

A New Unified, Localized, and Efficient Approach to Integral Equations

Dr. Muralidhara SubbaRao

Dept. of Electrical and Computer Engg., State University of New York at Stony Brook

<http://www.integralresearch.net> , rao@integralresearch.net

“ ... [this research] has applications. It is guaranteed to produce doctoral dissertations.”

--An expert reviewer of this research for the U.S. Government.

Overview

A new approach is presented for solving integral equations. It is a fundamental computational and theoretical advance that provides a unified, fully localized, and computationally efficient solution in closed-form that is useful in both symbolic and numeric computations. The approach is naturally suited for fine-grain parallel implementation. In practical problems such as shift-variant image deblurring, the new approach offers significant computational speed-up in comparison with standard current techniques. Since Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs) can be reformulated as integral equations by taking boundary conditions into account, the new approach is also relevant to solving ODEs and PDEs. It is also useful in image/signal filtering, and the analysis and modeling of linear and non-linear systems and processes, in engineering and applied sciences.

1. Linear Integral Equations

Consider a conventional linear integral equation [2,3,4,] that includes Fredholm and Volterra Integral Equations of both the First and the Second Kind:

$$g(x) = c f(x) + \int_r^s k(x,t) f(t) dt \quad (1.1)$$

Here x and t are real variables, $f(x)$ is an unknown real valued function that we need to solve for, and $g(x)$ and $k(x,t)$ are known (or given) real valued functions. $k(x,t)$ represents the original integration kernel. For simplicity and without loss of generality, the conventional practice of using a factor λ (corresponding to an eigenvalue parameter) in front of the integral sign is not used here. r and s may be real constants (including possibly $\pm\infty$) or one of them can be the real variable x (they may also be known functions of x , but we shall not deal with this case here). $c=0$ for *First Kind* and $c=1$ for *Second Kind* integral equations. $g(x) = 0$ for homogeneous integral equations. All functions here are assumed to be continuous, integrable, and differentiable up to some order as needed.

The above integral equation will be restated in an equivalent form and without loss of any generality, using the change of variable

$$u = x - t . \quad (1.2)$$

A novel, unique, and innovative feature of this change of variable is that the new variable u is a function of both x and t instead of just t . This novel idea, although simple, has eluded researchers in the past, and it facilitates localizing the problem and deriving a closed-form solution. A more general change of variable such as $u = ax + bt + c$ for real known constants a , b , and c (and other non-linear functions) are briefly considered in [1], but will not be considered here. With the above change of variable, we obtain

$$g(x) = c f(x) + \int_{x-r}^{x-s} k(x,u) f(u) du \quad (1.3)$$

which can be written as

$$g(x) = c f(x) + \int_{x-s}^{x-r} k(x, x-t) f(x-t) dt \quad (1.4)$$

The above equation is named *Rao Transform* (RT) (of $f(x)$) or *Rao Integral Equation* (RIE). It can be reduced to a *differential equation* as below:

$$\begin{aligned} g(x) &= c f(x) + \int_{x-s}^{x-r} k(x, x-t) f(x-t) dt \\ &\approx c f(x) + \int_{x-s}^{x-r} k(x, x-t) \left(\sum_{n=0}^N \frac{(-1)^n}{n!} t^n \frac{d^n f(x)}{dx^n} \right) dt \quad (\text{Taylor-series expansion}) \\ &\approx c f(x) + \sum_{n=0}^N \frac{d^n f(x)}{dx^n} \frac{(-1)^n}{n!} \int_{x-s}^{x-r} t^n k(x, x-t) dt \quad (\text{change order of } \int \text{ and } \sum) \end{aligned}$$

Therefore

$$g(x) \approx c f(x) + \sum_{n=0}^N k_n(x) f^{(n)}(x) \quad (1.5)$$

where

$$\begin{aligned} k_n(x) &\equiv \frac{(-1)^n}{n!} \int_{x-s}^{x-r} t^n k(x, x-t) dt, \\ f^{(n)} &\equiv f^{(n)}(x) \equiv \frac{d^n f(x)}{dx^n} \end{aligned} \quad (1.6)$$

In practical applications, both $k_n(x)$ and $f^{(n)}$ typically approach zero with increasing n . Let

$$f^{(m)}(x) = 0 \quad \text{for } m > N. \quad (1.7)$$

Later we will consider the limit as $N \rightarrow \infty$. Now, using suitable notation, we can derive from Eq. (1.5) an expression for the m -th order derivative of $g(x)$ for $m=0,1,2,\dots,N$, as:

$$g^{(m)} \equiv g^{(m)}(x) \equiv \frac{d^m g(x)}{dx^m} = c f^{(m)}(x) + \sum_{n=0}^N \sum_{p=0}^m C_p^m k_n^{(m-p)}(x) f^{(n+p)}(x) T(n+p) \quad (1.8)$$

where

$$k_n^{(m-p)}(x) = \frac{d^{(m-p)}}{dx^{(m-p)}} k_n(x) \quad (1.9)$$

and the function

$$T(n+p) = \begin{cases} 1 & \text{for } n+p \leq N \\ 0 & \text{otherwise.} \end{cases} \quad (1.10)$$

ensures that terms with $f^{(n+p)}(x)$ in Eq. (1.8) are set to zero when $n+p > N$.

Eq. (1.8) above can be simplified by grouping terms with respect to $f^{(n)}(x)$ to obtain

$$g^{(m)}(x) = c f^{(m)}(x) + \sum_{n=0}^N k_{m,n}(x) f^{(n)}(x) \quad \text{for } m = 0, 1, 2, \dots, N \quad (1.11)$$

where

$$k_{m,n}(x) = \sum_{p=0}^{\min(n,m)} C_p^m k_{n-p}^{(m-p)}(x). \quad (1.12)$$

Equations (1.11) and (1.12) can be derived by noting the relation between the terms of $g^{(m)}(x)$ derived from $g^{(m-1)}(x)$ by applying the derivative operator for $m = 1, 2, \dots, N$.

Eq. (1.11) can be written in vector-matrix form as:

$$\begin{bmatrix} g^{(0)} \\ g^{(1)} \\ \vdots \\ g^{(N)} \end{bmatrix} = c \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} f^{(0)} \\ f^{(1)} \\ \vdots \\ f^{(N)} \end{bmatrix} + \begin{bmatrix} k_{00} & k_{01} & \cdots & k_{0N} \\ k_{10} & k_{11} & \cdots & k_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N0} & k_{N1} & \cdots & k_{NN} \end{bmatrix} \begin{bmatrix} f^{(0)} \\ f^{(1)} \\ \vdots \\ f^{(N)} \end{bmatrix} \quad (1.13)$$

The above equation can be written in a more compact form using vector and matrix variables as:

$$\mathbf{g}_x = (c\mathbf{I} + \mathbf{K}_x) \mathbf{f}_x = \mathbf{J}_x \mathbf{f}_x. \quad (1.14)$$

The subscript x in the above equation makes explicit the dependence of the vector/matrix on x . In this equation, note that the components of the column vectors are not the sampled values of $g(x)$ and $f(x)$ at different points x , but they are the different order derivatives of $g(x)$ and $f(x)$ at the same point x . Therefore, in practical applications, the size of the matrices will be small (3 to 6), but this matrix equation will have to be solved many times, once at value of x . At each point x , \mathbf{g}_x and \mathbf{K}_x will have to be computed from known data around the point x .

We can obtain a symbolic solution for $f(x) = f^{(0)}(x)$ from Eq. (1.13) or (1.14) through a successive Gaussian elimination and back substitution method. This symbolic solution is useful in practical applications as the value of N is typically very small, around 3 to 6. Assuming that the matrix $\mathbf{J}_x = (c\mathbf{I} + \mathbf{K}_x)$ is non-singular, the solution can be written in matrix form as:

$$\mathbf{f}_x = \mathbf{J}_x^{-1} \mathbf{g}_x = \mathbf{K}'_x \mathbf{g}_x \quad (1.15)$$

In explicit algebraic form, denoting the elements of $\mathbf{J}_x^{-1} = \mathbf{K}'_x$ by k'_{ij} , we may write the solution as

$$f^{(0)} = f^{(0)}(x) = f(x) = \lim_{N \rightarrow \infty} \sum_{n=0}^N k'_{0n}(x) g^{(n)}(x) \approx \sum_{n=0}^N k'_{0n}(x) g^{(n)}(x) \quad (1.16)$$

In practical applications, in order to reduce the effects of noisy data on the solution, a regularization technique can be used to solve for \mathbf{f}_x . For example, a spectral filtering technique based on the Singular Value Decomposition (SVD) of $\mathbf{J}_x = (c\mathbf{I} + \mathbf{K}_x)$ such as the Truncated SVD or Tikhonov regularization [6] can be used.

See [1] for more details on extension of this approach to multi-dimensional cases and specific application examples, including two-dimensional shift-variant image deblurring. This approach is an extension of the author's work on depth-from-defocus in computer vision [5].

2. Non-linear Integral Equations

A class of non-linear integral equations is represented by

$$g(x) = c f(x) + \int_r^s k(x,t, f(t)) dt. \quad (2.1)$$

Using the same change of variable $u = x - t$, the above equation can be written as:

$$g(x) = c f(x) + \int_{x-s}^{x-r} k(x, x-t, f(x-t)) dt \quad (2.2)$$

The method of solution in this case is similar to that of linear integral equations described earlier up to the step of deriving expressions for the derivatives of $g(x)$ up to order N . Then the next step will be to solve a system of non-linear algebraic equations, which is, in general, complicated. Some details on this problem can be found in [1].

3. Differential Equations

Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs) can be reformulated as integral equations by taking boundary conditions into account [2,3,4]. Solving such integral equations using the new approach presented here will solve the ODEs and PDEs..

4. Advantages

The new approach provides a *localized solution* in the following sense—the solution $f(x)$ at a point x is expressed in terms of the values of the known function $g(x)$ at the same point x , and also the characteristics (or moments with respect to α) of the *localized kernel at the same point x* . In problems such as shift-variant image deblurring, this facilitates rapid convergence and accuracy of the series expansion in Eq. (1.16) that provides the solution. For the same reason, it permits a fine-grain parallel computation. The solutions at different points x can be computed in parallel using local data of the observed/measured function $g(x)$ (whose local derivatives are used). At each point x , elements of a small (3x3 to 6x6 in practice, depending on data noise and solution accuracy) matrix \mathbf{J}_x are computed and inverted. In practical applications, this matrix \mathbf{J}_x often has a very simple form and can be computed very efficiently. For example, in shift-variant image deblurring, it is upper triangular and can be computed fast, either using analytic expressions, or, because the kernel is symmetric, i.e. $k(x, x-t) = k(x, x+t)$, and has compact support with respect to t , i.e. it satisfies the condition $k(x, x-t) = 0$ for $|t| \geq A$ for some constant A which is much smaller than the size of the domain of integration (this is the case when the blurring model is based on geometric optics).

The solution at each point x will be *compatible* with the solutions at near-by points (up to some order derivatives of the solution). Therefore the *local solutions* can all be synthesized *seamlessly* (without glitches or discontinuities) at the borders (e.g. image deblurring) to obtain a *global solution* to the original problem. The new method provides an analytic solution suitable for both theoretical analysis and numerical implementation. It provides a unified theory and a unified computational framework for a large and diverse class of both linear and non-linear integral equations. Numerical methods based on the new method are likely to offer significant computational savings [1]. The new approach can be naturally extended from one-dimensional to multi-dimensional problems. An important future research topic is to compare the computational performance of the new approach and its variations (e.g. incorporating TSVD or Tikhonov regularization) with existing approaches in practical applications.

5. References

1. <http://www.integralresearch.net>
Book: "*Rao Transforms: Theory and Applications*", by M. Subbarao (Rao), U.S. Copyright Registration No. TX 6-195-821, June 1, 2005. Self-published book, 120 pages.
Second Edition: "*Rao Transforms: A New Approach to Integral and Differential Equations*", May 2007. Purchase at <http://www.integralresearch.net>.
White paper: <http://www.integralresearch.net/wps.pdf>.
Links for Patent Applications, Seminar Slides, and other info: www.integralresearch.net.
2. Kanwal, R. P., *Linear Integral Equations: Theory and Technique*, (2nd Ed.), Birkhauser Publishers, Boston, 1997.
3. M. Masujima, *Applied Mathematical Methods in Theoretical Physics*, WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim, 2005. ISBN 3-527-40534-8.
4. Polyanin, A. D., and Manzhirov, A. V., *Handbook of Integral Equations*, Boca Raton, FL: CRC Press, 1998.
5. M. Subbarao and G. Surya, "Depth from Defocus: A Spatial Domain Approach," *International Journal of Computer Vision*, Vol. 13, No. 3, pp. 271-294, 1994.
6. P. C. Hansen, J.G. Nagy, and D. P. O'Leary, "Deblurring Images: Matrices, Spectra, and Filtering", book published by SIAM, 2006, ISBN 0-89871-618-7.

Rao Transforms: A New Approach to Integral and Differential Equations

Dr. Muralidhara SubbaRao

<http://www.ece.sunysb.edu/~murali/> , murali@ece.sunysb.edu

<http://www.integralresearch.net> , rao@integralresearch.net

1. Summary

Rao Transforms (RTs) provide a brand new approach to the century old problem of solving linear and non-linear integral equations. The solution is *completely localized* and therefore permits extremely fine-grained parallel implementation on a computer. The approach is *simple* (easy to implement and comprehend) and *unified* as it solves a large class of diverse problems. Therefore RTs have both *computational* and *theoretical* advantages in practical applications. RTs are also useful in solving *differential equations* (DEs) as both ordinary and partial differential equations (ODE/PDEs) can be converted to integral equations that incorporate boundary conditions. Since many fundamental laws of physics are stated using ODE/PDEs, RTs are expected to have wide applications in science and engineering. In particular, RTs are useful in the design and analysis of linear and non-linear systems as they can be modeled by integral equations.

The *Rao Transform* (RT) theory for solving linear integral equations can be summarized as follows. *Rao Localization Transform/Equation* (RLT/RLE) is defined as

$$h(x, \alpha) = k(x + \alpha, x) \quad (\text{RLT})$$

It transforms a conventional integration kernel $k(x, \alpha)$ which is a “global form” description of the kernel into $h(x, \alpha)$ which is a “local form” description of the kernel. This transformation can be used to define a *Rao Transform* (RT) or *Rao Integral Equation* (RIE) as

$$g(x) = \int_{x-s}^{x-r} h(x - \alpha, \alpha) f(x - \alpha) d\alpha \quad (\text{RT})$$

which is, according to *Rao's First Theorem* (RFT), exactly equivalent to a conventional Integral Equation (IE) or Integral Transform (IT) of the type

$$g(x) = \int_r^s k(x, \alpha) f(\alpha) d\alpha. \quad (\text{IT})$$

In these equations, r and s may be constants or one of them can be equal to the variable x . Therefore, the above integral equation (IT) could be either Fredholm Integral Equation of the First Kind, or Volterra Integral Equation of the First Kind.

A *unified, analytic, and localized* (and therefore computationally efficient, accurate, and permits fine-grain parallel implementation) solution of RIE (and therefore the IE) can be obtained as follows.

First, in the RT defined above, substitute truncated Taylor series expansions of $h(x - \alpha, \alpha)$ around (x, α) and $f(x - \alpha)$ around x , and simplify to obtain a series expansion of $g(x)$ in terms of the derivatives of $f(x)$ and moments of derivatives of $h(x, \alpha)$.

Next, derive a system of N linear algebraic equations by taking the derivatives of $g(x)$ upto a required order N where N can be an arbitrarily large integer (but in practical applications, acceptable numerical accuracy can be obtained for small values of N , typically around 4). This system of equations will in general involve different numerical coefficients at different points x where we wish to compute a numerical solution for $f(x)$.

Finally, solve this system of N linear equations to obtain a solution for the derivatives of $f(x)$ at x , and in particular the zero-th derivative $f(x)$ which is of interest. In solving this system of equations, assume that derivatives of $f(x)$ of order more than N are zero. This step involves inverting an $N \times N$ matrix (or equivalent computations) where N is a small integer (around 4). This matrix inversion or equivalent computations will need to be done repeatedly, once at each point x where a numerical solution for $f(x)$ is desired. A symbolic closed-form solution for $f(x)$ could be obtained for small values of N as an expression involving the derivatives of $g(x)$ upto order N . Numerical accuracy of the solution for $f(x)$ at a point x is maintained because it is computed using known data $g(x)$ and the kernel $h(x, \alpha)$ at and around the same point x . The data of $g(x)$ and the kernel at and around x is needed to estimate their derivatives at x . This last step in computing the solution is often misunderstood by readers and therefore extra attention is helpful. Otherwise, the readers are likely to conclude incorrectly that the method of RT is far more numerically unstable than it actually is.

The resulting solution appears to be both theoretically and computationally superior to classical solutions provided by Fredholm's determinants, Volterra's iterated kernels, and orthonormal series expansions. Numerically, the new solution method is thought to be comparable or superior to classical techniques such as *Collocation* methods, *Galerkin* methods, and *Nystrom* methods. In particular, the new solution method is *non-iterative* and the computations are *highly localized*, thus making it ideal for implementation on massively parallel/distributed computing hardware.

The results of RTs for linear integral equations can be extended to non-linear integral equations with similar advantages. In this case, a conventional "global form" kernel $k(x, \alpha, f(\alpha))$ is transformed to a "local form" kernel $h(x, \alpha, f(x))$ by defining

$$h(x, \alpha, f(x)) = k(x + \alpha, x, f(x)) \quad . \quad (\text{GRLT})$$

The concept of "localizing" the integration kernel can be extended to more general transformations of the kernel. For example, the conventional integral equation

$$g(x) = \int_r^s k(x, \alpha) f(\alpha) d\alpha$$

can be restated without loss of generality by an exactly equivalent integral equation of the form

$$g(x) = \int_{\beta(x,r)}^{\beta(x,s)} k(x, \beta(x, \alpha)) f(\beta(x, \alpha)) d\beta$$

where $\beta = \beta(x, \alpha)$ is a one-to-one and onto function with respect to α . A simple example of $\beta(x, \alpha)$ is:

$$\beta(x, \alpha) = ax + b\alpha + c \quad \text{where } a, b, \text{ and } c \text{ are known scalar constants.}$$

Note that setting $a=1$, $b=-1$, and $c=0$ results in “localizing” the problem where $\beta(x, \alpha) = x - \alpha$.

Rao Transforms are an extension of the Spatial-Domain Convolution/Deconvolution Transform (S Transform) invented about 15 years ago by this author. S transform has proved to be superior to the Fourier transform in some image processing and computer vision applications such as image restoration/filtering and depth-from-defocus. Rao Transforms are expected to have both theoretical and computational advantages similar to S transform in comparison with conventional methods for solving integral equations.

2. Background

A *Spatial-domain Convolution/Deconvolution Transform* or *S Transform* was invented by this author in 1989 for the restoration of defocused images [1,2,7,15]. It provided a direct and localized spatial-domain method as opposed to the conventional Fourier domain method for image *deconvolution*, or the solution of a *Convolution Integral Equation* (CIE). CIE is a special case of the Fredholm Integral Equation of the First Kind where the integration kernel is *shift-invariant*. During the last 15 years, the practical applications and advantages of S transform has been successfully demonstrated for image restoration, depth-from-defocus, and autofocusing problems in modern digital cameras [2,7]. In particular, S transform based methods have been demonstrated to be superior to the Fourier transform based methods for depth-from-defocus and autofocusing applications, both in terms of computational speed and the accuracy of depth recovery and autofocusing. The main reason for this better performance seems to be that while inverse S transform provides *local* deconvolution, inverse Fourier transform provides global deconvolution.

In the Summer of 2004, Rao Transforms [1,16,17] were invented by this author. Initially, Rao Transforms were derived by extending the S transform to solve the linear *shift-variant* image restoration or deblurring problem. In this case, the blurring kernel varies with image position, i.e. the kernel is shift-variant. This problem is the same as the problem of solving the Fredholm Integral Equation of the First Kind. A search of the research literature revealed that Rao Transforms were novel. In addition, it was found that Rao Transforms could be generalized to solve a large class of both linear and non-linear integral equations including Fredholm type, Volterra type, Urysohn type, Hammerstein type, etc. Since Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs) can be reformulated as Integral Equations by taking boundary conditions into account, Rao Transforms are also relevant to solving ODEs and PDEs. The theoretical and computational advantages of S transform are expected to carry over to Rao Transforms in solving integral and differential equations as the two transforms are related.

3. Current State of the Art

In the current research literature [4,5,6,8,9,10,11,12], there is no unified theory for solving general integral equations. Solution methods for different cases are disconnected, lacking a common theoretical or computational framework. The computational requirements of many solution methods are so large that they are of little use in practical applications. Fredholm’s method of determinants and Volterra’s method of iterated kernels are two examples. Some solution methods rely on numerical and iterative techniques that contribute little towards a theoretical understanding of the problem. Three examples are Galerkin, Collocation, and

Nystrom methods. Numerical stability and accuracy of solutions are two desirable characteristics lacking in many methods. Methods based on ortho-normal series expansions may fall into this category. An international expert and a program director of a US federal research funding agency reviewed a brief preliminary proposal on RTs. During the review process, he offered his expert opinion on the unsatisfactory state of the current state of the art in solving integral equations as follows:

"Numerical analysts hate solving integral equations because the resulting matrix approximations are usually full. This means that it takes $O(n^2)$ operations to evaluate a matrix vector multiply and this is too much for large n . To get around this, applied mathematicians have looked for transformations which increase the sparsity of the matrix. Unfortunately, these type of transformations only work for matrices with special structure. The classical example is the Fourier transformation which reduces a circulant matrix to a diagonal matrix. These ideas go back at least to Cauchy. Recent examples are the Fast Multipole algorithm and some Wavelet transforms that I haven't paid much attention to. These methods are wonderful if your application has the right structure and are useless if they don't. [My agency] seems to mostly have problems in the category for which these methods are useless."

Clearly, the current state of the art is unsatisfactory. A quick search of literature revealed that Multipole algorithms and Wavelet transforms were not directly relevant to RTs. After a careful review of the brief preproposal, RTs were thought to be promising and a full proposal was invited. The full proposal received a very good technical review:

"In summary the proposed research appears to be valid and has applications. It is guaranteed to produce doctoral dissertations."

4. Advantages of Rao Transforms

Rao Transforms [1] provide a localized (permitting very fine-grain parallel implementation) analytic solution suitable for both theoretical analysis and numerical implementation. They provide a unified theory and a unified computational framework for a large class of both linear and non-linear integral equations. Numerical methods based on RTs are likely to offer significant computational savings. RTs can be naturally extended from the case of one dimensional problems to multi-dimensional cases. They can also be extended to linear combinations of standard form integral equations, and systems of simultaneous integral equations. Since RTs are novel, their complete list of advantages and disadvantages are yet to be determined through further research.

5. Rao Transform Theory

Early results on the basic theory of RTs and their application to solve a large class of linear/non-linear integral/integro-differential equations including many classical equations (e.g. Fredholm, Volterra, Urysohn, and Hammerstein) are presented in the following book [1]:

M. SubbaRao, *Rao Transforms: Theory and Applications*, U.S. Copyright Registration No. TX 6-195-821, June 1, 2005, self-published book, 120 pages, can be purchased at <http://www.integralresearch.net>.

Second Edition: M. SubbaRao, *Rao Transforms: A New Approach to Integral and Differential Equations*, May, 2007, self-published book, 126 pages, can be purchased at <http://www.integralresearch.net>.

In the above book, several specific examples are solved in complete detail. Here we present only a brief summary.

5.1 Rao Transform (RT) and General Rao Transform (GRT)

The basic ideas underlying Rao Transform theory is introduced here with an example of a linear integral equation involving one-dimensional real valued variables and functions. These ideas can be extended to non-linear and multi-dimensional cases as in the book listed above [1]. First we state some of the main RTs:

$$g(x) = \int_r^s k(x, \alpha) f(\alpha) d\alpha \quad \text{Integral Transform/Equation (IT/IE)} \quad (1)$$

$$h(x, \alpha) = k(x + \alpha, x) \quad \text{Rao Localization Transform/Equation (RLT/RLE)} \quad (2)$$

$$g(x) = \int_{x-s}^{x-r} h(x - \alpha, \alpha) f(x - \alpha) d\alpha \quad \text{Rao Transform (RT) / Rao Integral Equation (RIE)} \quad (3)$$

Here x and α are real variables, $f(x)$ is an unknown real valued function that we need to solve for, $g(x)$, $k(x, \alpha)$, and $h(x, \alpha)$ are known (or given) real valued functions. r and s may be real constants or one of them can be the real variable x . All functions here are assumed to be continuous, integrable, and differentiable. $k(x, \alpha)$ and $h(x, \alpha)$ are referred to as *global* and *local kernel functions* respectively. Using this notation and taking some liberty to use ambiguous notation (regarding g) and language in the interest of brevity, a crucial theorem is presented below.

Rao's First Theorem (RFT): *If $h(x, \alpha)$ is defined as in Eq. (2) above (i.e. RLT is true), then IT in Eq. (1) and RT in Eq. (3) above are exactly equivalent with no loss of generality, i.e. $g(x)$ and $f(x)$ in the two equations are the same.*

Proof:

$$\begin{aligned} \text{R.H.S. of RT (Eq. 3)} &= \int_{x-s}^{x-r} h(x - \alpha, \alpha) f(x - \alpha) d\alpha \\ &= \int_{x-s}^{x-r} k((x - \alpha) + \alpha, (x - \alpha)) f(x - \alpha) d\alpha \quad \text{from Eq. (2)} \\ &= \int_{x-s}^{x-r} k(x, x - \alpha) f(x - \alpha) d\alpha \end{aligned}$$

Change the variable of integration to β where

$$\beta = x - \alpha, \quad \text{and} \quad d\beta = -d\alpha.$$

The new limits of integration are

$$\alpha = x - s \Rightarrow \beta = s, \quad \alpha = x - r \Rightarrow \beta = r$$

Therefore,

$$\begin{aligned}
\text{R.H.S. of RT (Eq. 3)} &= \int_r^s k(x, \beta) f(\beta) d\beta \\
&= \int_r^s k(x, \alpha) f(\alpha) d\alpha \\
&= \text{R.H.S. of IT (Eq. 1)}.
\end{aligned}$$

The above theorem (RFT) is simple but has apparently eluded all the researchers so far. The author of this report derived it after he invented a new method for shift-variant image deblurring based on a Rao Transform and then tried to connect/relate his method to existing methods for solving integral equations.

Rao Transform essentially restates, without loss of any generality, the same original problem of solving an integral equation in a localized form. In the new form, a localized solution can be obtained as follows:

$$\begin{aligned}
g(x) &= \int_{x-s}^{x-r} h(x-\alpha, \alpha) f(x-\alpha) d\alpha \\
&\approx \int_{x-s}^{x-r} h(x-\alpha, \alpha) \left(\sum_{n=0}^N a_n \alpha^n f^{(n)}(x) \right) d\alpha \quad (\text{Taylor-series expansion up to } N \text{ terms}) \\
&\approx \sum_{n=0}^N a_n f^{(n)}(x) \left(\int_{x-s}^{x-r} \alpha^n h(x-\alpha, \alpha) d\alpha \right) \\
&\approx \sum_{n=0}^N a_n f^{(n)}(x) h_n(x)
\end{aligned}$$

where

$$\begin{aligned}
h_n(x) &\equiv \int_{x-s}^{x-r} \alpha^n h(x-\alpha, \alpha) d\alpha \\
&= \int_{x-s}^{x-r} \alpha^n k(x, x-\alpha) d\alpha \equiv k_n(x)
\end{aligned}$$

Therefore, we may also write

$$g(x) \approx \sum_{n=0}^N a_n f^{(n)}(x) k_n(x).$$

Let us define

$$h_n^{(m)}(x) \equiv \frac{d^m h_n(x)}{dx^m} = k_n^{(m)}(x) \equiv \frac{d^m k_n(x)}{dx^m}.$$

We can derive an expression for the m -th derivative of $g(x)$ assuming that the m -th derivative of $f(x)$ for $m > N$ is zero, for $m=0, 1, 2, \dots, N$, as:

$$f^{(m)}(x) = 0 \quad \text{for } m > N.$$

$$\begin{aligned} g^{(m)}(x) &\equiv \frac{d^m g(x)}{dx^m} \\ &= \sum_{p=0}^m C_p^m \sum_{n=0}^{N-p} a_n f^{(n+p)}(x) h_n^{(m-p)}(x) \\ &= \sum_{n=0}^N h_{m,n}(x) f^{(n)}(x) \quad \text{for } m = 0, 1, 2, \dots, N \quad (h_{m,n}(x) \equiv h_n^{(m)}(x)) \\ &= \sum_{n=0}^N k_{m,n}(x) f^{(n)}(x) \quad (k_{m,n}(x) \equiv k_n^{(m)}(x)) \end{aligned}$$

The above system of equations can be written in vector-matrix form as:

$$\begin{bmatrix} g^{(0)} \\ g^{(1)} \\ \vdots \\ g^{(N)} \end{bmatrix} = \begin{bmatrix} h_{00} & h_{01} & \cdots & h_{0N} \\ h_{10} & h_{11} & \cdots & h_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N0} & h_{N1} & \cdots & h_{NN} \end{bmatrix} \begin{bmatrix} f^{(0)} \\ f^{(1)} \\ \vdots \\ f^{(N)} \end{bmatrix}$$

or

$$\begin{bmatrix} g^{(0)} \\ g^{(1)} \\ \vdots \\ g^{(N)} \end{bmatrix} = \begin{bmatrix} k_{00} & k_{01} & \cdots & k_{0N} \\ k_{10} & k_{11} & \cdots & k_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N0} & k_{N1} & \cdots & k_{NN} \end{bmatrix} \begin{bmatrix} f^{(0)} \\ f^{(1)} \\ \vdots \\ f^{(N)} \end{bmatrix}$$

The above equation can be written in a more compact form using vector and matrix variables as:

$$\mathbf{g}_x = \mathbf{H}_x \mathbf{f}_x \quad \text{or} \quad \mathbf{g}_x = \mathbf{K}_x \mathbf{f}_x.$$

The subscript x in the above equation makes explicit the dependence of the variables on x . Note that, unlike some conventional formulations of the problem, the elements of the column vectors in the above matrix equations are the derivatives of $g(x)$ and $f(x)$, not the sampled values of $g(x)$ and $f(x)$ at discrete points. In practical applications, the derivatives of a function at a point are usually less correlated with each other than the sampled values of the function near the point. Also, derivatives at single point x are *fully local parameters* of a function as opposed to discretely sampled values of the function in an interval. Therefore this representation of the problem is an improvement and helps to keep the size of the matrices and computations to be small and more accurate. This advantage becomes more pronounced in the case of higher dimensional problems such as two-dimensional image deblurring and three-dimensional video data processing. Classical techniques such as Singular Value Decomposition (SVD) [3,14] for shift-variant image deblurring are global (non-local) approaches and they do not exploit spatial correlation between sampled image data. Instead the sampled data are treated as independent variables, and they rearrange the two-dimensional image data as a huge one-dimensional column vector by vertically stacking image matrix columns. Physically speaking, this is imposition of an unnatural structure on the data to fit the SVD formulation of the problem. In comparison, the RT

formulation of the problem naturally exploits the spatial correlation between image data and such multi-dimensional data by treating the local derivatives of the data as independent variables instead of the data itself. Also, the RT approach may provide a more graceful way to deal with image deblurring at the boundaries in comparison with the SVD approach which uses image reflection or image wrap-around.

We can obtain a symbolic solution for $f(x) = f^{(0)}(x)$ through recursive substitution or successive elimination as in Gaussian elimination. This symbolic solution is useful in practical applications as the value of N is very small, typically 2 to 6. Assuming that the matrix \mathbf{H}_x (or \mathbf{K}_x) is non-singular, the solution can be written in matrix form as:

$$\mathbf{f}_x = \mathbf{H}_x^{-1} \mathbf{g}_x \quad \text{or} \quad \mathbf{f}_x = \mathbf{K}_x^{-1} \mathbf{g}_x$$

In explicit algebraic form, we may write the solution as

$$f(x) \approx \sum_{n=0}^N a_n g^{(n)}(x) h'_n(x) = \sum_{n=0}^N a_n g^{(n)}(x) k'_n(x)$$

We have above a closed-form solution for $f(x)$. Here $f(x)$ is expressed as the weighted sum of the derivatives of $g(x)$ at x where the weight coefficients change with x . Therefore, in the numerical computation of the solution, we will have to recompute these coefficients at each point x where we need the solution $f(x)$. Computing the coefficients involves inverting or equivalent computations on the $N \times N$ \mathbf{H}_x matrix at each point x . As N is usually a small integer (around 4) in practical applications, the computational requirement is small. Numerical accuracy of the solution for $f(x)$ at a point x is maintained because it is computed using known data $g(x)$ and the kernel $h(x, \alpha)$ at and around the same point x . The data of $g(x)$ and the kernel at and around x is needed to estimate their derivatives at x . This step in computing the solution is often misunderstood by readers and therefore extra attention is helpful. Otherwise, the readers are likely to conclude incorrectly that the method of RT is far more numerically unstable than it actually is.

We can define a resolvent kernel \mathbf{h}' (or \mathbf{k}') which is determined by \mathbf{H}_x^{-1} (the inverse of \mathbf{H}_x , or \mathbf{K}_x^{-1} the inverse of \mathbf{K}_x) so that the following equation is satisfied in the limit as $N \rightarrow \infty$:

$$\begin{aligned} f(x) &= \lim_{N \rightarrow \infty} \sum_{n=0}^N a_n g^{(n)}(x) h'_n(x) \\ &= \int_{x-s}^{x-r} h'(x-\alpha, \alpha) g(x-\alpha) d\alpha \\ &= \int_{x-s}^{x-r} k'(x, x-\alpha) g(x-\alpha) d\alpha \end{aligned}$$

This method of solving linear integral equations based on the above theory is shown in Fig. 1. The key step is the use of RLT to convert a given integral equation with a *global kernel* to an exactly equivalent *Rao Integral Equation (RIE)* with a *local kernel*. Replacing the global kernel with a local kernel is referred to as *refunctionalization* of the equation or *change of function*. This idea or its variants may find applications in solving other functional equations. In terms of computational solution, this step of replacing a *global kernel* with a *local kernel* may be thought of as dividing one complex/global problem into infinitely many local/simple problems which can

be solved efficiently and in parallel at all points. The solutions obtained at different points or locations will all be compatible with each other, and therefore these local solutions can all be synthesized seamlessly to obtain a global solution to the original problem. In particular, the local problems are naturally suited for solving on a fine-grain parallel or distributed computer. In addition to “dividing” the original problem with respect to the independent variable x , the method of solution in Fig. 1 involves converting the integral equation to a differential equation and taking its derivatives. In this sense, RTs may be *the ultimate divide-and-conquer strategy* for solving integral equations.

Second Kind Integral Equations

The solution method described above is applicable to both Fredholm and Volterra Integral Equations of the First Kind. Extension of this method to the Second Kind Integral Equations is easy. The following equation can represent a Second Kind Integral Equation of either Fredholm (in which case r and s are both constants) or Volterra (r is a constant, but $s=x$):

$$g(x) = f(x) + \int_r^s k(x, \alpha) f(\alpha) d\alpha .$$

Using the same kernel localization technique as for the First Kind equations above, i.e.

$$h(x, \alpha) = k(x + \alpha, x)$$

we obtain the following Rao Integral Equation which is exactly equivalent to the Fredholm/Volterra Integral Equations of the Second Kind:

$$g(x) = f(x) + \int_{x-s}^{x-r} h(x - \alpha, \alpha) f(x - \alpha) d\alpha$$

We can derive the following solution by following steps similar to those earlier:

$$\mathbf{g}_x = (\mathbf{H}_x + \mathbf{I})\mathbf{f}_x ,$$

$$\mathbf{f}_x = (\mathbf{H}_x + \mathbf{I})^{-1} \mathbf{g}_x .$$

where \mathbf{I} denotes the $N \times N$ identity matrix. We see that, depending on the magnitude of the diagonal elements of \mathbf{H}_x , the numerical stability of the solution method for the second kind equations may be either significantly improved or only marginally improved in comparison with the First Kind equations. Numerical stability could be improved by using a smaller value of N , or by pre-smoothing $g(x)$, or using a regularization technique, or a combination of these, all at the cost of some accuracy. Further investigation of these issues is needed.

Non-linear Integral Equations

RTs are useful in solving linear integral equations such as Fredholm and Volterra Integral Equations of the First and Second kind. GRT defined in Eq. (6) below is useful in solving many types of non-linear integral equations. A theorem similar to RFT above can be proved in this case (see [1]). The method of solution is similar to that in Figure 1. In this case, the final step involves solving a system of non-linear algebraic equations instead of linear equations in Figure 1. Details on this method and the methods for solving other types of non-linear integral equations can be found in the book [1] mentioned above.

$$g(x) = \int_r^s k(x, \alpha, f(\alpha)) d\alpha \quad \text{General Integral Transform/Equation (GIT/GIE)} \quad (4)$$

$$h(x, \alpha, f(x)) = k(x + \alpha, x, f(x)) \quad \text{General Rao Localization Transform/Eq. (GRLT/GRLE)} \quad (5)$$

$$g(x) = \int_{x-s}^{x-r} h(x - \alpha, \alpha, f(x - \alpha)) d\alpha \quad \text{General Rao Transform/Equation (GRT/GRIE)} \quad (6)$$

5.2 Types of Integral Equations

A short list of the different types of Integral Equations that can be solved using Rao Transforms is given below. Many other types which can be solved using RTs but are not listed here can be found in [1]. In the following equations, x and α are real variables, $f(x)$ is an unknown real valued function that we need to solve for, $g(x)$ and $k(x, \alpha)$ (global kernel) are known (or given) real valued functions. All functions here are assumed to be continuous, integrable, and differentiable. c_n , a , and b , all denote real constants (possibly infinity), and $f^{(n)}$ denotes the n -th derivative of $f(x)$ at x with respect to x .

$$1. \text{ Fredholm Integral Equation of the First Kind} \quad g(x) = \int_a^b k(x, \alpha) f(\alpha) d\alpha$$

$$2. \text{ Fredholm Integral Equation of the Second Kind} \quad g(x) = f(x) + \int_a^b k(x, \alpha) f(\alpha) d\alpha .$$

$$3. \text{ Volterra Integral Equation of the First Kind} \quad g(x) = \int_a^x k(x, \alpha) f(\alpha) d\alpha$$

$$4. \text{ Volterra Integral Equation of the Second Kind} \quad g(x) = f(x) + \int_a^x k(x, \alpha) f(\alpha) d\alpha$$

$$5. \text{ Urysohn Integral Equation of the First Kind} \quad g(x) = \int_a^b k(x, \alpha, f(\alpha)) d\alpha$$

$$6. \text{ Urysohn Integral Equation of the Second Kind} \quad g(x) = f(x) + \int_a^b k(x, \alpha, f(\alpha)) d\alpha$$

$$7. \text{ Urysohn-Volterra Integral Equation of the First Kind} \quad g(x) = \int_a^x k(x, \alpha, f(\alpha)) d\alpha$$

$$8. \text{ Urysohn-Volterra Integral Equation of the Second Kind} \quad g(x) = f(x) + \int_a^x k(x, \alpha, f(\alpha)) d\alpha$$

9. Linear integro-differential equations (Fredholm Type)

$$g(x) = \sum_{n=0}^N c_n f^{(n)}(x) + \int_a^b k(x, \alpha) f(\alpha) d\alpha$$

10. Linear integro-differential equations (Volterra Type)

$$g(x) = \sum_{n=0}^N c_n f^{(n)}(x) + \int_a^x k(x, \alpha) f(\alpha) d\alpha$$

When any integral equation expressed in a conventional form such as above is expressed in an equivalent localized form using RFT and RLT, then the resulting equation is referred to as the Rao Type Integral Equation (RTIE).

Rao Transforms: A New Approach to Integral and Differential Equations

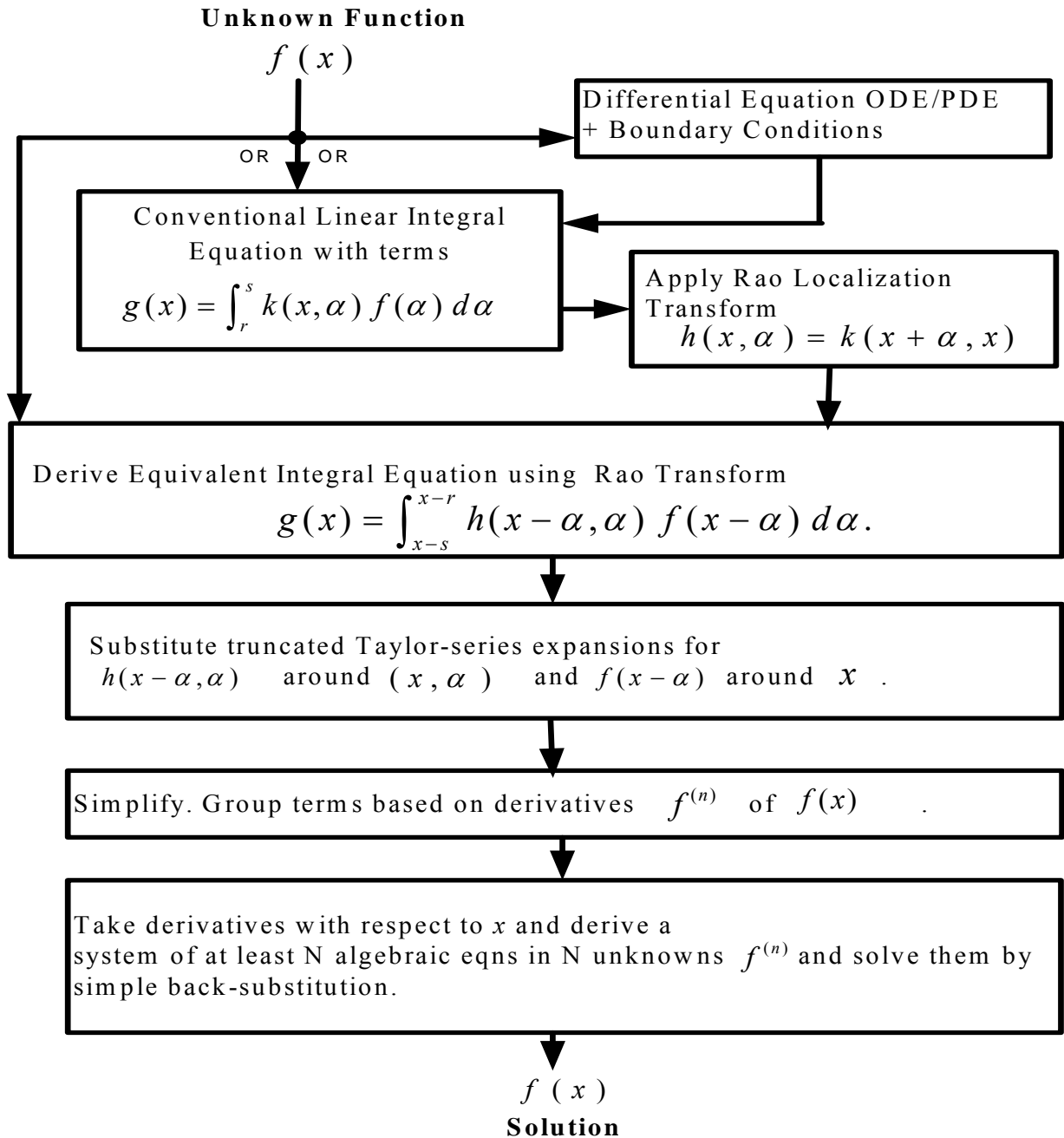


Figure 1. Solving Linear Integral and Differential Equations

[Prof. Murali SubbaRao, <http://www.ece.sunysb.edu/~murali/>, murali@ece.sunysb.edu]
[Dr. Muralidhara SubbaRao, rao@integralresearch.net, <http://www.integralresearch.net>]

6. Example: Restoration of Shift-Variant Blurred Images

6.1 Summary

A simple and easily verifiable example of the theory and application of RTs is presented here. A practical problem in shift-variant image restoration is considered which involves solving a Fredholm Integral Equation of the First Kind. A solution based on RTs is shown to provide a closed-form expression which is theoretically novel and interesting. This theoretical result is shown to extend to all linear integral equations in [1]. Similar ideas are used to derive the solution of non-linear integral equations as the solution of a set of non-linear algebraic equations in [1]. Computationally, the solution provided by the RTs for the example considered here is estimated to be about 1000 times faster for an image row with 1000 pixels compared to a conventional method of image restoration based on Singular Value Decomposition (SVD [3,14]) by inverting a large (1000x1000) matrix. This computational advantage is shown to extend to the special case of shift-invariant image blurring where a convolution integral equation is solved. In this special case, the computational advantage is comparable to the famous Fast Fourier Transform (FFT) algorithm. If this computational advantage extends to the general integral equations which can be solved by RTs, a possibility in at least some cases, it would be a remarkable result. Further research is needed to confirm or contradict this possibility. As differential equations can be solved by reformulating them as integral equations [5,11], RTs are relevant to solving them as well.

6.2 Shift-variant Image Restoration

We describe here a concrete one-dimensional example of the application of Rao Transforms. One practical problem for which this example is relevant is the restoration or deblurring of shift-variant motion blurred images. When a photograph is captured by a moving camera with a finite exposure period, say 0.1 second, objects nearer to the camera will have larger motion blur than farther objects. This situation can arise when the camera is on a moving platform such as a car or a robot. A similar situation, i.e. a shift-variant defocus blur, can arise in a laser barcode scanner for one-dimensional barcodes. When the plane of the barcode is slanted instead of being perpendicular to the direction of scan/view, the barcode image will be blurred by a shift-variant point spread function.

Let the original focused image be $f(x)$, and the corresponding blurred image be $g(x)$. The conventional image blurring model in this case uses a shift-variant point spread function or kernel $k(x, \alpha)$. The blurred image $g(x)$ and the kernel $k(x, \alpha)$ are assumed to be given and the problem is to solve for the focused image $f(x)$. The conventional blurring model is an integral equation of the form:

$$g(x) = \int_{-\infty}^{\infty} k(x, \alpha) f(\alpha) d\alpha \quad (6.1)$$

It is a one-dimensional Fredholm Integral Equation of the First Kind. The above model of blurring has two problems. First, it is difficult to find a closed-form inversion formula that is explicit and numerically stable. For example, the well-known Singular Value Decomposition (SVD) [3,14] technique is computationally expensive and unstable. Second, the above model does not capture the physical blurring process in a natural way. It seems to impose mathematical simplicity at the cost of direct natural modeling of the physical blurring process.

We model the blurred image $g(x)$ measured at x as the sum or integral over all possible point sources of the contribution due to each point source located at $x-\alpha$. The contribution is given by the product of the strength of the signal point source $f(x-\alpha)$ and the value of a new *localized* shift-variant point spread function $h(x-\alpha, \alpha)$. The new model of blurring is:

$$g(x) = \int_{-\infty}^{\infty} h(x-\alpha, \alpha) f(x-\alpha) d\alpha \quad (\text{RT}) \quad (6.2)$$

where

$$h(x, \alpha) = k(x+\alpha, x) \quad \text{and} \quad (\text{RLT}) \quad (6.3)$$

$$k(x, \alpha) = h(\alpha, x-\alpha) \quad (\text{IRLT}) \quad (6.4)$$

The new model above is an integral equation that is exactly equivalent to the original integral equation (6.1) (see RFT for proof). Equation (6.2) is referred to as the *Rao Integral Equation* (RIE) and defines the *Rao Transform* (RT). Equation (6.3) defines the *Rao Localization Transform* (RLT).

Now the m -th order partial derivative of $h(x, \alpha)$ at (x, α) with respect to x is denoted by

$$h^{(m)} = h^{(m)}(x, \alpha) = \frac{\partial^m h(x, \alpha)}{\partial x^m}. \quad (6.5)$$

The n -th derivative of $f(x)$ at x with respect to x will be denoted by

$$f^{(n)} = f^{(n)}(x) = \frac{d^n f(x)}{dx^n}. \quad (6.6)$$

The n -th *moment* of the m -th derivative of h is defined by

$$h_n^{(m)} = h_n^{(m)}(x) = \int_{-\infty}^{\infty} \alpha^n \frac{\partial^m h(x, \alpha)}{\partial x^m} d\alpha \quad (6.7)$$

Note that the derivative is with respect to x and the moment is with respect to α . The original signal $f(x)$ will be taken to be smooth or analytic so that it can be expanded in a Taylor series. The Taylor series expansion of $f(x-\alpha)$ around the point x up to order N is

$$f(x-\alpha) = \sum_{n=0}^N a_n \alpha^n f^{(n)}(x) \quad (6.8)$$

where

$$a_n = \frac{(-1)^n}{n!}. \quad (6.9)$$

The above equation is exact and free of any approximation error when f itself is a polynomial of degree less than or equal to N . In this case, the derivatives of f of order greater than N are all zero. When f has non-zero derivatives of order greater than N , then the above equation will have an approximation error corresponding to the residual term of the Taylor series expansion. This approximation error usually converges rapidly to zero as N increases. In the limit as N tends to infinity, the above series expansion becomes exact and complete.

Similarly, the Taylor series expansion of $h(x-\alpha, \alpha)$ around the point (x, α) up to order M is

$$h(x-\alpha, \alpha) = \sum_{m=0}^M a_m \alpha^m h^{(m)}(x, \alpha) \quad (6.10)$$

where a_m are as in Eq. (6.9). Using a truncated Taylor series expansion as above gives very accurate approximations in many practical applications. One such example is image deblurring.

In this case the kernel function usually changes smoothly and slowly with respect to x . If an analytic expression is given for $h(x, \alpha)$, then it may be possible to avoid the approximation introduced by truncation of the Taylor series above in Eq. (6.10). In this case, we would substitute Eq. (6.8) and the following analytic expression in Eq. (6.2) directly for obtaining a series expression for RT:

$$h_n(x) = \int_{-\infty}^{\infty} \alpha^n h(x - \alpha, \alpha) d\alpha .$$

Let $N=2$ and $M=1$, and let the original global kernel $k(x, \alpha)$ be a Gaussian, that is

$$k(x, \alpha) = \frac{1}{\sqrt{2\pi}\sigma(\alpha)} \exp\left(-\frac{(x-\alpha)^2}{2\sigma^2(\alpha)}\right) \quad (6.11)$$

where $\exp(x) = e^x$. The kernel above is a “global” kernel. It is localized using the Rao Localization Transform (RLT) to define a new “local” kernel h as in Eq. (6.3), i.e.

$h(x, \alpha) = k(x + \alpha, x)$, which becomes

$$h(x, \alpha) = \frac{1}{\sqrt{2\pi}\sigma(x)} \exp\left(-\frac{\alpha^2}{2\sigma^2(x)}\right) \quad (6.12)$$

For notational convenience, we denote

$$\rho(x) = \frac{1}{\sigma(x)}. \quad (6.13)$$

Therefore

$$h(x, \alpha) = \frac{\rho(x)}{\sqrt{2\pi}} \exp\left(-\frac{\alpha^2 \rho^2(x)}{2}\right) \quad (6.14)$$

The Taylor series expansion of $h(x - \alpha, \alpha)$ around the point (x, α) up to order $M = 1$ is

$$h(x - \alpha, \alpha) = h^{(0)}(x, \alpha) + h^{(1)}(x, \alpha) (-\alpha) . \quad (6.15)$$

It can be shown that, when h is as defined in Eq. (6.14),

$$h^{(1)}(x, \alpha) = \frac{\partial h(x, \alpha)}{\partial x} = h(x, \alpha) \frac{\rho_x(x)}{\rho(x)} (1 - \alpha^2 \rho^2(x)) \quad (6.16)$$

where $\rho_x(x)$ is the derivative of $\rho(x)$ with respect to x .

Note that the above function is an *even* function of α as it involves only α^2 . This function is symmetric with respect to α , i.e. $h^{(l)}(x, \alpha) = h^{(l)}(x, -\alpha)$. Therefore, all odd moments of $h^{(l)}$ with respect to α will be zero, and with $M=1$ and $N=2$, the RT becomes

$$g(x) = \int_{-\infty}^{\infty} (f^{(0)} - \alpha f^{(1)} + \frac{\alpha^2}{2} f^{(2)}) (h^{(0)} - \alpha h^{(1)}) d\alpha . \quad (6.17)$$

Simplifying, we get

$$g^{(0)} = f^{(0)}(h_0^{(0)} - h_1^{(1)}) + f^{(1)}(h_2^{(1)} - h_1^{(0)}) + f^{(2)} \frac{1}{2}(h_2^{(0)} - h_3^{(1)}) \quad (6.18)$$

Since all odd moments of $h^{(0)}$ and $h^{(1)}$ are zero, we set $h_1^{(0)} = h_1^{(1)} = h_3^{(1)} = 0$. This simplifies the problem. Further, we have for this case, $h_0^{(0)} = 1$ and both first $h_0^{(1)}$ and second $h_0^{(2)}$ (and

all higher) derivatives of $h_0^{(0)}$ are always zero. Also the first and higher derivatives with respect to x of $h_1^{(0)}$, $h_1^{(1)}$, $h_2^{(1)}$, and $h_3^{(1)}$ are all zero. Only the first derivative of $h_2^{(0)}$ may not be zero. It will be denoted by $h_2^{(1)}$. Significant simplification of a mathematical problem such as here is likely in many practical applications. With the above simplifications, we get

$$g^{(0)} = f^{(0)} + f^{(1)}h_2^{(1)} + f^{(2)}\frac{1}{2}h_2^{(0)} \quad (6.19)$$

Taking derivatives of the above equation once and twice, we get

$$g^{(1)} = f^{(1)} + f^{(2)}h_2^{(1)} + f^{(2)}\frac{1}{2}h_2^{(1)} \quad (6.20)$$

and

$$g^{(2)} = f^{(2)}. \quad (6.21)$$

We treat the above equations (6.19 to 6.21) as three linear algebraic equations in the three unknowns $f^{(0)}$, $f^{(1)}$, and $f^{(2)}$. They can be easily solved through successive back substitution. In this particular example, which corresponds to a typical practical application, the process of solving becomes trivial. In a general case where the kernel is not symmetric, there will be a system of N linear equations in the N unknowns $f^{(n)}$. These equations can be solved symbolically/numerically by recursive substitution (e.g. Gaussian elimination) and back-substitution. In practical applications, N is very small, typically in the range 2 to 6, and so the computational requirement is limited. Here we solve for $f^{(0)} = f(x)$ to obtain

$$f(x) = f^{(0)} = g^{(0)} - g^{(1)}h_2^{(1)} - g^{(2)}\frac{1}{2}(h_2^{(0)} - 3h_2^{(1)} \cdot h_2^{(1)}) \quad (6.22)$$

The solution above can be further simplified by noting that

$$h_2^{(0)} = \sigma^2(x) \text{ and } h_2^{(1)} = \frac{\rho_x(x)}{\rho(x)} (\sigma^2(x) - \rho^2(x) 3\sigma^4(x)) = -2 \frac{\rho_x(x)}{\rho^3(x)} \quad (6.23)$$

In matrix notation, the forward and inverse RT for this case (Equations 6.19 to 6.22) can be written as

$$\begin{bmatrix} g^{(0)} \\ g^{(1)} \\ g^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & h_2^{(1)} & (1/2)h_2^{(0)} \\ 0 & 1 & (3/2)h_2^{(1)} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f^{(0)} \\ f^{(1)} \\ f^{(2)} \end{bmatrix}$$

$$\begin{bmatrix} f^{(0)} \\ f^{(1)} \\ f^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & -h_2^{(1)} & -(1/2)(h_2^{(0)} - 3h_2^{(1)} \cdot h_2^{(1)}) \\ 0 & 1 & -(3/2)h_2^{(1)} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g^{(0)} \\ g^{(1)} \\ g^{(2)} \end{bmatrix}$$

Thus we have obtained (in Eq. 6.22) the *Inverse Rao Transform* (IRT) for a case that is useful in practical applications. It is a closed-form solution up to second order terms. Solution up to any order N can be obtained similarly. As mentioned earlier, both matrix equations above involve upper triangular square matrices. Therefore simple back-substitution can be used to obtain both the forward and the inverse Rao Transforms. Here a solution for $f(x)$ is given in terms of the

derivatives of $g(x)$ at x , and moments with respect to α of derivatives with respect to x of the localized kernel h . In all our search of relevant research literature, we have never seen such a closed-form solution for the Fredholm Integral Equation of the First Kind. This is a “local” solution and converges rapidly for this particular example. In particular, unlike the conventional SVD method (3,14), the local solution above given by IRT can be implemented more easily on a parallel or distributed computer. The solution $f(x)$ is computed for each different value of x separately and independently. A few other more complicated examples of solutions of integral equations are presented in the book [1], but the simple example here illustrates the potential power of RTs.

Note that, in the process of solving for $f(x)$ at x , we also obtain solutions for $f^{(1)}$, and $f^{(2)}$. Therefore, a Taylor series expansion as in Equation (6.8) can be used to extend the solution at x approximately to a small interval surrounding x . This extended solution will be exact for all α if f itself is a polynomial of order $N \leq 2$ globally. Otherwise, the accuracy of the solution for $f(x-\alpha)$ will decrease as α increases.

6.3 Computational Complexity

Let a one-dimensional (1D) laser barcode scanner be characterized by a 1D Gaussian blurring kernel given by Eq. (6.11) or (6.12). Suppose that it is used to scan a two inch long barcode on a slanted plane at 1000 dots per inch resolution. Let the slant of the plane be such that the blur parameter σ increases smoothly from 1 pixel at the starting pixel to 10 pixels at the last pixel. These 1000 samples represent one row of the blurred image signal $g(x)$ of size $P=1000$ pixels. Let the support size of the kernel h be s where s is defined as

$$|h(x - \alpha, \alpha)| \approx 0 \text{ for } |\alpha| > s \text{ for all } x. \quad (6.24)$$

A localized kernel that satisfies the above condition (6.24) is said to have a compact support. In our example, this s is roughly proportional (by a factor of 2) to the blur parameter σ and has a value in the range of 2 to 20 pixels. The value of s is typically 1 to 2 orders of magnitude smaller than P . Although closed-form expressions as in Eq. (6.23) may exist for computing the moments of the derivatives of the kernel when the kernel is a standard mathematical function such as a Gaussian, for a general kernel function $O(s)$ operations would be needed to compute these moments at each pixel x . A conventional method of restoring the above image is as follows. A discrete version of Equation (6.1) is written with uniformly sampled $g(x)$ and $f(x)$ represented by vectors of size 1000 and the kernel k by a 1000x1000 sampled matrix. A solution for $f(x)$ is obtained by inverting the 1000x1000 matrix corresponding to the global kernel k . Inverting a $P \times P$ matrix has a computational complexity of $O(P^3)$ which is very high for $P=1000$. In the presence of noise, a Singular Value Decomposition method may be used in practice. It is possible to solve this problem by considering small intervals at time, for example by dividing the 1000 pixel interval into 10 intervals of 100 pixel each, and solving the equation separately in each interval. In this case, the computational complexity would be $10 \cdot O(100^3)$. However, this approach will reduce the accuracy of the solution, especially at the end points of the intervals since blurred pixels just outside the interval will affect measured $g(x)$ just inside the interval. In addition this approach permits only a very coarse grained parallel implementation on a computer.

In the new method based on RT, the computational complexity is $O(Pst)$ operations where t is the highest order derivative considered in the local Taylor series expansion of f or g . In our

blurring example, a typical value for t would be 4. Estimating the derivatives of g at each pixel x requires $O(t)$ operations using $O(t)$ data points. This t is typically 2 to 3 orders of magnitude smaller than P and a reasonable estimate may be $O(t) \leq O(\log P)$. t is also the number of terms on the right hand side of inverse Rao Transform (in Eq. 6.22). Therefore the computational speed up is $O(P^2/(st))$ which is typically over $O(P)$ or 1000. The method can be implemented in parallel with one processor for each pixel operating on $O(t)$ data points of $g(x)$ and $O(s)$ data points of the kernel $h(x - \alpha, \alpha)$ for $|\alpha| \leq s$ for all x . In the limiting case when truncation of the Taylor series expansions are not permitted in order to preserve full accuracy, $t=P$. Note that a P -th order polynomial will fit the P data points of g which can be expanded in a Taylor series with P terms with no approximation, and therefore there will be P terms in the solution of f . In this case, solution of f and all its P derivatives at a single point x can be used to compute f at all possible points x using a Taylor series expansion as in Equation (6.8). As the interval of support s of the kernel h increases to span the entire space of P points where g is defined, s approaches P . In this worst case scenario, the computational complexity of the RT method equals that of the conventional method or the SVD method [3] which is $O(P^3)$. If the blurring kernel is a standard mathematical function, then $O(s)$ computational operations needed for computing the moments of the kernel h can be avoided because an analytic expression can be used to compute them. This offers significant additional computational advantage.

It is interesting to note that when $M=0$ in Equation (6.10), the *shift-variant* blurring becomes *shift-invariant* which corresponds to a *convolution* operation. See Reference [2,7] for a practical application of this case. In this case, $O(s) = O(1)$ because there is no need to re-compute the moments of h at each point x as they remain the same at all points. The moments of h need to be computed only once. Therefore the computational complexity of the proposed method becomes $O(Pt)$. This compares very well with the Fast Fourier Transform (FFT) method which has a computational complexity of $O(P \log P)$. The computational speed up attained by RT as compared to FFT is $O(\log P/t)$ which may be significant in some practical applications (see [2]). Note that while RT can be naturally extended to the shift-variant problems, FFT cannot be extended to them. In addition, RT permits a more fine-grain parallel implementation than the FFT.

The example above is for a one-dimensional image. For an actual 2D image and other higher dimensional applications, the computational advantages would be much higher. In conclusion, Rao Transforms are theoretically interesting, and practically useful. Further research is needed to explore and exploit their full potential.

7. Solving Ordinary and Partial Differential Equations (ODEs/PDEs)

A differential equation by itself is inherently under-constrained in the absence of initial values or boundary conditions. A differential equation along with the initial values or boundary conditions can be compactly and explicitly represented by an integral equation. For example, the method of Green's Function [8] to solve differential equations often results in an integral equation in which the integration kernel is determined by the Green's function. In this integral representation, it becomes possible to solve the problem. It is said (e.g. [5]) that one of the most important achievements and applications of integral equation methods are in solving partial differential

equations (PDEs) of second order, such as the Laplace ($\nabla^2 f = 0$), Poisson ($\nabla^2 f = -4\pi g$; g is a given function of position), and Helmholtz (or the steady-state wave) ($(\nabla^2 + k^2)f = 0$; k is a number) equations. Boundary value problems for elliptic type PDEs can be reduced to Fredholm integral equations, and parabolic and hyperbolic PDEs lead to Volterra integral equations. For these reasons, Rao Transforms as a tool for solving Integral Equations are directly relevant to solving ODEs and PDEs. This is an area that remains completely unexplored. Applying RTs to solve ODEs and PDEs may lead to fundamentally new approaches to problems in theoretical physics and mechanics. It may also lead to a big leap in computational techniques for solving boundary value problems in many applied sciences, from the design of submarine hulls and missiles [19] to inverse optics [18]. In order to see how the application of RTs may change a conventional technique for solving differential equations, we shall consider a simple example here.

7.1 Example: n-th order initial value problem

Consider an n -th order linear differential equation and initial values:

$$\sum_{k=0}^n A_k(x) \frac{d^{(n-k)}y}{dx^{(n-k)}} = F(x), \quad \text{with} \quad (7.1)$$

$$y^{(n-k)}(r) = q_{n-k} \text{ as initial conditions for } k = 0, 1, \dots, n-1, \quad (7.2)$$

where A_k, F are defined and continuous in the interval $r \leq x \leq s$ and q_k are given constants. Assume, without loss of generality that $A_0(x) = 1$. Define $dy^n/dx^n = f(x)$. This initial value problem can be converted to the following Volterra Integral Equation of the second kind [5] (pages 62,63):

$$g(x) = f(x) - \int_r^x K(x, \alpha) f(\alpha) d\alpha, \quad \text{where} \quad (7.3)$$

$$K(x, \alpha) = - \sum_{k=1}^n A_k(x) \frac{(x-\alpha)^{k-1}}{(k-1)!} \quad \text{and} \quad (7.4)$$

$$g(x) = F(x) - q_{n-1} A_1(x) - [(x-r)q_{n-1} + q_{n-2}] A_2(x) - \dots - \left\{ [(x-r)^{n-1}/(n-1)!]q_{n-1} + \dots + (x-r)q_1 + q_0 \right\} A_n(x) \quad (7.5)$$

After solving the integral equation above for $dy^n/dx^n = f(x)$, the solution for y can be obtained by integrating it n times as in [5]. The solution is given by

$$y(x) = \int_r^x \frac{(x-\alpha)^{n-1}}{(n-1)!} f(\alpha) d\alpha + \sum_{k=0}^{n-1} \frac{(x-r)^k}{k!} q_k. \quad (7.6)$$

Applying the Rao Transform technique, we obtain the following equivalent Rao Type integral equation for the above Volterra Integral Equation of the second kind (7.3):

$$g(x) = f(x) - \int_0^{x-r} H(x-\alpha, \alpha) f(x-\alpha) d\alpha, \quad \text{where} \quad (7.7)$$

$$H(x, \alpha) = - \sum_{k=1}^n A_k(x+\alpha) \frac{\alpha^{k-1}}{(k-1)!}. \quad (7.8)$$

Solution of the above integral equation is obtained by following steps similar to those in Figure 1. In short, expand $H(x-\alpha, \alpha)$ in a truncated Taylor series around the point (x, α) , expand $f(x-\alpha)$ in a truncated Taylor series around (x) , simplify the integral equation by grouping

terms based on the derivatives of f with respect to x , consider various order derivatives of g with respect to x to obtain a system of linear algebraic equations, solve them through a successive elimination and back substitution procedure. The theoretical and computational advantages of this new method needs to be investigated as part of future research.

8. Generalization of Rao Transform Theory

The theory of RTs can be generalized further for more general transformations of the integration kernel than just localizing it. Consider the conventional linear integral equation:

$$g(x) = \int_r^s k(x, \alpha) f(\alpha) d\alpha.$$

In the above equation, note that integration is with respect to α , and therefore α is a dummy variable. It can be replaced by another suitable dummy variable β that is a function of both α and x defined by:

$$\beta = \beta(x, \alpha).$$

where $\beta(x, \alpha)$ is a one to one and onto function with respect to α , i.e., for any x , for each α , $\beta(x, \alpha)$ is unique, distinct, and can be inverted to obtain α . With this change of variable, the integral equation becomes:

$$g(x) = \int_{\beta(x,r)}^{\beta(x,s)} k(x, \beta(x, \alpha)) f(\beta(x, \alpha)) d\beta.$$

A simple example of $\beta(x, \alpha)$ is:

$$\beta(x, \alpha) = ax + b\alpha + c \text{ where } a, b, \text{ and } c \text{ are known scalar constants.}$$

Note that setting $a=1$, $b=-1$, and $c=0$ results in the original definition of Rao Transforms where $\beta(x, \alpha) = x - \alpha$. In an extended theory of RTs, $\beta = \beta(x, \alpha)$ can be a general function, but satisfying one-to-one and onto (bijective) property with respect to α for any x . In the discussion below we will only consider the case where $\beta(x, \alpha) = ax + b\alpha + c$.

The case when $\beta(x, \alpha)$ is a general linear function of x and α includes translation and scaling of $f(x)$ with respect to x and α . In this case, the resulting integral equation can be solved easily by following steps similar to deriving the original inverse RT. In this case, the Taylor series expansion of $f(ax + b\alpha + c)$ can be done in two ways. The first way is in terms of the derivatives of $f(ax + c)$ and powers of $b\alpha$, and the second way is in terms of the derivatives of $f(ax)$ and powers of $b\alpha + c$. The choice depends on the problem at hand. In both cases, this Taylor series expansion can be denoted symbolically as

$$f(\beta(x, \alpha)) = f(ax + b\alpha + c) \approx \sum_{n=0}^N a_n(\alpha) f_{(x)}^{(n)}.$$

Therefore, given $\beta(x, \alpha) = ax + b\alpha + c$, we can derive

$$\begin{aligned}
g(x) &= \int_{ax+br+c}^{ax+bs+c} k(x, ax+b\alpha+c) f(ax+b\alpha+c) b d\alpha \\
&\approx \int_{ax+br+c}^{ax+bs+c} k(x, ax+b\alpha+c) \left(\sum_{n=0}^N \frac{(b\alpha)^n}{n!} \frac{d^n f(ax+c)}{dx^n} \right) b d\alpha \quad (\text{Taylor series expansion}) \\
&\approx \sum_{n=0}^N \frac{d^n f(ax+c)}{dx^n} \int_{ax+br+c}^{ax+bs+c} \frac{(b\alpha)^n}{n!} k(x, ax+b\alpha+c) b d\alpha \\
&\approx \sum_{n=0}^N f_{(x)}^{(n)} k_n(x)
\end{aligned}$$

In the above derivation, the key step is the factorization of the terms in the series expansion of $f(\beta(x, \alpha))$ where one term depends only on α and the other term depends only on x . After we derive the expression

$$g(x) \approx \sum_{n=0}^N f_{(x)}^{(n)} k_n(x),$$

the subsequent steps for solving for $f(x)$ is similar to the original RT. They involve deriving expressions for derivatives of $g(x)$ assuming

$$f_{(x)}^{(m)} = 0 \quad \text{for } m > N$$

and solving a system of linear equations in the unknowns $f_{(x)}^{(m)}$.

The above generalization of the RT theory for *linear integral equations* can be extended in a similar manner to the case of *non-linear integral equations*, and non-linear cases of $\beta(x, \alpha)$. This generalization will be considered in the future.

8.1 Discrete Case

In the discrete case, using matrices, vectors, and integer index variables that correspond to the continuous case, we may write

$$g[x] = \sum_{\alpha=r}^{\alpha=s} k[x, \alpha] f[\alpha]$$

In the above equation, note that summation is done with respect to the index α . As summation can be done in any order as long as the terms of summation remain the same, we may use a new index of summation β that is a uniquely invertible function with respect to α of both α and x defined by:

$$\beta = \beta[x, \alpha].$$

The summation can be done in any order we desire by appropriately choosing the bijective mapping between α and β . Compare this with the continuous case where a small change in α corresponds to a small change in β . Now the summation becomes:

$$g[x] = \sum_{\beta=\beta(x,r)}^{\beta(x,s)} k[x, \beta[x, \alpha]] f[\beta[x, \alpha]].$$

Depending on the application, it may be possible to speed up the above summation, or equivalently a matrix-vector product computation, by appropriately choosing $\beta = \beta[x, \alpha]$ and

grouping terms suitably and factoring common terms. For example, in FFT computation, odd and even index entries of $f[i]$ are grouped separately at each recursion level, and their FFTs are computed. Speed-up is also possible if $f[\beta[x, \alpha]]$ can be expressed in a rapidly convergent series expansion (e.g. Taylor series) where the terms are separable with respect to x and α as:

$$f[\beta(x, \alpha)] \approx \sum_{n=0}^N e_n[\alpha] f_{[x]}^{[n]}$$

In this case, computation is reduced by truncating the above series expansion. This is permissible provided the error is bounded and acceptable. The expression for $g[x]$ becomes:

$$\begin{aligned} g[x] &\approx \sum_{\beta=\beta(x,r)}^{\beta(x,s)} k[x, \beta[x, \alpha]] \sum_{n=0}^N f_{[x]}^{[n]} e_n[\alpha] \\ &\approx \sum_{n=0}^N f_{[x]}^{[n]} \sum_{\beta=\beta(x,r)}^{\beta(x,s)} k[x, \beta[x, \alpha]] e_n[\alpha] \\ &\approx \sum_{n=0}^N f_{[x]}^{[n]} k_{[n]}[x] \end{aligned}$$

9. Future Research Topics

Except for numerical stability and convergence characteristics of solutions, all other theoretical characteristics such as existence and uniqueness of solutions should be the same for both conventional integral equations (CIEs) and their equivalent Rao integral equations (RIEs). This is a consequence of the fact that the domain of definition of the kernels in the two cases are exactly the same. However, while rederiving existing theoretical results on CIEs for RIEs directly may not provide new results, such rederivations may be simpler/unifying which would reduce efforts in teaching/learning the subject matter, and perhaps provide new insights. Therefore such theoretical study would be of interest as a future research topic.

The most important achievements of Rao Transforms may not be that they provide unified theoretical and computational frameworks for solving a large class of integral equations, but it may be that they provide computationally more parallel, efficient, accurate, and stable results than current state-of-the-art methods, at least in some cases, particularly those cases which are of practical importance. Finding such cases where RTs have advantages and applying them would be a useful research topic.

Comparing the computational performance of RTs with current state-of-the-art methods should be a future research topic. Problems of practical importance in engineering (e.g. shift-variant image/signal restoration) and science (e.g. inverse optics, potential theory, initial/boundary value problems) should be investigated for solutions based on RTs. Relative performance of RTs in terms of computational resources used, accuracy, numerical stability, and convergence characteristics, need to be evaluated. Fast computational algorithms should be developed for RTs. Special attention should be paid to solving classical integral equations of Fredholm, Volterra, Urysohn, and Hammerstein types. Extension of Rao Transform theory to functions of complex variables, and extension to discrete or matrix equations are also future research topics of interest. For example, RT can be used to define the concept of “local eigen value/function” and the discrete version of Rao Transform equation can be used to define “local matrix/tensor”

product. Presentation slides accompanying this report at www.integralresearch.net provide some additional details on this aspect.

An expert reviewer of a research proposal based on RTs commented as follows on the overall potential of future research on RTs:

“If certain class of differential equations and boundary value problems could be solved more effectively, it is likely to have application in problems of computational and finite element fluid mechanics. Specifically could help in the hull design of underwater vehicles. Solving integral equation will also likely impact computational physics”.

Perhaps the latest book on integral equations is by Dr. Michio Masujima [8], a Mathematical Physicist from MIT. It was published in 2005, and according to the Preface there, it is based on courses on integral equations taught in the Department of Mathematics at MIT, and at Harvard. On the back cover of the book is the claim *“All there is to know about functional analysis, integral equations, and calculus of variations in a single volume”*. On page 39 of the book we find the following discussion.

It is in general easy to transform a differential equation into an integral equation but it says on page 39 *“we never solve a differential equation by such a transformation.”* It then continues *“Indeed, an integral equation is much more difficult to solve than a differential equation in a closed form. Only very rarely can this be done. Therefore, whenever it is possible to transform an integral equation into a differential equation, it is a good idea to do so [and solve it]”*. It should be noted that RTs (always as opposed to rarely) transform an integral equation into a differential equation that is easily solved in closed-form using a truncated Taylor series expansion. Equation (6.18) is a simple example of such a differential equation. However, there is a crucial difference because a conventional differential equation uses boundary conditions specified as separate equations, but the differential equation derived by RT already includes the boundary conditions (see [1] for a more general example of applying RTs to solve a Volterra Integral Equation of the Second Kind). Further research is needed to investigate the relation between RTs and other conventional techniques for solving differential equations.

10. References

1. ***“Rao Transforms: Theory and Applications”***, by **M. Subbarao (Rao)**, U.S. Copyright Registration No. TX 6-195-821, June 1, 2005. Self-published book, 120 pages. Purchase at <http://www.integralresearch.net>.
Second Edition: M. SubbaRao, *Rao Transforms: A New Approach to Integral and Differential Equations*, May, 2007, self-published book, 126 pages, can be purchased at <http://www.integralresearch.net>.
2. M. Subbarao, T. Wei, and G. Surya, "Focused Image Recovery from Two Defocused Images Recorded with Different Camera Settings," *IEEE Transactions on Image Processing*, Vol. 4, No. 12, 16 pages, Dec. 1995.
3. A. K. Jain, *Fundamentals of Digital Image Processing*, Section 8.9, pages 299 to 301, Prentice-Hall, Inc., 1989, ISBN 0-13-336165-9.

4. Polyanin, A. D., and Manzhirov, A. V., *Handbook of Integral Equations*, Boca Raton, FL: CRC Press, 1998.
5. Kanwal, R. P., *Linear Integral Equations: Theory and Technique*, (2nd Ed.), Birkhauser Publishers, Boston, 1997.
6. K. E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, 1997.
7. M. Subbarao and G. Surya, "Depth from Defocus: A Spatial Domain Approach," *International Journal of Computer Vision*, Vol. 13, No. 3, pp. 271-294, 1994.
8. M. Masujima, *Applied Mathematical Methods in Theoretical Physics*, WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim, 2005. ISBN 3-527-40534-8.
9. Eric W. Weisstein. "Integral Equation." From *Mathworld*, A Wolfram Web Resource. <http://mathworld.wolfram.com/IntegralEquation.html>
10. Corduneanu, C. *Integral Equations and Applications*, Cambridge, England: Cambridge University Press, 1991.
11. Kondo, J. *Integral Equations*, Oxford, England: Clarendon Press, 1992.
12. Delves, L.M. and Mohamed, J.L. *Computational Methods for Integral Equations*, Cambridge University Press, 1985.
13. M. Subbarao, "Passive ranging and rapid autofocusing," U.S. patent No. 5,148,209, Sept. 15, 1992.
14. W. K. Pratt, *Digital Image Processing*, Second Edition, Section 12.3, pages 376 to 382, John Wiley and Sons, 1991, ISBN 0-471-85766-1.
15. M. Subbarao, *A method and apparatus for restoring convolution degraded images and signals*, U.S. patent application serial no. 07/548,287, filing date 07/03/90 (abandoned).
16. M. Subbarao (Rao), "Methods and apparatus for computing the input and output signals of a linear shift-variant system", United States Patent and Trademark Office, Provisional patent application filed on 11/08/2004, Ref. No. 16800 U.S. PTO. 60/626028. Full patent application filed on 9/26/2005. No. 113010 U.S. PTO, 11/235724.
17. (a) M. Subbarao (Rao), "Method and Apparatus for Solving Linear and Non-Linear Integral and Integro-Differential Equations", United States Patent and Trademark Office, Provisional patent application filed on 11/23/2004, Ref. No. 22151 U.S. PTO. 60/630395. Combined full patent application filed on 10/03/2005, No. 112991 U.S. PTO, 11/242191.

(b) M. Subbarao (Rao), "Unified Method and Apparatus for Solving Linear and Non-Linear Integral, Integro-Differential, and Differential Equations", Provisional patent application filed in the United States Patent and Trademark Office, 11/29/2004, Ref. No. 19249 U.S. PTO, 60/631555. Combined full patent application filed on 10/03/2005, No. 112991 U.S. PTO, 11/242191.
18. M. Subbarao (Rao), "Direct Vision Sensor for 3D Computer Vision, Digital Imaging, and Digital Video", Provisional patent application filed in the United States Patent and Trademark Office, June 18, 2005, Ref. No. 113013 U.S. PTO, 60/691358. Full patent application filed on 6/10/2006. No. 112959 U.S. PTO, 11/450024.
19. P. M. Guevel and P. R. Bardey, "Air or submarine engine with improved contour", U. S. Patent No. 5024396, June 18, 1991.

Acknowledgements

I am thankful to the people who took interest in this research and offered their comments. They include a Professor of Mathematics, a Professor of Engineering, a Program Director of a US Federal Research Funding Agency, and reviewers of the agency.

Dr. Muralidhara SubbaRao (Rao) graduated with a B. Tech. degree in Electrical Engineering from the Indian Institute of Technology, Madras, and an M.S. and a Ph.D., both in Computer Science, from the University of Maryland at College Park. He is a Professor of Electrical and Computer Engineering at SUNY Stony Brook University. His teaching and research interests are Computer Vision and Digital Image Processing. He has been the Principal Investigator of research grants from industry and the National Science Foundation. He has authored one book (research monograph other than that on RTs), published over 50 papers in professional journals and conferences, and is the inventor of 4 U.S. patents that have been licensed to industry. He is a pioneer researcher in the field of Computer Vision who invented the **Depth-from-Defocus** technique that uses arbitrarily defocused images (without requirement of any focused image) for three-dimensional shape recovery. Ten students have completed their Ph.D. thesis research under his supervision. He was a principal member and the Chief Computer Scientist of a high-tech start-up company for online image management in 2000-2001.