IIR System:

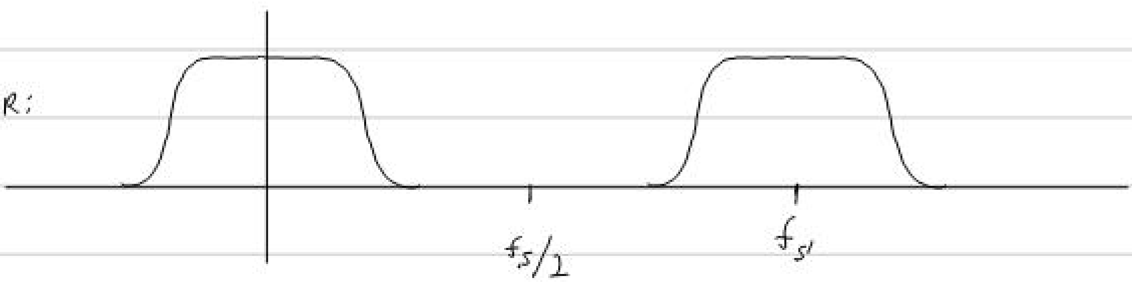
$$y_n = b_0 u_n + b_1 u_{n-1} + \dots + b_M u_{n-M} - a_1 y_{n-1} - a_2 y_{n-2} - \dots - a_N y_{n-N}$$

$u_n = e^{in\theta}$ & $y_n = K(\theta)\, e^{in\theta}$

$$\Rightarrow K(\theta)\, e^{in\theta} = b_0 e^{in\theta} + b_1 e^{i(n-1)\theta} + \dots + b_M e^{i(n-M)\theta}$$
$$- K(\theta)\left(a_1 e^{i(n-1)\theta} + a_2 e^{i(n-2)\theta} + \dots + a_N e^{i(n-N)\theta}\right)$$

$$\Rightarrow K(\theta) e^{in\theta}\left(1 + a_1 e^{-i\theta} + \dots + a_N e^{-iN\theta}\right) = e^{in\theta}\left(b_0 + b_1 e^{-i\theta} + \dots + b_M e^{-iM\theta}\right)$$

DSP LOW-PASS FILTER:



INVERSE DTFT:

A: ORTHOGONALITY OF COMPLEX SINUSOIDS:

IF k IS AN INTEGER,

$$\int_{-\pi}^{\pi} e^{ik\theta}\, d\theta = 2\pi\,\delta_k = \begin{cases} 2\pi & k=0 \\ 0 & k\neq 0 \end{cases}$$

PROOF: $k=0 \Rightarrow \int_{-\pi}^{\pi} e^{ik\theta}\, d\theta = \int_{-\pi}^{\pi} 1 . d\theta = 2\pi$

$k \neq 0 \Rightarrow \int_{-\pi}^{\pi} e^{ik\theta}\, d\theta = \frac{1}{ik}\left[e^{ik\pi} - e^{-ik\pi}\right]$

$$= \frac{1}{ik}\left[(-1)^k - (-1)^k\right] = 0$$

ORTHOGONALITY: IF m & n ARE INTEGERS,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\left(e^{im\theta}\right)^* d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)\theta}\, d\theta = \delta_{n-m}$$

INVERSE DTFT.

$$\int_{-\pi}^{\pi} G(\theta)\, e^{in\theta}\, d\theta = \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} g_k e^{-ik\theta} e^{in\theta}\, d\theta$$

$$= \sum_{k=-\infty}^{\infty} g_k \int_{-\pi}^{\pi} e^{i(n-k)\theta}\, d\theta$$

$$= \sum_{k=-\infty}^{\infty} g_k 2\pi \delta_{n-k}$$

$$= 2\pi g_n$$

$$\Rightarrow \quad g_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\theta)\, e^{in\theta}\, d\theta.$$

2-SIDED Z-TRANSFORM PROPERTIES:

SHIFTING:

IF $p_n = q_{n-k}$ FOR ALL n, & A FIXED k

THEN

$$P(z) = \mathcal{Z}\{p_n\} = \sum_{l=-\infty}^{\infty} p_l z^{-l} = \sum_{l=-\infty}^{\infty} q_{l-k} z^{-l}$$

$$\left[\begin{array}{l} m = l-k \\ \Rightarrow l = m+k \end{array}\right] \qquad = \sum_{m=-\infty}^{\infty} q_m z^{-(m+k)}$$

$$= z^{-k} \sum_{m=-\infty}^{\infty} q_m z^{m} = z^{-k} Q(z)$$

CONVOLUTION: IF $(p_n) = (f_n) * (g_n)$

THEN $P(z) = F(z)G(z)$

PROOF: $P(z) = \sum_{n=-\infty}^{\infty} p_n z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f_k g_{n-k} z^{-n}$

$$= \sum_{k=-\infty}^{\infty} f_k \sum_{n=-\infty}^{\infty} g_{n-k} z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} f_k z^{-k}\, G(z) \qquad \text{(SHIFTING)}$$

$$= F(z)G(z)$$

2-SIDED DTFT EXAMPLES:

1: $g_n = a^n u_n = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$ WITH $|a| < 1$ & a COMPLEX.

$G(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} (az^{-1})^k = \frac{1}{1 - az^{-1}}$ FOR $|az^{-1}| < 1$ i.e, $|z| > |a|$

(ROC : $\{z \mid |z| > |a|\}$; INCLUDES UNIT CIRCLE)

2: $g_n = -a^n u_{-n-1} = \begin{cases} -a^n & n < 0 \\ 0 & n \geq 0 \end{cases}$ WITH $|a| > 1$ & a COMPLEX.

$G(z) = -\sum_{k=-\infty}^{-1} a^k z^{-k} = -\sum_{l=1}^{\infty} a^{-l} z^l = -\sum_{l=1}^{\infty} \left(\frac{z}{a}\right)^l = \frac{-z/a}{1 - z/a}$

So $G(z) = \frac{-z/a}{1 - \frac{z}{a}}$ FOR $\left|\frac{z}{a}\right| < 1$, i.e., $|z| < |a|$
(INCLUDES UNIT CIRCLE)

$= \frac{-1}{az^{-1} - 1}$

$= \frac{1}{1 - az^{-1}}$ FOR $|z| < |a|$

So FOR THE 2-SIDED Z-TRANSFORM,

$$\mathcal{Z}^{-1}\left\{\frac{1}{1 - az^{-1}}\right\} = \begin{cases} a^n u_n & \text{IF } |a| = |z_p| < 1 \\ -a^n u_{-n-1} & \text{IF } |a| = |z_p| > 1 \end{cases}$$

3: DECAYING SINUSOIDS:

A) $g_n = r^n \cos(n\theta_0) U_n$ WITH $|r| < 1$

$$G(z) = \sum_{n=0}^{\infty} (r^n \cos n\theta_0) z^{-n}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} r^n e^{jn\theta_0} z^{-n} + \sum_{n=0}^{\infty} r^n e^{-jn\theta_0} z^{-n}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} (r e^{i\theta_0} z^{-1})^n + \sum_{n=0}^{\infty} (r e^{-i\theta_0} z^{-1})^n\right)$$

$$= \frac{1}{2}\left(\frac{1}{1-re^{i\theta_0}z^{-1}} + \frac{1}{1-re^{-i\theta_0}z^{-1}}\right) \qquad |z| > |r|$$

$$= \frac{1}{2}\,\frac{2-(2r\cos\theta_0)z^{-1}}{1-(2r\cos\theta_0)z^{-1}+r^2z^{-2}}$$

$$= \frac{1-(r\cos\theta_0)z^{-1}}{1-(2r\cos\theta_0)z^{-1}+r^2z^{-2}} \qquad |z| > |r|$$

B) $g_n = r^n \sin(n\theta_0) U_n$ WITH $|r| < 1$

$$G(z) = \frac{1}{2j}\left(\frac{1}{1-re^{i\theta_0}z^{-1}} - \frac{1}{1-re^{-i\theta_0}z^{-1}}\right) \qquad \text{FOR } |z| > |r|$$

$$= \frac{(r\sin\theta_0)z^{-1}}{1-(2r\cos\theta_0)z^{-1}+r^2z^{-2}} \qquad \text{FOR } |z| > |r|$$

NOTE: VALID FOR $|z| > |r|$, i.e., $|z| > |z_{p_1}|$ & $|z| > |z_{p_2}|$

POLES ARE: $z_{p_1} = re^{i\theta_0}$ & $z_{p_2} = re^{-i\theta_0}$

NOTE. FOR $g_n = -r^n \substack{\cos \\ \sin}(n\theta_0) U_{n-1}$ WITH $|r| > 1$

WE GET THE SAME FUNCTIONS, VALID FOR $|z| < |z_{p_1}| = |z_{p_2}| = |r|$

1-SIDED DTFT

EXAMPLES:

1: $g_n = a^n u_n = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$ WITH a COMPLEX.

$$G(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} (az^{-1})^k = \frac{1}{1-az^{-1}}$$ FOR $|az^{-1}| < 1$ i.e. $|z| > |a|$

(ROC: $\{z \mid |z| > |a|\}$)

NOTE: WORKS FOR ALL VALUES OF a.

SIMILARLY FOR $\left(r^n \begin{smallmatrix}\sin \\ \cos\end{smallmatrix}(n\theta_0)\right)$ FOR ALL r & θ_0:
SAME FUNCTIONS AS 2-SIDED TRANSFORM, FOR ALL z WITH $|z| > |r|$

CONSEQUENCE:

$$Z^{-1}\left\{\frac{1}{1-az^{-1}}\right\} = \begin{cases} a^n u_n & \text{FOR ONE-SIDED} \\ \begin{Bmatrix} a^n u_n & \text{IF } |a|<1 \\ -a^n u_{-n-1} & \text{IF } |a|>1 \end{Bmatrix} & \text{FOR TWO-SIDED.} \end{cases}$$