow \rho(\mathbf{C}_{n-p+1}) = \rho([B, AB, \ldots, A^{n-p}B]) = n$, where n is the dimension of A, and $p = \rho(B)$.

Theorem 6.2: Controllability is preserved under an equivalence transformation: $\bar{B} = PB$, and $\bar{A} = PAP^{-1}$ gives $\bar{\mathbf{C}} = P\mathbf{C}$.

Theorem 6.3: The controllability indices are invariant under equivalence transformation and reordering of the columns of B — so intrinsic property of system.

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Notes:

6.1 Observability (Sec. 6.3):

Definition: A set of state equations is observable if, for any unknown initial state $\mathbf{x}(0)$, there is a finite time t_1 such that knowledge of $\mathbf{u}(t)$ and $\mathbf{y}(t)$ over the interval $[0 : t_1]$ determines $\mathbf{x}(0)$.

Since

$$\mathbf{y}(t) = Ce^{At}\mathbf{x}(0) + C\int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)\,d\tau + D\mathbf{u}(t)$$

the only unknown part is $Ce^{At}\mathbf{x}(0) = \bar{\mathbf{y}}(t)$; we will work with this, and define observability to mean that $\mathbf{x}(0)$ can be determined from the zero-input response $\mathbf{y}(t) = Ce^{At}\mathbf{x}(0)$ over a finite time.

Theorem 6.4: Observability is equivalent to

$$W_o(t_1) = \int_0^{t_1} e^{A'\tau}C'Ce^{A\tau}\,d\tau$$

being nonsingular for any $t_1 > 0$.

Notes:

Observability (continued):

Proof:

$$\left(\int_0^{t_1} e^{A'\tau} C' C e^{A\tau}\, d\tau\right) \mathbf{x}(0) = \int_0^{t_1} e^{A'\tau} C' \mathbf{y}(\tau) d\tau$$

and so if W_o is nonsingular, we can solve immediately for $\mathbf{x}(0)$.

Conversely, if W_o is singular, there is a vector $\mathbf{x}_0 \neq \mathbf{0}$ such that $0 = \mathbf{x}_0' W_o(t_1)\mathbf{x}_0$.

Therefore

$$\begin{aligned} 0 &= \int_0^{t_1} \mathbf{x}_0' e^{A'\tau} C' C e^{A\tau} \mathbf{x}_0\, d\tau \\ &= \int_0^{t_1} ||C e^{A\tau} \mathbf{x}_0||^2 d\tau \end{aligned}$$

and so $Ce^{At}\mathbf{x}_0 = \mathbf{0}$ for all t in $[0, t_1]$. Then $\mathbf{y}(t)$ is zero for $\mathbf{x}(0) = \mathbf{x}_0$ and for $\mathbf{x}(0) = \mathbf{0}$, and so $\mathbf{x}(0)$ can not be determined from $\mathbf{y}(t)$.

Theorem 6.5 (Duality:) (A, B) controllable $\Longleftrightarrow$ (A', B') observable.

Proof: Take transpose.

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Notes:

Observability (continued):

Therefore all the corresponding criteria hold for observability, and observability indices can be defined and have the same properties as controllability indices.

In particular, the system is observable if and only if the matrix

$$\mathbf{O}_m = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix}$$

has rank n for some m. For $m = n$, this matrix is called the *observability matrix*, and is denoted by $\mathbf{O}$.

As before, the Cayley-Hamilton theorem implies that the rank can not increase for $m > n$.

In the single-output case, the rank can not be n for $m < n$, and so we always take $n = m$, but in the multi-output case, the rank may be n for some $m < n$.

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Notes:

Observability (continued):

The observation problem can also be solved in principle by taking successive derivatives: then the equation becomes

$$\mathbf{O}_m \mathbf{x}(0) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix} \mathbf{x}(0) = \begin{bmatrix} \mathbf{y}(0) \\ \mathbf{y}'(0) \\ \vdots \\ \mathbf{y}^{(m-1)}(0) \end{bmatrix}$$

and so $\mathbf{x}(0)$ can be determined uniquely from $\mathbf{y}^{(k)}(0)$ for $0 \leq k \leq n-1$ provided $\rho(\mathbf{O}_n) = n$

— not useful in practice.

Notes:

6.2 Canonical Decomposition (Sec. 6.4):

Recall: If $\bar{\mathbf{x}} = P\mathbf{x}$, then $\bar{A} = PAP^{-1}$; $\bar{B} = PB$, $\bar{C} = CP^{-1}$, and $\bar{\mathbf{C}} = P\mathbf{C}$, and $\bar{\mathbf{O}} = \mathbf{O}P^{-1}$.

Theorem 6.6: If

$$\rho(\mathbf{C}) = \rho([B, AB, \ldots A^{n-1}B]) = n_1 < n$$

then by choosing $P^{-1} = [\mathbf{q}_1, \mathbf{q}_2, \ldots \mathbf{q}_{n_1}, \mathbf{q}_{n_1+1}, \ldots \mathbf{q}_n]$ where the first n_1 columns are linearly independent columns of $\mathbf{C}$, and the remaining columns are arbitrary, except for invertibility of P, we get a system of the form:

$$\bar{A} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix}$$

and the n_1- dimensional equation

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A_c\mathbf{x}(t) + B_c\mathbf{u}(t) \\ \mathbf{y}(t) &= C_c\mathbf{x}(t) + D\mathbf{u}(t) \end{aligned}$$

is controllable and has the same transfer function matrix as the original system.

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Notes:

Canonical Decomposition continued

Proof: By assumption, all of the columns of $\mathbf{C}$ are linear combinations of $[\mathbf{q}_1, \mathbf{q}_2, \ldots \mathbf{q}_{n_1}]$. Also, the k-th column of $\bar{A}$ is given by the representation of $A\mathbf{q}_k$ in terms of $[\mathbf{q}_1, \mathbf{q}_2, \ldots \mathbf{q}_n]$.

Now, for $1 \leq k \leq n_1$, each $A\mathbf{q}_k$ is linearly dependent on $[\mathbf{q}_1, \ldots \mathbf{q}_{n_1}]$, and linearly independent of $[\mathbf{q}_{n_1+1}, \ldots \mathbf{q}_n]$.

To see this, first assume that $\mathbf{q}_k$ is a column from the matrix $[B, AB, \ldots A^{n-2}B]$; then $A\mathbf{q}_k$ is a column from the matrix $[AB, A^2B, \ldots A^{n-1}B]$, and all of these columns are linear combinations of $[\mathbf{q}_1, \mathbf{q}_2, \ldots \mathbf{q}_{n_1}]$ by construction.

Otherwise, $\mathbf{q}_k$ is a column from the matrix $[A^{n-1}B]$; then $A\mathbf{q}_k$ is a column from the matrix $[A^nB]$, all of whose columns are linear combinations of the columns of $\mathbf{C}$ by the Cayley-Hamilton theorem, and so of $[\mathbf{q}_1, \ldots \mathbf{q}_{n_1}]$.

It follows that the first n_1 columns of $\bar{A}$ have zeros in the last $n - n_1$ rows, which gives the form of $\bar{A}$.

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Notes:

Canonical Decomposition continued

The columns of $\bar{B}$ are the representations of the columns of B in terms of the $\mathbf{q}_i$; but the columns of B are columns of $\mathbf{C}$, and so are linear combinations $[\mathbf{q}_1, \mathbf{q}_2, \ldots \mathbf{q}_{n_1}]$. As before, this implies that the last $n - n_1$ rows of B are zero, which gives the form of $\bar{B}$.

Next, the smaller system is controllable: by direct calculation, we get

$$\begin{aligned}\bar{\mathbf{C}} &= \begin{bmatrix} B_c & A_c B_c & \ldots & A_c^{n-1} B_c \\ 0 & 0 & \ldots & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{C}_c & A_c^{n_1} B_c & \ldots & A_c^{n-1} B_c \\ 0 & 0 & \ldots & 0 \end{bmatrix}\end{aligned}$$

Notes:

Canonical Decomposition continued

Now, by the Cayley-Hamilton theorem, the columns of $A_c^k B_c$ for $k \geq n_1$ are linearly dependent on the columns of $\mathbf{C}_c$, and so

$$\begin{aligned}
\rho(\mathbf{C}_c) &= \rho\left(\begin{bmatrix} \mathbf{C}_c & A_c^{n_1} B_c & \dots & A_c^{n-1} B_c \\ 0 & 0 & \dots & 0 \end{bmatrix}\right) \\
&= \rho(\bar{\mathbf{C}}) \\
&= \rho(\mathbf{C}) \\
&= n_1
\end{aligned}$$

and since the dimension of the smaller system is n_1, the smaller system is controllable.

Equality of the transfer function matrix follows by direct calculation, using the fact that the inverse of a block upper triangular matrix is block upper triangular, with the block inverses on the diagonal.

Theorem 6.O6: Corresponding observability theorem.

Do example 6.8.

Notes:

Canonical (Kalman) Decomposition continued

Theorem 6.7: Every state-space system can be transformed by an equivalence transform, into a system of the form (Kalman decomposition):

$$\begin{bmatrix} \dot{\mathbf{x}}_{co} \\ \dot{\mathbf{x}}_{c\bar{o}} \\ \dot{\mathbf{x}}_{\bar{c}o} \\ \dot{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} = A \begin{bmatrix} \mathbf{x}_{co} \\ \mathbf{x}_{c\bar{o}} \\ \mathbf{x}_{\bar{c}o} \\ \mathbf{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{co} \\ \mathbf{x}_{c\bar{o}} \\ \mathbf{x}_{\bar{c}o} \\ \mathbf{x}_{\bar{c}\bar{o}} \end{bmatrix} + D\mathbf{u}$$

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Notes:

Canonical (Kalman) Decomposition continued

where

$$A = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix}$$

and the vector $\mathbf{x}_{co}$ is controllable and observable, etc..

Also, the state equation is zero-state equivalent to the controllable and observable (minimal) equation:

$$\begin{aligned} \dot{\mathbf{x}}_{co} &= A_{co}\mathbf{x}_{co} + B_{co}\mathbf{u} \\ \mathbf{y} &= C_{co}\mathbf{x}_{co} + D\mathbf{u} \end{aligned}$$

and has the same transfer matrix.

Proof: put previous two theorems together.

Look at

- diagram on p. 163
- MATLAB function *minreal* on p. 164; calculates the above decomposition.
- example 6.9

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Notes:

7 Chapter 7: Minimality and Coprimeness:

Will work with single-input, single-output system, and fourth-order example.

Definition: A system is *minimal* if there is no system of lower dimension with the same transfer function.

Assume a strictly proper transfer function:

$$g(s) = \frac{N(s)}{D(s)} = \frac{b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

Let $v(s) = D^{-1}(s)u(s)$; then $D(s)v(s) = u(s)$ and $y(s) = N(s)v(s)$.

Define state variables by:

$$\mathbf{x}(t) = \begin{bmatrix} v'''(t) \\ v''(t) \\ v'(t) \\ v(t) \end{bmatrix}$$

and so

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Notes:

Minimality and Coprimeness continued:

$$\mathbf{X}(s) = \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix} v(s)$$

Then, by direct substitution,

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ &= \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{u} \\ \mathbf{y} &= C\mathbf{x} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} \mathbf{x} \end{aligned}$$

Direct calculation shows controllability matrix is upper triangular with ones on the diagonal; so this realization is always controllable — controllable canonical form.

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Notes:

Minimality and Coprimeness continued:

Theorem 7.1: This realization is observable if, and only if, $N(s)$ and $D(s)$ are coprime.

Proof: If not coprime, let s_0 be a common root. Let $\mathbf{x} = [s_0^3, s_0^2, s_0, 1]'$. Then $N(s_0) = C\mathbf{x} = 0$. Also, $A\mathbf{x} = s_0\mathbf{x}$, and so $A^k\mathbf{x} = s_0^k\mathbf{x}$, for all k. Then

$$\mathbf{O}\mathbf{x} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} C \\ s_0 C \\ s_0^2 C \\ s_0^3 C \end{bmatrix} \mathbf{x} = 0$$

So the system is not observable.

Converse: assume the system is not observable. Then there is an eigenvalue s_0 and eigenvector $\mathbf{x}$ with

$$\begin{bmatrix} A - s_0 I \\ C \end{bmatrix} \mathbf{x} = 0$$

and so $A\mathbf{x} = s_0\mathbf{x}$ which implies that $\mathbf{x} = [s_0^3, s_0^2, s_0, 1]$, and so $C\mathbf{x} = N(s_0) = 0$. Also, $D(s_0) = \Delta(s_0) = 0$, and so $N(s)$ and $D(s)$ are not coprime.

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Notes:

Minimality and Coprimeness continued:

Theorem 7.2: A system is minimal if, and only if, it is controllable and observable, and then $\deg(g) = \dim(A)$.

Proof: If the system is uncontrollable or unobservable, then it is not minimal, from chapter 6.

Conversely, given a realization of order n with $\rho(\mathbf{O}) = \rho(\mathbf{C}) = n$, $\mathbf{OC}$ is invertible. Assume there is a realization of order $m < n$. Then CA^kB is the same for both realizations, and so $\mathbf{OC}$ is the same for both. But for an m-dimensional realization $\rho(\mathbf{OC}) \leq m$ — a contradiction.

For the degree of g: $N(s)$ and $D(s)$ are coprime if, and only if, the controllable canonical form is observable; since all minimal realizations are equivalent to the controllable canonical form (next theorem), the result follows.

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Notes:

Minimal Realizations:

Theorem 7.3: All minimal realizations of $H(s)$ are equivalent to each other, and so to the controllable canonical form.

Proof: given two minimal realizations,

$$\begin{aligned} \mathbf{OC} &= \bar{\mathbf{O}}\bar{\mathbf{C}} \\ \mathbf{OAC} &= \bar{\mathbf{O}}\bar{\mathbf{A}}\bar{\mathbf{C}} \end{aligned}$$

and $\mathbf{O}$, $\bar{\mathbf{O}}$, $\bar{\mathbf{C}}$, and $\mathbf{C}$ are invertible, since both are controllable and observable.

Let

$$P = \bar{\mathbf{O}}^{-1}\mathbf{O} = \bar{\mathbf{C}}\mathbf{C}^{-1}$$

then

$$P^{-1} = \mathbf{O}^{-1}\bar{\mathbf{O}} = \mathbf{C}\bar{\mathbf{C}}^{-1}$$

Therefore $\bar{\mathbf{C}} = P\mathbf{C}$, and the first column of this equation gives $\bar{B} = PB$. Similarly, $\bar{\mathbf{O}} = \mathbf{O}P^{-1}$ gives $\bar{C} = CP^{-1}$.

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Notes:

Minimal Realizations continued:

Since

$$\mathbf{O}A\mathbf{C} = \bar{\mathbf{O}}\bar{A}\bar{\mathbf{C}}$$

it follows that

$$\bar{A} = \bar{\mathbf{O}}^{-1}\mathbf{O}A\mathbf{C}\bar{\mathbf{C}}^{-1} = PAP^{-1}$$

as required.

Note: $H = \mathbf{OC}$ and $H_1 = \mathbf{O}A\mathbf{C}$ involve only the transfer function (next slide), and every minimal realization defines a factorization of H such that $A = \mathbf{O}^{-1}H_1\mathbf{C}^{-1}$. The above proof implies that every minimal realization comes from such a factorization.

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Notes:

7.1 Markov Parameters and Hankel Matrices:

Recall that, if $h(s)$ is the transfer function of a single-input, single-output state-variable system, then

$$\begin{aligned} h(s) &= C(sI - A)^{-1}B + d \\ &= d + \sum_{k=1}^{\infty} CA^{k-1}B/s^k \\ &= \sum_{k=0}^{\infty} h_k/s^k \end{aligned}$$

where $h_0 = d$ and, for $k \geq 1$

$$h_k = CA^{k-1}B$$

The h_k are called the **Markov parameters** of the system and, in the discrete-time case, are the impulse response.

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Notes:

Markov Parameters continued

If we now define the Hankel matrix:

$$H_{i,j} = \begin{bmatrix} h_i & h_{i+1} & h_{i+2} & \dots & h_j \\ h_{i+1} & h_{i+2} & h_{i+3} & \dots & h_{j+1} \\ h_{i+2} & h_{i+3} & \ddots & \dots & h_{j+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_j & h_{j+1} & h_{j+2} & \dots & h_{2j-i} \end{bmatrix}$$

it then follows that

$$\mathbf{OC} = H_{1,n}$$

and

$$\mathbf{O}A\mathbf{C} = H_{2,n+1}$$

It follows that a SISO system has a finite-dimensional realization (or that $h(s)$ is a proper rational function) if, and only if, $\rho(H_{1,j})$ is bounded for all j, and that the dimension n of the minimal realization is this upper bound, which occurs for $H_{1,n}$

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Notes:

Markov Parameters continued

Also, all minimal realizations can be found by factoring $H_{1,n}$:

$$H_{1,n} = M_1 M_2$$

and then

$$\begin{bmatrix} d & C \\ B & A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & M_1^{-1} \end{bmatrix} H_{0,n} \begin{bmatrix} 1 & 0 \\ 0 & M_2^{-1} \end{bmatrix}$$

and $M_1 = \mathbf{O}$ and $M_2 = \mathbf{C}$.

Examples:

1. The factorization $H_{1,n} = IH_{1,n}$ gives the observability canonical, or companion, form.
2. The factorization $H_{1,n} = H_{1,n}I$ gives the controllability canonical, or companion, form.
3. The factorization $H_{1,n} = LU$ (when it exists) gives a tridiagonal form.
4. Singular-value decomposition gives a “balanced” form.

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Notes:

8 State Feedback (Chapter 8):

Single-input linear state feedback is defined by replacing the input u by $u = r - u_1$, where r is the external input, and $u_1 = \mathbf{k}\mathbf{x}$, where $\mathbf{k}$ is a row vector, is the feedback. Then the system becomes (assuming $d = 0$)

$$\begin{aligned}\dot{\mathbf{x}} &= (A - \mathbf{b}\mathbf{k})\mathbf{x} + \mathbf{b}r \\ y &= C\mathbf{x}\end{aligned}$$

Theorem 8.1: $(A_f, \mathbf{b}) = (A - \mathbf{b}\mathbf{k}, \mathbf{b})$ is controllable if, and only if, $(A, \mathbf{b})$ is controllable.

Proof (for $n = 4$): If $\mathbf{C}$ is the controllability matrix for $(A, \mathbf{b})$, the controllability matrix for $(A_f, \mathbf{b})$ is

$$\begin{aligned}\mathbf{C}_f &= [\mathbf{b}, (A - \mathbf{b}\mathbf{k})\mathbf{b}, (A - \mathbf{b}\mathbf{k})^2\mathbf{b}, (A - \mathbf{b}\mathbf{k})^3\mathbf{b}] \\ &= \mathbf{C}\begin{bmatrix} 1 & -\mathbf{k}\mathbf{b} & -\mathbf{k}A_f\mathbf{b} & -\mathbf{k}A_f^2\mathbf{b} \\ 0 & 1 & -\mathbf{k}\mathbf{b} & -\mathbf{k}A_f\mathbf{b} \\ 0 & 0 & 1 & -\mathbf{k}\mathbf{b} \\ 0 & 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

from which controllability follows.

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Notes:

State Feedback continued:

Example 8.1: State feedback does not preserve observability.

Theorem 8.2: If a system is controllable it can be transformed into Controllable Canonical Form by the transformation $\bar{\mathbf{x}} = P\mathbf{x}$ with

$$Q = P^{-1}$$

$$= \begin{bmatrix} \mathbf{b} & A\mathbf{b} & A^2\mathbf{b} & A^3\mathbf{b} \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

giving

$$A_c = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} ; \mathbf{b}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(Here the characteristic polynomial of A is $\Delta(s) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4$)

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Notes:

State Feedback continued:

Also

$$\hat{g}(s) = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4}$$

where

$$\mathbf{c}_c = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix}$$

Proof: Let Q be as above; then the first column of Q is $\mathbf{b}$, and so the representation of $\mathbf{b}$ in terms of $[\mathbf{q}_1, \mathbf{q}_2, \ldots \mathbf{q}_n]$ is $[1, 0, 0 \ldots 0]'$. The k-th column of A is given by the representation of $A\mathbf{q}_k$ in terms of the $\mathbf{q}_j$. But the columns of Q satisfy:

$$\begin{aligned} \mathbf{q}_1 &= \mathbf{b} \\ \mathbf{q}_2 &= \alpha_1 \mathbf{b} + A\mathbf{b} \\ \mathbf{q}_3 &= \alpha_2 \mathbf{b} + \alpha_1 A\mathbf{b} + A^2\mathbf{b} \\ & \ldots \end{aligned}$$

So $A\mathbf{q}_2 = \mathbf{q}_3 - \alpha_2 \mathbf{b} = \mathbf{q}_3 - \alpha_2 \mathbf{q}_1$, etc., to give the Controllable Canonical Form.

Transfer function: done previously, in realization discussion.

Notes:

State Feedback continued:

Note: $P = \bar{\mathbf{C}}\mathbf{C}^{-1}$, where $\bar{\mathbf{C}}$ is the controllability matrix of the Controllable Canonical Form, and $\mathbf{C}$ is the controllability matrix of the original system; thus $Q = \mathbf{C}\bar{\mathbf{C}}^{-1}$, and so the matrix

$$\begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

must be the inverse of the controllability matrix of the Controllable Canonical Form.

Notes:

Pole Placement:

Theorem 8.3 (Pole placement): If a system is controllable, then state feedback can assign the eigenvalues of A arbitrarily (subject to complex conjugates), i.e., by choosing $\mathbf{k}$, the characteristic polynomial of $A - \mathbf{b}\mathbf{k}$ can be made to have any value.

Proof: Transform the original system to CCF by $\mathbf{x}_c = P\mathbf{x}$. Then $A_c = PAP^{-1}$ and $\mathbf{b}_c = P\mathbf{b}$. Also $u = r - \mathbf{k}P^{-1}\mathbf{x}_c = r - \mathbf{k}_c\mathbf{x}_c$.

Therefore $A_c - \mathbf{b}_c\mathbf{k}_c = P(A - \mathbf{b}\mathbf{k})P^{-1}$, and so $A_c - \mathbf{b}_c\mathbf{k}_c$ and $P(A - \mathbf{b}\mathbf{k})P^{-1}$ have the have the same eigenvalues and characteristic polynomial.

Now, in the CCF, given a desired characteristic polynomial

$$s^4 + \alpha_{d,1}s^3 + \alpha_{d,2}s^2 \ldots \alpha_{d,4}$$

let

$$\mathbf{k}_c = \begin{bmatrix} \alpha_{d,1} - \alpha_1 & \alpha_{d,2} - \alpha_2 & \ldots & \alpha_{d,4} - \alpha_4 \end{bmatrix}$$

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Notes:

Pole Placement continued:

Then $A_c - \mathbf{b}_c\mathbf{k}_c$ is also in CCF, with first row $[-\alpha_{d,1} \quad -\alpha_{d,2} \quad -\alpha_{d,3} \quad -\alpha_{d,4}]$.

Therefore, the denominator of the transfer function is

$$s^4 + \alpha_{d,1}s^3 + \alpha_{d,2}s^2 + \alpha_{d,3}s + \alpha_{d,4}$$

as required.

This also gives an algorithm for finding the feedback vector: if $\mathbf{a}' = \begin{bmatrix} \alpha_{d,1} - \alpha_1 & \dots & \alpha_{d,n} - \alpha_n \end{bmatrix}$ is the difference between the desired denominator coefficients and the original coefficients, the feedback vector is

$$\mathbf{k} = \mathbf{a}' \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 1 & \alpha_1 & \cdots & \alpha_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^{-1} \mathbf{C}^{-1}$$

where $\mathbf{C}$ is the controllability matrix of the original system.

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Notes:

Pole Placement continued:

Note:

1. Since the numerator coefficients of the transfer function are given by the output coupling vector of the CCF (that is, by C_c), and C_c is unaffected by state feedback, it follows that the numerator, or equivalently, the zero-locations, of the transfer function can not be changed by state feedback.
2. The feedback gain vector can also be found by using a Lyapunov equation, provided the desired polynomial has no roots in common with the original polynomial (procedure 8.1 and theorem 8.4).
3. The choice of pole locations is usually a compromise between many criteria such as response speed, overshoot, driving energy, saturation of states, etc.; it usually requires both exerience and trial-and-error.

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Notes:

8.1 State Estimation (Sec. 8.4):

If a system is observable, then, in principle, the state can be determined from output.

How, in real time?

— State estimator.

Essential if we can not observe all state variables, or if we want to use output feedback.

Estimator Idea: Set up a system to match the original system, and then set up a feedback loop to correct for deviations (closed-loop estimator, figure 8.6);

State equation for estimator system:

$$\begin{aligned}\dot{\mathbf{x}}_e &= A\mathbf{x}_e + \mathbf{b}u + \mathbf{l}(y - \mathbf{c}\mathbf{x}_e) \\ &= (A - \mathbf{l}\mathbf{c})\mathbf{x}_e + \mathbf{b}u + \mathbf{l}\mathbf{c}\mathbf{x} \\ &= (A - \mathbf{l}\mathbf{c})\mathbf{x}_e + \mathbf{b}u + \mathbf{l}y\end{aligned}$$

— dual to state feedback.

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Notes:

State Estimation continued:

The error is:

$$\mathbf{e} = \mathbf{x} - \mathbf{x}_e$$

Differentiate and substitute to get

$$\dot{\mathbf{e}} = (A - \mathbf{l}\mathbf{c})\mathbf{e}$$

So, provided the system is observable, we can get arbitrary eigenvalues for the error dynamics, and so can tailor its dynamic response; the initial unknown error can be made to decay arbitrarily rapidly (subject to real-world constraints). If the estimator time-constants are much faster than the time-constants of the original system, the estimated state will track the actual state well.

Again, this estimation vector $\mathbf{l}$ can be found using a Lyapunov equation, provided there are no common roots.

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Notes:

8.2 Separation Theorem (Sec. 8.5):

Theorem: Using the output of a state estimator, instead of the actual state, as input for state feedback changes neither the transfer function nor the eigenvalues of the state feedback system.

Proof: The state equations for the full system, including the estimator, are:

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} - \mathbf{b}\mathbf{k}\mathbf{x}_e + \mathbf{b}r \\ \dot{\mathbf{x}}_e &= (A - \mathbf{l}\mathbf{c})\mathbf{x}_e + \mathbf{b}(r - \mathbf{k}\mathbf{x}_e) + \mathbf{l}\mathbf{c}\mathbf{x} \end{aligned}$$

These can be combined as

$$\begin{aligned} \dot{\mathbf{x}}_{tot} &= \begin{bmatrix} A & -\mathbf{b}\mathbf{k} \\ \mathbf{l}\mathbf{c} & A - \mathbf{l}\mathbf{c} - \mathbf{b}\mathbf{k} \end{bmatrix} \mathbf{x}_{tot} + \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix} r \\ y &= \begin{bmatrix} \mathbf{c} & 0 \end{bmatrix} \mathbf{x}_{tot} \end{aligned}$$

Notes:

Separation Theorem continued:

The eigenvalues are not obvious in this form, so use the equivalence transformation:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{x} - \mathbf{x}_e \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_e \end{bmatrix}$$

Then

$$P = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = P^{-1}$$

and the transformed state equations are:

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} &= \begin{bmatrix} A - \mathbf{bk} & \mathbf{bk} \\ 0 & A - \mathbf{lc} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} r \\ y &= \begin{bmatrix} \mathbf{c} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \end{aligned}$$

Notes:

Separation Theorem continued:

Since the A matrix of the overall system is now block triangular, its eigenvalues are the union of the eigenvalues of the diagonal blocks, and so the eigenvalues of the state feedback system ($A - \mathbf{bk}$), and the state estimator ($A - \mathbf{lc}$) are unchanged from the values obtained by the separate designs.

In addition, the system is now in the form for decomposition into controllable and uncontrollable subsystems, and so has the same transfer function as the controllable subsystem, which is identical to the original state feedback system, with transfer function

$$g_f(s) = \mathbf{c}(sI - A + \mathbf{bk})^{-1}\mathbf{b}$$

Note that the overall system is never controllable or minimal; the estimation error is the uncontrollable component.

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Notes:

9 Output Feedback using Coprime Fractional Representations:

Note: Chapters 7 & 9 in text, but not following text.

Assume that we have a vector space, G, of systems, and a linear subspace of H of "stable" systems, both of which are closed under multiplication, with a multiplicative identity denoted by 1. We denote by H_I the subset of elements of H which have inverses in G (i.e., inverses which might be unstable), and by H_M the subset of elements of H with stable inverses ("minimum-phase").

Main Example:

- G is the set of $n \times n$ transfer function matrices
- H is the set of $n \times n$ stable transfer function matrices
- H_I is the set of $n \times n$ stable transfer function matrices whose determinant is not identically zero
- H_M is the subset of H_I whose inverses are stable.

Notes:

Here, stable can have the usual meaning, but the theory can be applied without change if we interpret “stable” as having no poles in any prescribed region of the complex plane. (It can also be applied to distributed and time-varying linear systems, if the coprime fractional representations can be found.)

We then define:

1. $g \in G$ has a *right fractional representation* if $g = n_r d_r^{-1}$, with $n_r \in H$, and $d_r \in H_I$.
2. The fractional representation is *right coprime* if there are u_r and $v_r \in H$ with $u_r n_r + v_r d_r = 1$
3. $g \in G$ has a *left fractional representation* if $g = d_l^{-1} n_l$, with $n_l \in H$, and $d_l \in H_I$.
4. The fractional representation is *left coprime* if there are u_l and $v_l \in H$ with $n_l u_l + d_l v_l = 1$

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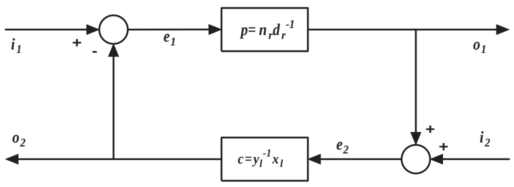
Notes:

9.1 Analysis

We want to analyze the following feedback system:



where $n_r d_r^{-1}$ is a right coprime factorization of the known system $p(s)$, and $y_l^{-1} x_l$ is a left coprime factorization of the feedback compensator to be designed, $c(s)$.

If h_{ij} denotes the transfer function from input j to output i, the gains are given by:

$$
\begin{aligned}
h_{11} &= (1+pc)^{-1}p \\
h_{12} &= -(1+pc)^{-1}pc \\
h_{21} &= (1+cp)^{-1}cp \\
h_{22} &= (1+cp)^{-1}c
\end{aligned}
$$

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Notes:

Analysis continued:

If we assume that the plant p has a partial state $z = d_r^{-1} e_1$, then we get

$$\begin{aligned} e_1 &= d_r z \\ o_1 &= n_r z \\ o_2 &= i_1 - e_1 = i_1 - d_r z \\ e_2 &= i_2 + o_1 = i_2 + n_r z \end{aligned}$$

and so, if the response z is stable for all stable input pairs i_1 and i_2, then all signals in the loop will be stable.

We can find the stability condition for z as follows:

$$d_r z = e_1 = i_1 - y_l^{-1} x_l e_2$$

and so

$$\begin{aligned} y_l d_r z &= y_l i_1 - x_l e_2 \\ &= y_l i_1 - x_l i_2 - x_l o_1 \\ &= y_l i_1 - x_l i_2 - x_l n_r z \end{aligned}$$

so that

$$(x_l n_r + y_l d_r) z = y_l i_1 - x_l i_2$$

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Notes:

Analysis continued:

Therefore, the response z will be stable if $(x_l n_r + y_l d_r)$ has a stable inverse.

Conversely, it can be shown that if z is a stable response for every stable input pair i_1 and i_2, then $(x_l n_r + y_l d_r)$ has a stable inverse.

Finally, if $(x_l n_r + y_l d_r) = r$, where r has a stable inverse, then we can find the same compensator c with a different coprime factorization. To see this, let

$$\hat{y}_l = r^{-1} y_l \quad \text{and} \quad \hat{x}_l = r^{-1} x_l$$

Then

$$c = y_l^{-1} x_l = y_l^{-1} r r^{-1} x_l = \hat{y}_l^{-1} \hat{x}_l$$

and so $c = \hat{y}_l^{-1} \hat{x}_l$ is also a left factorization of c.

It is also left coprime, since

$$\hat{x}_l n_r + \hat{y}_l d_r = r^{-1}(x_l n_r + y_l d_r) = r^{-1}$$

and so

$$\hat{x}_l(n_r r) + \hat{y}_l(d_r) = 1$$

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Notes:

9.2 Design:

From the previous analysis we conclude:

If the system has a right coprime factorization $p = n_r d_r^{-1}$, then the feedback loop is stable if, and only if, the controller has a left coprime factorization $c = y_l^{-1} x_l$ such that

$$(x_l n_r + y_l d_r) = 1$$

Therefore, to find all stabilizing controllers, we need to find all solutions to this equation.

These solutions are given by

$$y_l = (v_r + w n_l) \quad \text{and} \quad x_l = (u_r - w d_l)$$

and so all stabilizing compensators are given by

$$c = (v_r + w n_l)^{-1}(u_r - w d_l)$$

where $p = d_l^{-1} n_l$ is a *left* coprime representation of p, and w is an arbitrary stable transfer function,

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Notes:

Design continued:

9.2.1 Solution derivation:

We already know one solution, since the coprime factorization of p gives us $u_r n_r + v_r d_r = 1$; then all other solutions must be of the form

$$y_l = v_r + y_h \quad \text{and} \quad x_l = u_r + x_h$$

where y_h and x_h are solutions of the homogeneous equation

$$y_h d_r + x_h n_r = 0$$

But if $p = d_l^{-1} n_l$ is the *left* coprime representation of p, then

$$d_l^{-1} n_l = n_r d_r^{-1} \quad \Longrightarrow \quad n_l d_r - d_l n_r = 0$$

and so

$$y_h = w n_l \quad \text{and} \quad x_h = -w d_l$$

are homogeneous solutions for all $w \in H$.

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Notes:

Design continued:

Conversely, if y_h and d_h are arbitrary solutions of $y_h d_r + x_h n_r = 0$, let

$$w = -x_h d_l^{-1}$$

Then

$$x_h = -w d_l$$

and $y_h d_r = -x_h n_r$ implies that

$$\begin{aligned} y_h &= (-x_h)(n_r d_r^{-1}) \\ &= (w d_l)(d_l^{-1} n_l) \\ &= w n_l \end{aligned}$$

To show that $w \in H$, use the left coprimeness:

$$\begin{aligned} w &= w(n_l u_l + d_l v_l) \\ &= y_h u_l - x_h v_l \in H \end{aligned}$$

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Notes:

9.2.2 Conclusions:

- For any stable w such that $v_r + wn_l$ is invertible, $c = (v_r + wn_l)^{-1}(u_r - wd_l)$ stabilizes the loop, and all stabilizing compensators are of this form.
- All of the gains are linear functions of w. From the analysis:

$$\begin{aligned} e_1 &= d_r z \quad \text{and} \quad o_1 = n_r z \\ o_2 &= i_1 - e_1 = i_1 - d_r z \\ e_2 &= i_2 + o_1 = i_2 + n_r z \end{aligned}$$

and since $(y_l d_r + x_l n_r) = 1$

$$z = y_l i_1 - x_l i_2$$

It then follows that

$$\begin{aligned} h_{11} &= n_r y_l = n_r v_r + n_r w n_l \\ h_{12} &= -n_r x_l = -n_r u_r + n_r w d_l \\ h_{21} &= 1 - d_r y_l = 1 - d_r v_r - d_r w n_l \\ h_{22} &= d_r x_l = d_r u_r - d_r w d_l \end{aligned}$$

Notes:

9.3 Example:

Let

$$p(s) = \frac{s}{s^2-1} = \frac{s}{(s+1)^2} \Big/ \frac{s-1}{s+1} = \frac{n(s)}{d(s)}$$

Then we have

$$1 = 4\frac{s}{(s+1)^2} + \frac{s-1}{s+1}\frac{s-1}{s+1} = u(s)n(s) + v(s)d(s)$$

$$\begin{aligned} c(s) &= \left(4 - \frac{s-1}{s+1}w(s)\right) \Big/ \left(\frac{s-1}{s+1} + \frac{s}{(s+1)^2}w(s)\right) \\ &= \frac{4s^2+8s+4-(s^2-1)w(s)}{s^2-1+sw(s)} \end{aligned}$$

and the input-output transfer function is

$$\begin{aligned} h_{11} &= \frac{s}{(s+1)^2} \cdot \frac{s-1}{s+1} + \frac{s^2}{(s+1)^4}w(s) \\ &= \frac{s^3-s+s^2w(s)}{(s+1)^4} \end{aligned}$$

where $w(s)$ is any stable transfer function.

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