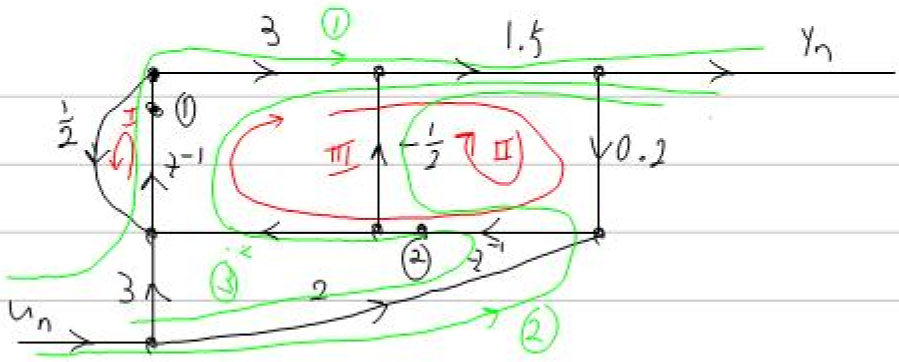


$P_1 = \frac{27}{2} z^{-1}$

$P_2 = -\frac{3}{2} z^{-1}$

$P_3 = 9 z^{-2}$

$L_I = \frac{1}{2} z^{-1}$

$L_{II} = -\frac{3}{20} z^{-1}$

$L_{III} = \frac{9}{10} z^{-2}$

TO IMPLEMENT:

① FIND STATE EQUATIONS:

$$x_1(n+1) = 3u_n + \frac{1}{2}x_1(n) + x_2(n)$$

$$x_2(n+1) = 2u_n + \frac{1}{5}\left(1.5\left(-\frac{1}{2}x_2(n) + 3x_1(n)\right)\right)$$

$$y_n = \frac{9}{2}x_1(n) - \frac{3}{4}x_2(n)$$

$$\underline{x}(n+1) = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{9}{10} & -\frac{3}{20} \end{pmatrix} \underline{x}(n) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} u_n$$

$$y_n = \begin{pmatrix} \frac{9}{2} & -\frac{3}{4} \end{pmatrix} \underline{x}(n)$$

② PSEUDOCODE:

```
LOOP FOREVER
    { INPUT U
        T1 = 1/2 * X1 + X2 + 3 * U
        T2 = 9/10 * X1 - 3/20 * X2 + 2 * U
        Y = 9/2 * X1 - 3/4 * X2
        OUTPUT Y
        X1 = T1
        X2 = T2
    }
```

TRANSFER FUNCTION:

$$H(z) = C(zI-A)^{-1}B + d$$

$$= \begin{pmatrix} \frac{9}{2} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} z-\frac{1}{2} & -1 \\ -\frac{9}{10} & z+\frac{3}{20} \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \frac{1}{z^2 - \frac{7}{20}z - \frac{39}{40}} \cdot \begin{pmatrix} \frac{9}{2} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} z+\frac{3}{20} & 1 \\ \frac{9}{10} & z-\frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \frac{1}{z^2 - \frac{7}{20}z - \frac{39}{40}} \cdot \begin{pmatrix} \frac{9}{2} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} 3z+\frac{49}{20} \\ 2z+\frac{17}{10} \end{pmatrix}$$

— FINISH AS EXERCISE

MASON'S FORMULA:

$$\Delta = 1 - \left(\frac{1}{2}z^{-1} - \frac{3}{20}z^{-1} + \frac{9}{10}z^{-2}\right) + \left(\frac{1}{2}z^{-1}\right)\left(-\frac{3}{20}z^{-1}\right)$$

$$\Delta_1 = 1$$

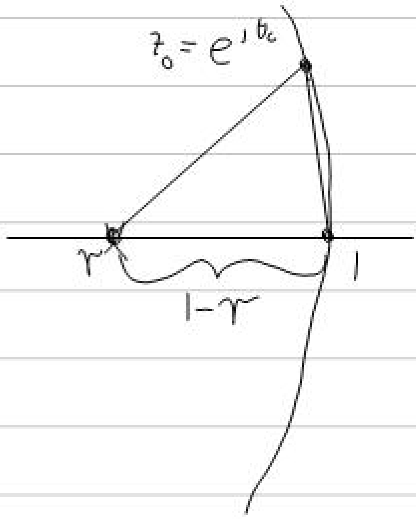
$$\Delta_2 = 1 - \left(\frac{1}{2}z^{-1}\right)$$

$$\Delta_3 = 1$$

So
$$H(z) = \frac{27/2\, z^{-1} - \frac{3}{2}z^{-1}\left(1-\frac{1}{2}z^{-1}\right) + 9z^{-2}}{1 - \frac{7}{20}z^{-1} - \frac{39}{40}z^{-2}}$$

EXAMPLE: AUDIO D.C. BLOCKING FILTER: HIGH-PASS, UNITY GAIN IN PASS-BAND; 3-dB FREQUENCY 25 Hz; $f_s = 50$ kHz

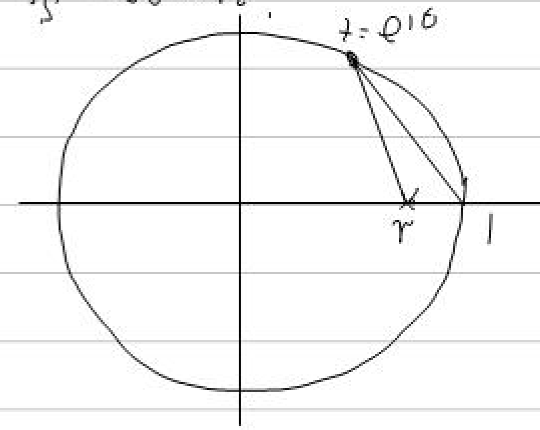
(FIRST-ORDER):



$$H(z) = \frac{1-z^{-1}}{1-rz^{-1}}$$

WITH 3-dB POINT

$$\theta_0 = 1-r$$

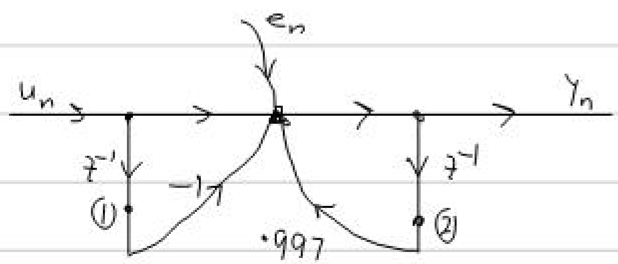


$$\theta_0 = \frac{2\pi f_0}{f_s} = \frac{\pi}{1000}$$

$$\Rightarrow r = 1-\theta_0 \approx .997$$

$$\text{So} \quad H(z) = \frac{1-z^{-1}}{1-.997z^{-1}}$$

DIRECT FORM I:



STATE EQUATIONS:

$$x_1(n+1) = u_n$$

$$x_2(n+1) = e_n + u_n - x_1(n) + .997x_2(n)$$

$$y_n = e_n + u_n - x_1(n) + .997x_2(n)$$

OR

$$\underline{x}(n+1) = \begin{pmatrix} 0 & 0 \\ -1 & .997 \end{pmatrix} \underline{x}(n) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u_n + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e_n$$

FOR $e_n = 0$:

$$H(z) = \begin{pmatrix} -1 & .997 \end{pmatrix} \begin{pmatrix} z & 0 \\ 1 & z-.997 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1$$

$$= \frac{1}{z(z-.997)} \begin{pmatrix} -1 & .997 \end{pmatrix} \begin{pmatrix} z-.997 & 0 \\ -1 & z \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1$$

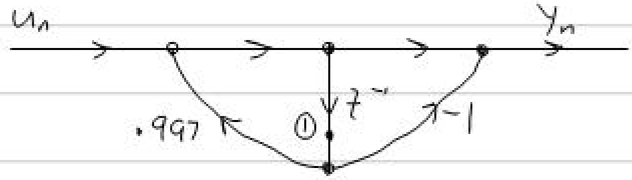
$$= \frac{1}{z(z-.997)} \begin{pmatrix} -1 & .997 \end{pmatrix} \begin{pmatrix} z-.997 \\ z-1 \end{pmatrix} + 1$$

SO THE STATE RESPONSE IS:

$$\underline{H}_s(z) = \frac{1}{z(z-.997)}\begin{pmatrix} z-.997 & 0 \\ -1 & z \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} z^{-1} \\ z^{-1}\dfrac{1-z^{-1}}{1-.997z^{-1}} \end{pmatrix}$$

DIRECT FORM II:



STATE EQUATION:

$$x(n+1) = u_n + .997\,x(n)$$

$$\Rightarrow \quad H_s(z) = \frac{1}{1-.997z^{-1}}$$

FOR LOW FREQUENCIES, $\left|H_{s'}(e^{i\theta})\right| \approx \frac{1000}{\pi} \approx 318$

— PROBABLE OVERFLOW!

DIRECT FORM I REVISITED:

$$\underline{x}(n+1) = \begin{pmatrix} 0 & 0 \\ -1 & .997 \end{pmatrix}\underline{x}(n) + \begin{pmatrix} 1 \\ 1 \end{pmatrix}u_n$$

INPUT TO STATE TRANSFER FUNCTION:

$$\left(zI - \begin{pmatrix} 0 & 0 \\ -1 & .997 \end{pmatrix}\right)\underline{X}(z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}U(z)$$

$$\Rightarrow \quad \underline{X}(z) = \underbrace{\begin{pmatrix} z & 0 \\ 1 & z-.997 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\underline{H}_s(z)} U(z)$$

$$= \frac{1}{z(z-.997)}\begin{pmatrix} z-.997 & 0 \\ -1 & z \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

DIRECT FORM I HIGH-PASS FILTER NOISE PERFORMANCE.

INPUT: e_n & OUTPUT: y_n.

$$\underline{x}(n+1) = \begin{pmatrix} 0 & 0 \\ -1 & .997 \end{pmatrix} \underline{x}_n + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e_n$$

$$y_n = \begin{pmatrix} -1 & .997 \end{pmatrix} \underline{x}_n + e_n$$

SO NOISE TO OUTPUT TRANSFER FUNCTION:

$$H_{ye}(z) = \begin{pmatrix} -1 & .997 \end{pmatrix} \begin{pmatrix} z & 0 \\ 1 & z-.997 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1$$

$$= \frac{1}{z(z-.997)} \begin{pmatrix} -1 & .997 \end{pmatrix} \begin{pmatrix} z-.997 & 0 \\ -1 & z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1$$

$$= \frac{1}{z(z-.997)} \begin{pmatrix} -1 & .997 \end{pmatrix} \begin{pmatrix} 0 \\ z \end{pmatrix} + 1$$

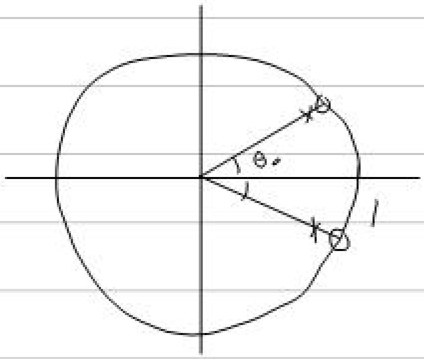
$$= \frac{.997}{z-.997} + 1$$

$$= \frac{.997z^{-1}}{1-.997z^{-1}} + 1$$

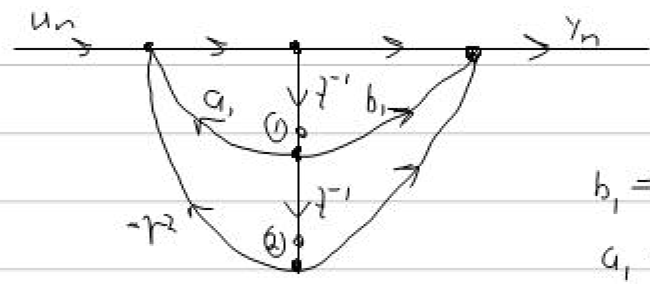
HIGH NOISE GAIN AT THE OUTPUT, AT LOW FREQUENCY

NOTCH FILTER: DIRECT FORM II.

$$H(z) = \frac{1-(2\cos\theta_0)z^{-1}+z^{-2}}{1-(2r\cos\theta_0)z^{-1}+r^2z^{-2}} = \frac{(1-e^{i\theta_0}z^{-1})\,(1-e^{-i\theta_0}z^{-1})}{(1-re^{i\theta_0}z^{-1})(1-re^{-i\theta_0}z^{-1})}$$



SFG:



$b_1 = -2\cos\theta_0$

$a_1 = 2r\cos\theta_0$

STATE EQUATIONS:

$$x_1(n+1) = u_n + a_1x_1(n) - r^2x_2(n)$$
$$x_2(n+1) = x_1(n)$$

$$\Rightarrow \quad \underline{x}(n+1) = \begin{pmatrix} a_1 & -r^2 \\ 1 & 0 \end{pmatrix}\underline{x}(n) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}u_n$$

$$\Rightarrow \quad \underline{X}(z) = \begin{pmatrix} z-a_1 & r^2 \\ -1 & z \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\bar{U}(z)$$

So

$$H_{\underline{x},u}(z) = \frac{1}{z^2-a_1z+r^2}\begin{pmatrix} z & -r^2 \\ 1 & z-a_1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{z}{z^2-a_1z+r^2} \\ \dfrac{1}{z^2-a_1z+r^2} \end{pmatrix}$$

AMPLITUDE RESPONSES TO BOTH STATES ARE EQUAL TO $\dfrac{1}{|e^{2i\theta}-a_1e^{i\theta}+r^2|}$

DENOMINATOR $\approx (1-r)\,2\sin\theta_0$ FOR $\theta = \pm\theta_0$

+ SO GAIN $\approx \dfrac{1}{(1-r)\,2\sin\theta_0}$ FOR $\theta = \pm\theta_0$ — OVERFLOW IF $1-r$ IS SMALL.