Rao Transforms: A New Approach to Integral and Differential Equations

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- **Rao Transforms (RTs) provide a brand new approach to the century old problem of Integral and Differential Equations in one and multiple dimensions.**
- **The new approach is *unified, localized, simplified,* and *computationally efficient.* Both symbolic and numerical solutions are provided by RTs.**
- **Differential Equations are first converted to Integral Equations by incorporating boundary conditions and then solved.**
- **Many fundamental laws of physics are stated using differential equations. Therefore RTs are expected to have wide applications in science and engineering** .
- **RTs are useful in shift-variant image/signal filtering, modeling linear/non-linear integral systems, optics, inverse optics, computer vision, mathematical software (e.g. Matlab, Mathematica), and medical image analysis.**
- **The RT approach is fully *localized* and therefore permits extremely fine-grained parallel implementation on a computer. Errors are minimized at boundaries.**
- **Localized structure of data is naturally exploited (e.g. a 2D image data is not restructured as a 1D vector for shift-variant deblurring).**
- **Patent applications are pending and a book has been self-published on RTs.**

Integral Transforms/Equations

In the definitions below, *x* and α are real variables, *f* is an unknown real valued function that we need to solve for, *g, k,* and *h* are known (or given) real valued functions. *r* and *s* may be real constants or one of them can be the real variable *x* .

1. Integral Transform (IT/E) $g(x) = \int_r^s k(x,\alpha)\, f(\alpha)\, d\alpha$

2. Rao Localization Equation (RLT/E) $h(x,\alpha) \equiv k(x+\alpha, x)$

3. Rao Transform (RT/E) $g(x) = \int_{x-s}^{x-r} h(x-\alpha,\alpha)\ \ f(x-\alpha)\, d\alpha$

x: point of measurement of "effect", alpha: position of "source"

Rao's First Theorem (RFT): *If h is defined by RLT in (2), then IT in (1) and RT in (3) above are exactly equivalent*, i.e., RHS(1)=RHS(3).

RT is inverted to solve for *f(x).* A system of *linear* algebraic equations will have to be solved. The derivatives of *f(x)* at *x* are the unknowns. The solution is given in terms of the derivatives of *g(x)* at *x*. Therefore the solution is a local solution at *x*.

RFT: Proof of Equivalence of RT and IT when RLT is used

$$\text{R.H.S. of RT} = \int_{x-s}^{x-r} h(x-\alpha,\alpha)\; f(x-\alpha)\,d\alpha$$

$$= \int_{x-s}^{x-r} k((x-\alpha)+\alpha,(x-\alpha))\; f(x-\alpha)\,d\alpha \qquad \text{from RLT}$$

$$= \int_{x-s}^{x-r} k(x,x-\alpha)\; f(x-\alpha)\,d\alpha$$

Change the variable of integration to β where

$$\beta = x-\alpha, \quad \text{and} \quad d\beta = -d\alpha.$$

The new limits of integration are

$$\alpha = x-s \;\Rightarrow\; \beta = s, \qquad \alpha = x-r \;\Rightarrow\; \beta = r$$

$$\text{R.H.S. of RT} = \int_{r}^{s} k(x,\beta)\; f(\beta)\,d\beta$$

$$= \int_{r}^{s} k(x,\alpha)\; f(\alpha)\,d\alpha$$

$$= \text{R.H.S. of IT.}$$

Series Expansion and Inversion of RT

$$
\begin{aligned}
g(x) &= \int_{x-s}^{x-r} h(x-\alpha,\alpha)\; f(x-\alpha)\, d\alpha \\
&\approx \int_{x-s}^{x-r} h(x-\alpha,\alpha) \left(\sum_{n=0}^{N} a_n\, \alpha^n\; f^{(n)}(x) \right) d\alpha \quad \text{(Taylor-series expansion)} \\
&\approx \sum_{n=0}^{N} a_n\; f^{(n)}(x) \left(\int_{x-s}^{x-r} \alpha^n\; h(x-\alpha,\alpha)\, d\alpha \right) \\
&\approx \sum_{n=0}^{N} a_n\; f^{(n)}(x)\, h_n(x)
\end{aligned}
$$

Derive symbolic expressions for the k-th derivative of g(x)

denoted by $g^{(k)}$ for k=1,2,…N, and recursively substitute in the above equation.

Assume $f^{(k)}(x) = 0 \quad \text{for } k > N.$

$$
\begin{bmatrix} g^{(0)} \\ g^{(1)} \\ \vdots \\ g^{(N)} \end{bmatrix} = \begin{bmatrix} h_{00} & h_{01} & \cdots & h_{0N} \\ h_{10} & h_{11} & \cdots & h_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{00} & h_{01} & \cdots & h_{0N} \end{bmatrix} \begin{bmatrix} f^{(0)} \\ f^{(1)} \\ \vdots \\ f^{(N)} \end{bmatrix} \qquad \mathbf{g}_x = \mathbf{H}_x \mathbf{f}_x
$$

Inverse RT: $\mathrm{f}_x = \mathbf{H}_x^{-1}\, \mathbf{g}_x$ $\qquad f(x) = \int_{x-s}^{x-r} h'(x-\alpha,\alpha)\, g(x-\alpha)\, d\alpha$

The resolvent kernel h’ is determined by the inverse of H.

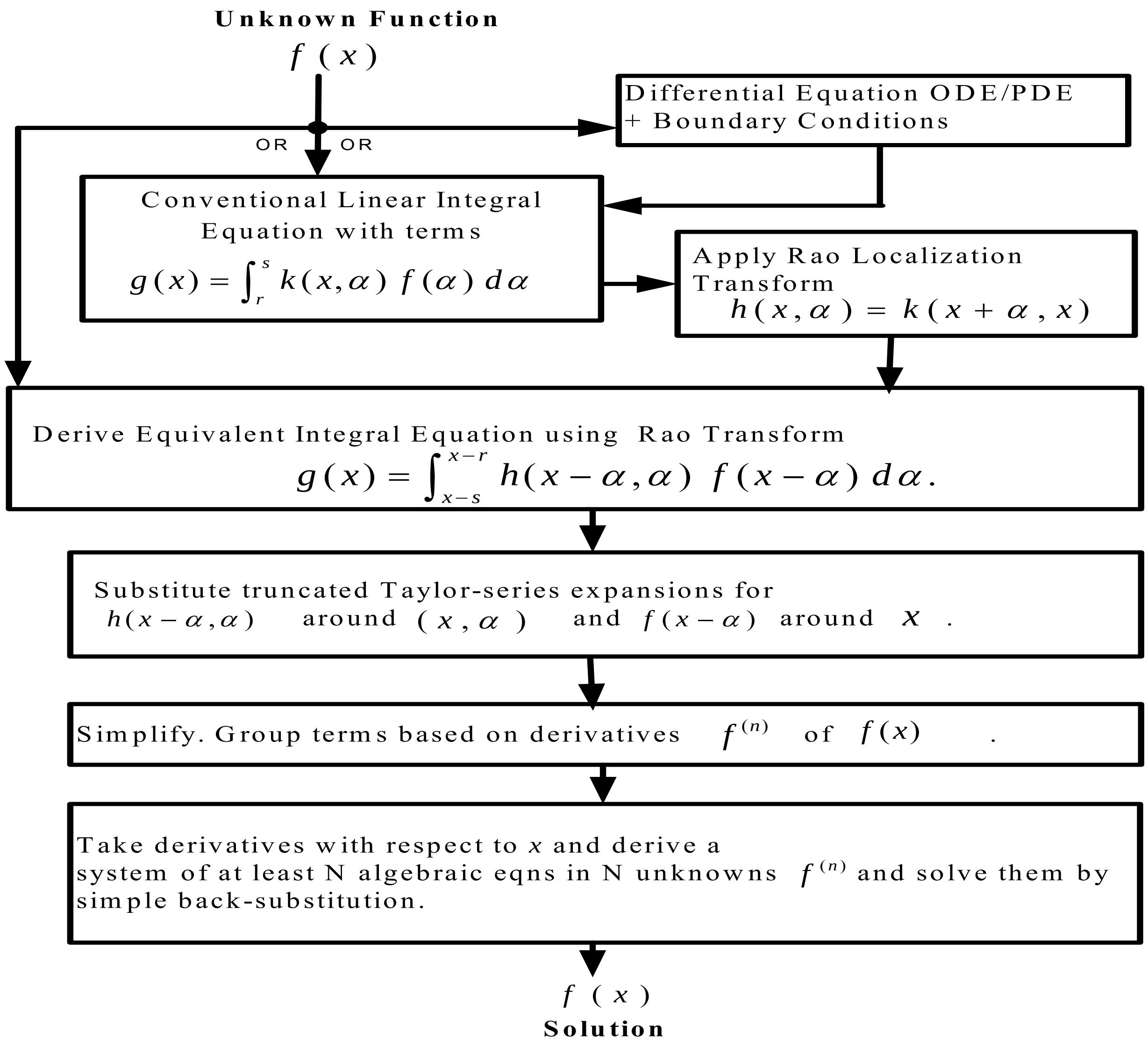


Figure 1. Solving Linear Integral and Differential Equations

General Integral Transforms/Equations
(Non-linear Integral Equations)

General IT (GIT/E) $$g(x) = \int_r^s k(x, \alpha, f(\alpha))\, d\alpha$$

General RLE (GRLT/E) $$h(x, \alpha, f(x)) \equiv k(x+\alpha, x, f(x))$$

General RT (GRT/E) $$g(x) = \int_{x-s}^{x-r} h(x-\alpha, \alpha, f(x-\alpha))\, d\alpha$$

Theorem: If *h* is defined as in GRLT, then GIT and GRT are exactly equivalent.

GRT is inverted to solve for *f(x).* A system of non-linear algebraic equations will have to be solved. The derivatives of *f(x)* at *x* are the unknowns. The solution is given in terms of the derivatives of *g(x)* at *x*. Therefore the solution is a local solution at *x*.

Application Examples and Advantages

- Many specific examples have been solved, including the Fredholm and Volterra integral equations of both the First and the Second kind.
- A general differential equation corresponding to an n-th order initial value problem has been solved.
- RTs have been applied to the problem of restoring images defocused by a shift-variant point spread function with significant computational savings.
- Additional problems are being investigated for solving them using RTs and comparing computational / theoretical advantages relative to existing methods.
- In comparison with existing methods, RTs have the unique feature of (1) completely localizing the problem, (2) solving the local problem accurately and efficiently, and then (3) synthesizing the local solutions *seamlessly* to obtain a complete or global solution.
- RTs are especially suited for practical applications due to the **"locality property"** of physical systems, i.e. the **influence** or **effect** of an **agent** or **source** is higher in its vicinity and generally decreases with increasing distance.
- Numerous future research topics are open for investigation. The approach of RTs seems to hold great promise. See www.integralresearch.net.

Applications: List of Useful Practical Problems

Most of the useful practical problems are covered by the following list where the variables may be one, two, or three dimensional vectors.

1. Fredholm Integral Equation of the First Kind

$$g(x) = \int_a^b k(x,\alpha)\ f(\alpha)\ d\alpha$$

2. Fredholm Integral Equation of the Second Kind

$$g(x) = f(x) + \int_a^b k(x,\alpha)\ f(\alpha)\ d\alpha$$

3. Volterra Integral Equation of the First Kind

$$g(x) = \int_a^x k(x,\alpha)\ f(\alpha)\ d\alpha$$

4. Volterra Integral Equation of the Second Kind

$$g(x) = f(x) + \int_a^x k(x,\alpha)\ f(\alpha)\ d\alpha$$

Solution method for all of the above problem types are the same in the new RT approach shown in Fig. 1 earlier.

Applications: RT Formulation and Solution

All the four linear integral equations (Fredholm/Volterra First/Second Kind) which account for most of the practical applications can be reformulated and solved as a single integral equation using RTs as follows:

$$g(x) = c\ f(x) + \int_{x-s}^{x-r} h(x-\alpha, \alpha)\ f(x-\alpha)\, d\alpha$$

In the above equation, setting c=0/1 results in First/Second Kind equations and setting s=constant/x results in Fredholm/Volterra integral.

Using the same notation as before, and denoting an identity matrix by **I ,**we obtain the solution in all cases to be:

$$\mathbf{g}_x = (c\mathbf{I} + \mathbf{H}_x)\, \mathbf{f}_x$$

$$\mathbf{f}_x = (c\mathbf{I} + \mathbf{H}_x)^{-1}\, \mathbf{g}_x$$

Existing Approaches (current state of the art)

- Approaches are different for different problem types (e.g. First/Second kind)
- Computational requirements are too high. Accuracy of solution and numerical stability are serious problems. Some methods are iterative.
- Computations are global and so not amenable to fine-grain parallel computations.
- When approximate localizations are employed by solving the problems separately in small intervals, the solutions at the boundaries will not be fully compatible. The solutions will have “noticeable seams” at the boundaries of intervals. So synthesizing approximate local solutions into a single “seamless” global solution is difficult.
- The current state of the art on solving integral equations is very unsatisfactory in practical applications. This was summarized as follows by an expert Program Director of a US Government Research Funding agency who reviewed a whitepaper on RTs:

  “[My agency] seems to mostly have problems in the category for which these [currently existing] methods are useless."

Examples of current approaches

- Current methods can be broadly classified as
  "Matrix Inversion" and "Matrix multiplication" methods.
- Matrix inversion methods correspond to inverting a matrix of size NxN which is proportional to the size of input data g sampled at a set of points. Examples of this method are Fredholm's method, Eigen vector/value methods such as Singular Value Decomposition, Collocation, Galerkin, and Nystrom.
- Matrix multiplication methods correspond to iteratively multiplying an NxN matrix with an Nx1 vector where N is the size of sampled input data g.
  Examples of this method are Volterra's iterated kernels, Neumann series, and Born approximation. These are mainly applicable to Second Kind equations.
- Typical computational complexity is O(N^3) which is too much.
- In comparison, the typical computational complexity of the new RT approach is around O(N^2) resulting in a speed up of O(N). This can be very significant in the case of 2D and 3D problems.

Application Example 1: Image Restoration

- This example is relevant to deblurring an image of a 1D barcode printed on a slanted plane or a curved surface and scanned by a laser scanner that uses a lens for imaging.
- Lenses have limited depth-of-field, and so the image will be blurred by a shift-variant point spread function (PSF).

Approximation: $|h(x-\alpha,\alpha)| \approx 0 \text{ for } |\alpha| > s \text{ for all } x.$

Computing the blurred image g(x) or $g^{(0)}$: Forward RT:

$$\begin{bmatrix} g^{(0)} \\ g^{(1)} \\ g^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & h_2^{(1)} & (1/2)h_2^{(0)} \\ 0 & 1 & (3/2)h_2^{(1)} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f^{(0)} \\ f^{(1)} \\ f^{(2)} \end{bmatrix} \qquad h_n^{(m)} = h_n^{(m)}(x) = \int_{-\infty}^{\infty} \alpha^n \frac{\partial^m h(x,\alpha)}{\partial x^m} d\alpha$$

Computing the focused image f(x) or $f^{(0)}$: Inverse RT:

$$\begin{bmatrix} f^{(0)} \\ f^{(1)} \\ f^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & -h_2^{(1)} & -(1/2)(h_2^{(0)} - 3h_2^{(1)} \cdot h_2^{(1)}) \\ 0 & 1 & -(3/2)h_2^{(1)} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g^{(0)} \\ g^{(1)} \\ g^{(2)} \end{bmatrix}$$

Intrepretation of condition on the kernel h

- In order to interpret the meaning of the condition on the kernel h for convergence, consider the example of a Gaussian blurring kernel in image defocusing of a slanted plane or a curved surface. In this case we have the following situation.
- Conventional (global) kernel of an IT:

$$k(x,\alpha) = \frac{1}{\sqrt{2\pi}\sigma(\alpha)} \exp\left(- \frac{(x-\alpha)^2}{2\sigma^2(\alpha)} \right)$$

- Standard assumption for conventional kernel in solving an IT:

$$|k(x,\alpha)| \approx 0 \text{ for } |x| > T \quad \text{for all } x.$$

- RT (local) kernel

$$h(x,\alpha) = \frac{1}{\sqrt{2\pi}\sigma(x)} \exp\left(- \frac{\alpha^2}{2\sigma^2(x)} \right) \qquad h(x,\alpha) = k(x+\alpha, x)$$

- Assumption for RT kernel: $|h(x-\alpha,\alpha)| \approx 0 \text{ for } |\alpha| > s \quad \text{for all } x.$

- Note that the two assumptions are similar, but the new assumption may be a little weaker and cleaner..

New Concepts: Local Eigen Values and Local Eigen Functions

- Set $g(x) = \lambda\,\phi(x)$ in RT as:

$$\begin{bmatrix} g^{(0)} \\ g^{(1)} \\ \vdots \\ g^{(N)} \end{bmatrix} = \begin{bmatrix} h_{00} & h_{01} & \cdots & h_{0N} \\ h_{10} & h_{11} & \cdots & h_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{00} & h_{01} & \cdots & h_{0N} \end{bmatrix} \begin{bmatrix} \phi^{(0)} \\ \phi^{(1)} \\ \vdots \\ \phi^{(N)} \end{bmatrix}$$

$$\mathbf{g}_x = \mathbf{H}_x \boldsymbol{\varphi}_x$$

$$\lambda \boldsymbol{\varphi}_x = \mathbf{H}_x \boldsymbol{\varphi}_x \quad \Rightarrow \quad (\mathbf{H}_x - \mathbf{I}\lambda)\boldsymbol{\varphi}_x = 0$$

Local Eigen Values are the roots of $\text{Det}\,(\mathbf{H}_x - \mathbf{I}\lambda) = 0$

Local Eigen Functions are determined by the solution of

$$(\mathbf{H}_x - \mathbf{I}\lambda_i)\boldsymbol{\varphi}_{xi} = 0 \quad , \quad \lambda_i = 0, 1, 2, \cdots, N$$

Actual Local Eigen Functions (N-th order) are:

$$\phi_i(x+\alpha) = \phi_i^{(0)}(x) + \alpha\,\phi_i^{(1)}(x) + \frac{\alpha^2}{2}\phi_i^{(2)}(x) + \cdots + \frac{\alpha^N}{N!}\phi_i^{(N)}(x) \quad , \quad i = 0, 1, 2, \cdots, N.$$

They approximate the Global Eigen Functions in a small interval around x.

As N tends to infinity, Local Eigen Function tend to the Global Eigen Functions.

Application Example 2: N-th Order Initial Value Problem

Ordinary Differential Equation:

$$\sum_{k=0}^{n} A_k(x) \frac{d^{(n-k)}y}{dx^{(n-k)}} = F(x), \qquad y^{(n-k)}(r) = q_{n-k} \qquad k = 0,1,\cdots,n-1$$

Solution is obtained by inverting the following RT:

$$g(x) = f(x) - \int_0^{x-r} h(x-\alpha,\alpha)\ f(x-\alpha)\, d\alpha$$

$$h(x,\alpha) = -\sum_{k=1}^{n} A_k(x+\alpha) \frac{\alpha^{k-1}}{(k-1)!}$$

$$g(x) = F(x) - q_{n-1}\ A_1(x) - [(x-r)\ q_{n-1} + q_{n-2}]\ A_2(x) - \cdots$$
$$- \left\{[(x-r)^{n-1}/(n-1)!]q_{n-1} + \cdots + (x-r)q_1 + q_0\right\} A_n(x)$$

$$y(x) = \int_r^x \frac{(x-\alpha)^{n-1}}{(n-1)!} f(\alpha)\, d\alpha \quad + \quad \sum_{k=0}^{n-1} \frac{(x-r)^k}{k!} q_k$$

Application Example 3: Direct Vision Sensing by Inverse Optics

1. Model the camera as a linear shift-variant system where the captured image g(x,y) is a function of camera parameters e=(s,F,D,λ), focused image f(x,y), shape z(x,y) with parameters distance Z0, slopes Zx, Zy, and curvatures if needed.

$$g_p(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x,y,\alpha,\beta,\mathbf{e}_p,\mathbf{z})\ f(\alpha,\beta)\,d\alpha\,d\beta$$

2. Localize the shift-variant point spread function (SV-PSF) using RLT:

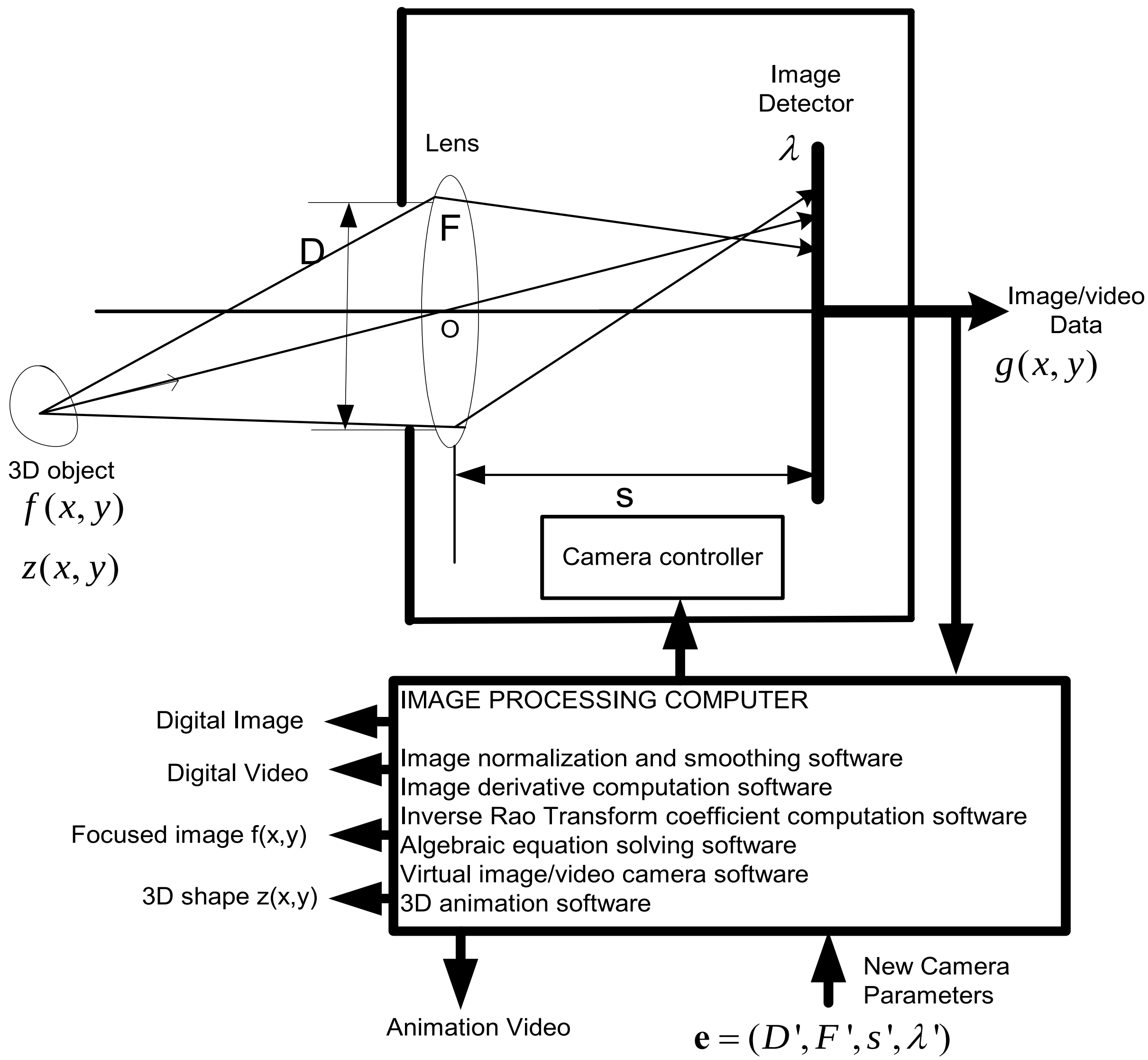
$$h(x,y,\alpha,\beta,\mathbf{e}_p,\mathbf{z}) = k(x+\alpha,y+\beta,x,y,\mathbf{e}_p,\mathbf{z})$$

3. Apply RFT to get a localized model of image formation

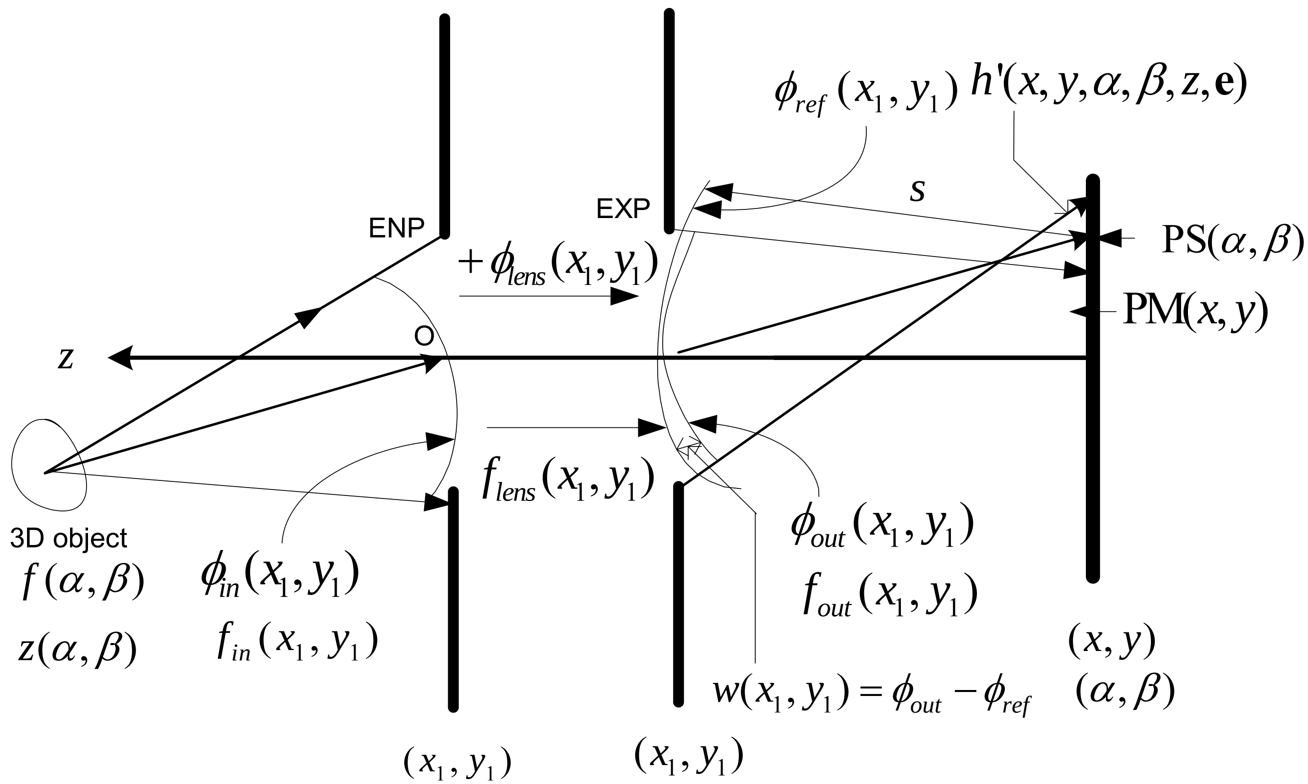
$$g_p(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\alpha,y-\beta,\alpha,\beta,\mathbf{e}_p,\mathbf{z})\ f(x-\alpha,y-\beta)\,d\alpha\,d\beta$$

4. Capture P images with different camera parameter settings for p=1,2,…,P.

$$\mathbf{e}_p = (D_p, s_p, f_p, \lambda_p)$$



f(x,y) : Focused image (input from scene to camera)
z(x,y): 3D shape of object surface (input from scene to camera)
g(x,y): blurred image recorded by camera (output from camera)


$\phi_{ref}(x_1, y_1)$
$h'(x, y, \alpha, \beta, z, \mathbf{e})$
s
EXP
ENP
$\mathrm{PS}(\alpha, \beta)$
$+\phi_{lens}(x_1, y_1)$
$\mathrm{PM}(x, y)$
O
z
$f_{lens}(x_1, y_1)$
3D object
$f(\alpha, \beta)$
$z(\alpha, \beta)$
$\phi_{in}(x_1, y_1)$
$f_{in}(x_1, y_1)$
$\phi_{out}(x_1, y_1)$
$f_{out}(x_1, y_1)$
(x, y)
(α, β)
$w(x_1, y_1) = \phi_{out} - \phi_{ref}$
(x_1, y_1)
(x_1, y_1)

Application Example 3: Direct Vision Sensing by Inverse Optics

5. Substitute truncated Taylor series expansions of f, h, and z, to obtain the inverse RT:

$$f(x,y) = f^{(0,0)} = \sum_{n=0}^{N} \sum_{i=0}^{n} S'_{p,n,i}\, g_p^{(n-i,i)}$$

In one specific case, we get:

$$\begin{aligned} f^{(0,0)} &= g_p^{(0,0)} - g_p^{(1,0)}\, h_{p,2,0}^{(1,0)} - g_p^{(0,1)}\, h_{p,0,2}^{(0,1)} \\ &+ g_p^{(2,0)} \left(\frac{3}{2} (h_{p,2,0}^{(1,0)})^2 + \frac{1}{2} h_{p,0,2}^{(0,1)}\, h_{p,2,0}^{(0,1)} - \frac{1}{2} h_{p,2,0}^{(0,0)} \right) \\ &+ g_p^{(0,2)} \left(\frac{3}{2} (h_{p,0,2}^{(0,1)})^2 + \frac{1}{2} h_{p,2,0}^{(1,0)}\, h_{p,0,2}^{(1,0)} - \frac{1}{2} h_{p,0,2}^{(0,0)} \right) \end{aligned}$$

6. Solve the system of P equations to obtain the P unknowns f(x,y), distance Z0, slopes Zx, Zy, and curvatures if needed.
7. Synthesize the solutions at all points (x,y) to get complete 3D shape and focused image f(x,y).

Generalization of RT: From Kernel "localization" to Kernel transformation

In the conventional integral equation

$$g(x) = c\ f(x) + \int_r^s k(x,\alpha)\ f(\alpha)\ d\alpha$$

we can replace the integration variable α with a one-to-one and onto function β=β(x,α) which is bijective with respect to α. For example β=x-α results in "localization" of the kernel seen earlier. Scaling and translation can be incorporated using β=ax+bα+d for constants a,b,d. In this case we obtain

$$\begin{aligned}
g(x) &= c\ f(x) + \int_{ax+br+d}^{ax+bs+d} k(x, ax+b\alpha+d)\ f(ax+b\alpha+d)\, b\, d\alpha \\
&\approx c\ f(x) + \int_{ax+br+d}^{ax+bs+d} k(x, ax+b\alpha+d)\left(\sum_{n=0}^{N} \frac{(b\alpha)^n}{n!}\frac{d^n f(ax+d)}{dx^n}\right) b\, d\alpha \qquad \text{(Taylor series expansion)} \\
&\approx c\ f(x) + \sum_{n=0}^{N} \frac{d^n f(ax+d)}{dx^n} \int_{ax+br+d}^{ax+bs+d} \frac{(b\alpha)^n}{n!} k(x, ax+b\alpha+d)\ b\, d\alpha \\
&\approx c\ f(x) + \sum_{n=0}^{N} f_{(x)}^{(n)}\ k_n(x)
\end{aligned}$$

Now, assuming $f_{(x)}^{(m)} = 0 \quad \text{for } m > N$ we can derive expressions for the various derivatives of g(x) as before and solve the integral equation as

$$\mathbf{g}_x = (c\mathbf{I} + \mathbf{K}_x)\ \mathbf{f}_x \qquad \text{and} \qquad \mathbf{f}_x = (c\mathbf{I} + \mathbf{K}_x)^{-1}\ \mathbf{g}_x$$

Future Research and Conclusions

- Investigate the application of RTs to other practical problems such as fluid mechanics and computational physics.
- Compare the relative performance of RTs with current state of the art techniques for practical problems in terms of accuracy, stability, and computational efficiency.
- Investigate the theoretical and computational aspects of RTs further because RTs provide an elegant way to convert an integral equation to a single localized differential equation that incorporates all the boundary conditions which is easily solved both symbolically and numerically.
- It may be possible to rederive previously known theoretical results on integral equations such as uniqueness and existence using RTs in a simpler and more unifying way that provides additional insights into problems of practical importance,
- The basic idea of localization may be extensible to other inverse problems of practical importance. This should be investigated.
- Application of RTs to solve medical image analysis problems such as computer tomography (x-ray, optical, MRI, etc.) should be investigated.