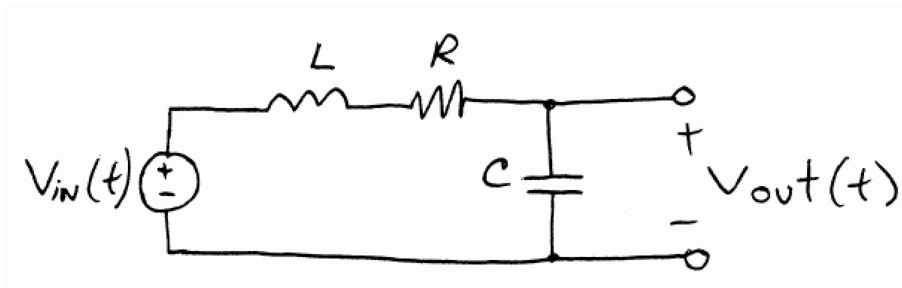
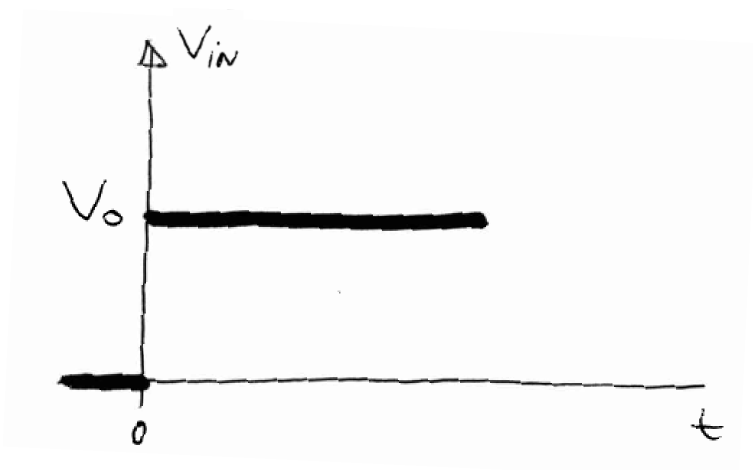
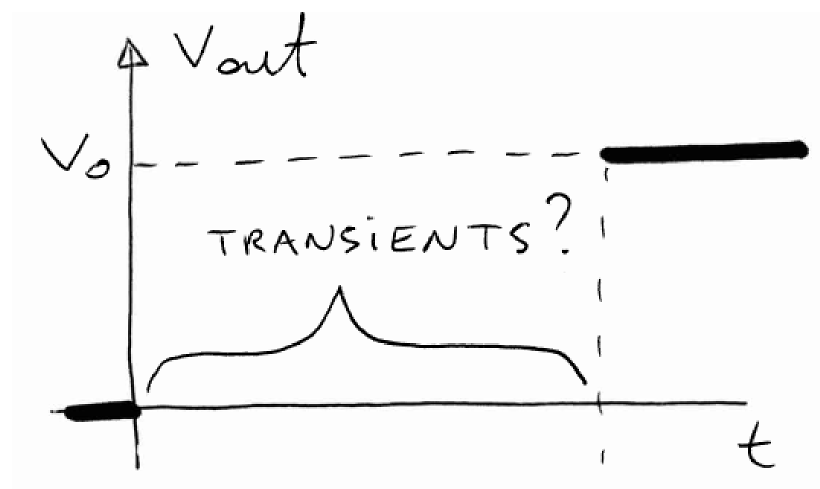
Step response of series RLC circuit - output taken across capacitor.



$$V_{in}(t) = V_o \cdot h(t)$$

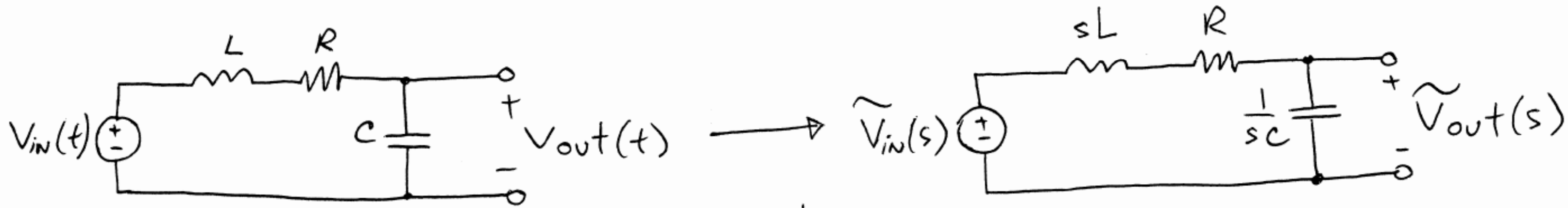
$$V_{out}(t) = ?$$





What happens during transient period from initial steady state to final steady state?

Transfer function of series RLC - output taken across capacitor.



$$T(s) = \left.\frac{\tilde{V}_{out}(s)}{\tilde{V}_{in}(s)}\right|_{\substack{\text{ZERO}\\ \text{IC}}} = \frac{\frac{1}{sC}}{sL + R + \frac{1}{sC}} = \frac{1}{LC} \cdot \frac{1}{s^2 + s\cdot\frac{R}{L} + \frac{1}{LC}}$$

Poles: $s^2 + s\cdot\frac{R}{L} + \frac{1}{LC} = 0$ $\qquad p_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$

Case 1: $\frac{R^2}{4L^2} > \frac{1}{LC}$, i.e. $R > 2\sqrt{\frac{L}{C}}$ ***- two different real poles***

Case 2: $\frac{R^2}{4L^2} = \frac{1}{LC}$, i.e. $p_1 = p_2 = -\frac{R}{2L}$ ***- two identical real poles***

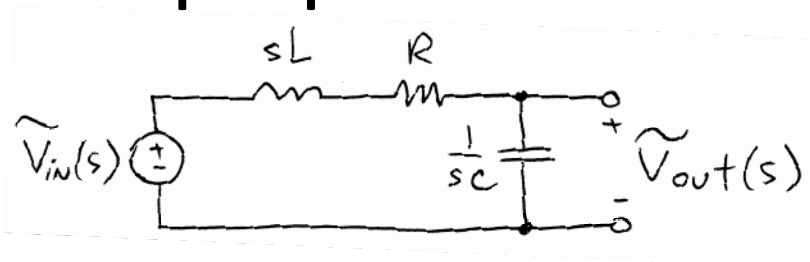
Case 3: $\frac{R^2}{4L^2} < \frac{1}{LC}$, i.e. $R < 2\sqrt{\frac{L}{C}}$ ***- complex conjugate poles***

$$s^2 + \frac{R}{L}\cdot s + \frac{1}{LC} = s^2 + 2\cdot\gamma\cdot\omega_n\cdot s + \omega_n^2$$

$$\omega_n = \frac{1}{\sqrt{LC}} \quad \& \quad \gamma = \frac{R}{2}\sqrt{\frac{C}{L}} < 1$$

Case 1: two different real poles.

Step response of series RLC - output taken across capacitor.



$$p_1 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$p_2 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$R > 2\sqrt{L/C}$$

$$p_1 < p_2 < 0 \quad \& \quad |p_1| > |p_2|$$

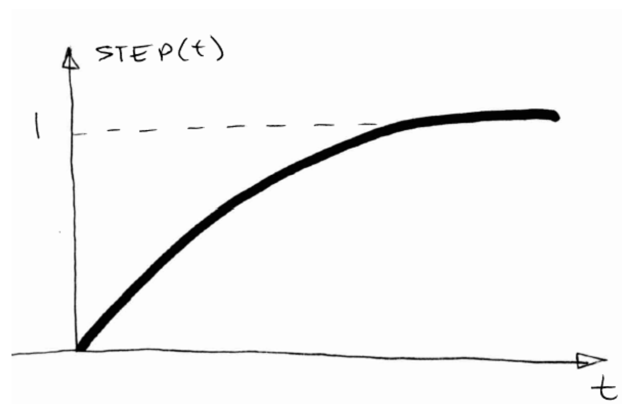
$$T(s) = \frac{1}{LC} \cdot \frac{1}{(s-p_1)(s-p_2)}$$

$$STEP(t) = L^{-1}\left[\frac{T(s)}{s}\right] = \frac{1}{LC} \cdot L^{-1}\left[\frac{1}{s(s-p_1)(s-p_2)}\right]$$

$$\frac{1}{s(s-p_1)(s-p_2)} = \frac{A}{s} + \frac{B}{s-p_1} + \frac{C}{s-p_2}$$

$$A = \frac{1}{p_1 \cdot p_2} \quad ; \quad B = \frac{1}{p_1(p_1-p_2)} \quad ; \quad C = \frac{1}{p_2(p_2-p_1)}$$

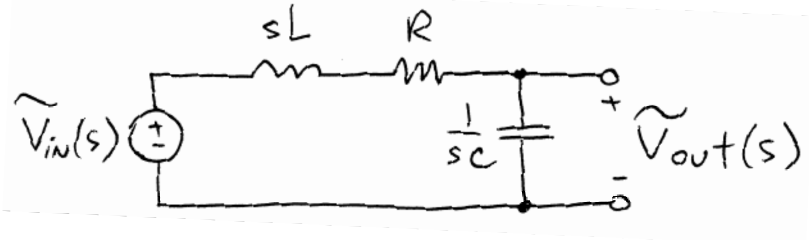
$$STEP(t) = \frac{1}{LC}\left[A \cdot h(t) + B \cdot e^{p_1 t} \cdot h(t) + C \cdot e^{p_2 t} \cdot h(t)\right]$$



Overdamped case – the circuit demonstrates relatively slow transient response.

Case 1: two different real poles.

Frequency response of series RLC - output taken across capacitor.



$$p_1 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

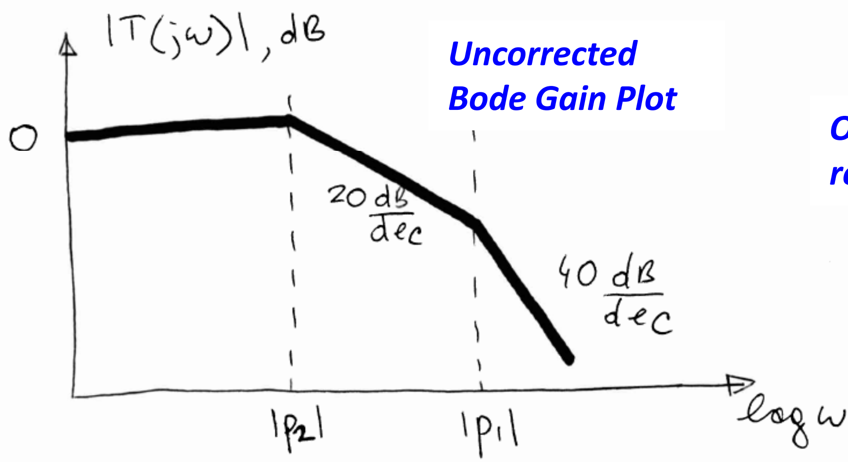
$$p_2 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$R > 2\sqrt{L/C}$$

$$p_1 < p_2 < 0 \ \& \ |p_1| > |p_2|$$

$$T(j\omega) = \frac{1}{LC} \cdot \frac{1}{(j\omega - p_1)(j\omega - p_2)}$$

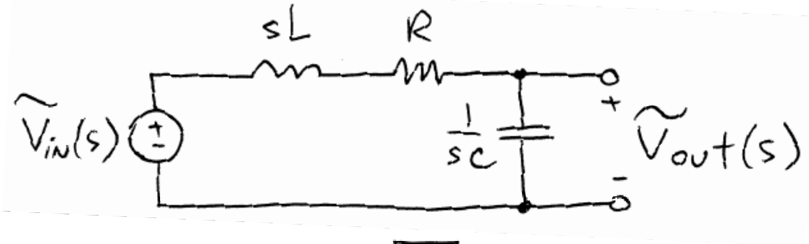
$$20\log|T(j\omega)| = 20\log\frac{1}{LC} - 20\log\sqrt{\omega^2 + p_1^2} - 20\log\sqrt{\omega^2 + p_2^2}$$



Uncorrected Bode Gain Plot

Overdamped case – the circuit demonstrates relatively limited bandwidth

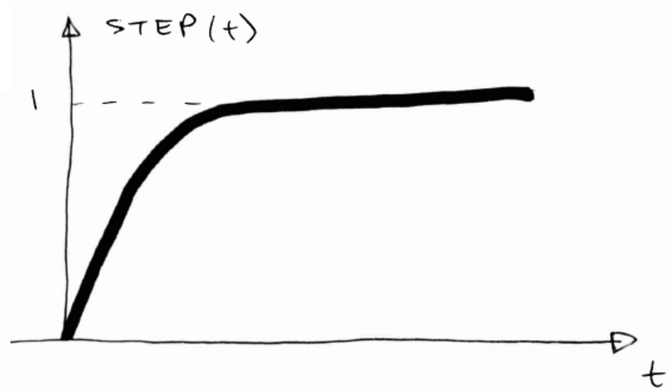
Case 2: two identical real poles.
Step response of series RLC - output taken across capacitor.



$$R = 2\sqrt{\frac{L}{C}}$$

$$p = p_1 = p_2 = -\frac{R}{2L} < 0$$

$$T(s) = \frac{1}{LC} \cdot \frac{1}{(s-p)^2}$$



$$\text{STEP}(t) = L^{-1}\left[\frac{T(s)}{s}\right] = \frac{1}{LC} L^{-1}\left[\frac{1}{s(s-p)^2}\right]$$

$$\frac{1}{s(s-p)^2} = \frac{A}{s} + \frac{B}{(s-p)^2} + \frac{C}{s-p}$$

$$A = \frac{1}{p^2}; \quad B = \frac{1}{p}; \quad C = -\frac{1}{p^2} \quad \& \quad p = -\frac{R}{2L} = -\frac{1}{\sqrt{LC}}$$

$$\text{step}(t) = \frac{1}{LC}\left[A \cdot h(t) + B \cdot t\, e^{pt} \cdot h(t) + C \cdot e^{pt} \cdot h(t)\right] =$$

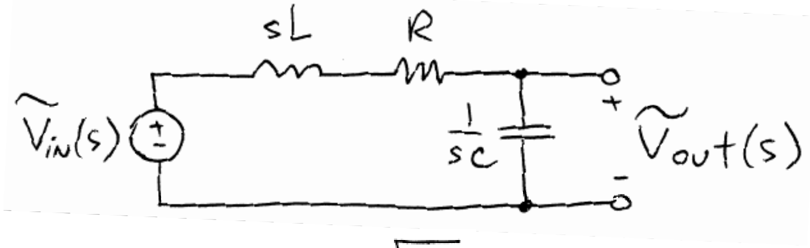
$$= h(t) + \left(-\frac{t}{\sqrt{LC}}\right) e^{-\frac{t}{\sqrt{LC}}} \cdot h(t) - e^{-\frac{t}{\sqrt{LC}}} \cdot h(t) =$$

$$= h(t) - \left[\frac{t}{\sqrt{LC}} + 1\right] e^{-\frac{t}{\sqrt{LC}}} \cdot h(t)$$

Critically damped case – the circuit demonstrates the shortest possible rise time without overshoot.

Case 2: two identical real poles.

Frequency response of series RLC - output taken across capacitor.



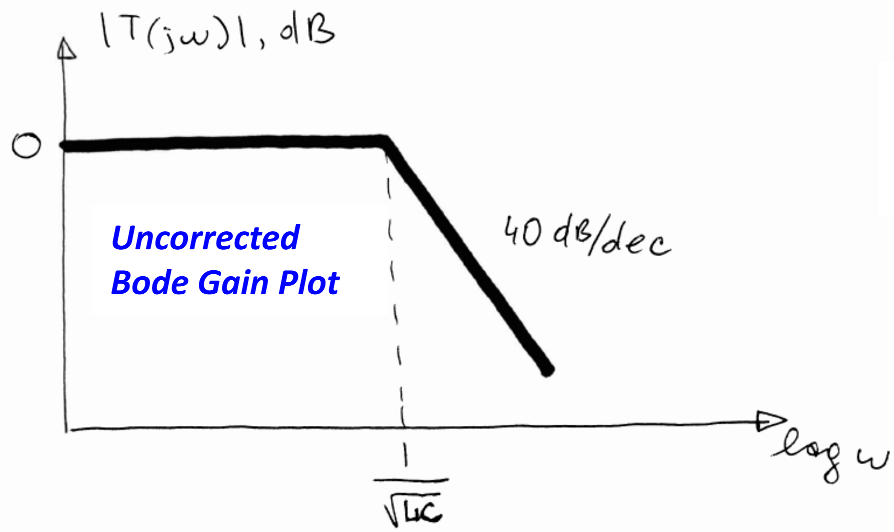
$$R = 2\sqrt{\frac{L}{C}}$$

$$p = p_1 = p_2 = -\frac{R}{2L} < 0$$

$$T(s) = \frac{1}{LC}\cdot\frac{1}{(s-p)^2}$$

$$T(j\omega) = \frac{1}{LC}\cdot\frac{1}{(j\omega-p)^2} \quad ; \quad p = -\frac{1}{\sqrt{LC}}$$

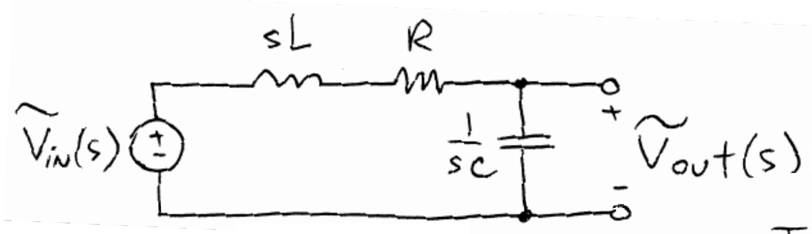
$$20\log|T(j\omega)| = 20\log\frac{1}{LC} - 20\cdot 2\cdot \log\sqrt{\omega^2+p^2}$$



Critically damped case
– the circuit demonstrates the widest bandwidth without apparent resonance.

Case 3: two complex poles.

Step response of series RLC - output taken across capacitor.



$$R < 2\sqrt{\frac{L}{C}}$$

$$p_{1,2} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$T(s) = \frac{1}{LC}\cdot\frac{1}{s^2 + 2\gamma\omega_n s + \omega_n^2} = \frac{1}{LC}\cdot\frac{1}{(s-p)(s-p^*)}$$

$$\text{STEP}(t) = \mathcal{L}^{-1}\left[\frac{T(s)}{s}\right] = \frac{1}{LC}\cdot\mathcal{L}^{-1}\left[\frac{1}{s(s-p)(s-p^*)}\right]$$

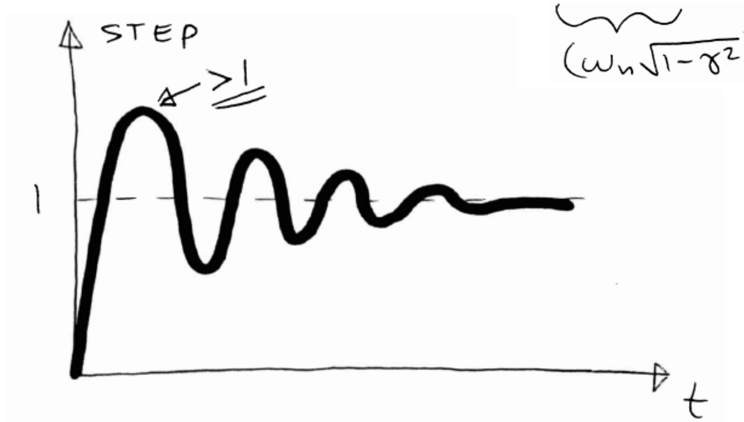
$$\frac{1}{s(s-p)(s-p^*)} = \frac{A}{s} + \frac{B}{s-p} + \frac{C}{s-p^*}$$

$$A = \frac{1}{p\cdot p^*} = LC\ ;\quad B = \frac{1}{p(p-p^*)}\ ;\quad C = \frac{1}{p^*(p-p^*)} = B^*$$

$$\text{STEP}(t) = h(t) + h(t)\cdot\underbrace{2\cdot\frac{|B|}{LC}}_{(\omega_n\sqrt{1-\gamma^2})^{-1}}\cdot e^{-\frac{R}{2L}t}\cdot\cos\left(\omega_n\sqrt{1-\gamma^2}\cdot t + \varphi_B\right)$$

$$\omega_n = \frac{1}{\sqrt{LC}}$$

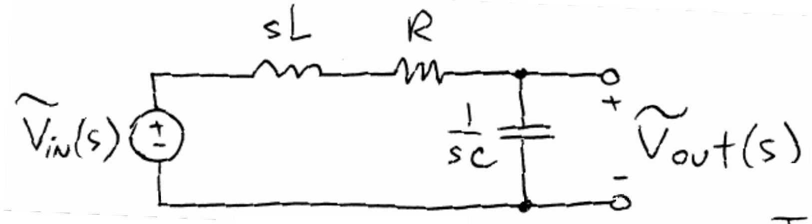
$$\gamma = \frac{R}{2}\sqrt{\frac{C}{L}} < 1$$



Underdamped case – the circuit oscillates.

Case 3: two complex poles.

Frequency response of series RLC - output taken across capacitor.



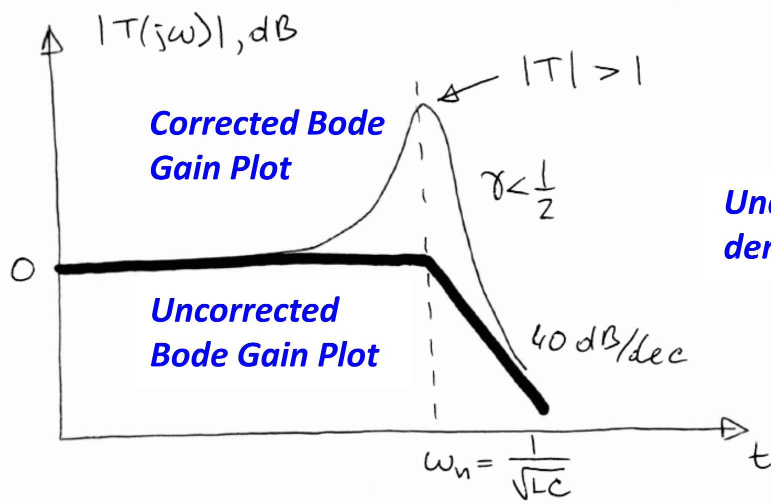
$$R < 2\sqrt{\frac{L}{C}}$$

$$p_{1,2} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$T(s) = \frac{1}{LC} \cdot \frac{1}{s^2 + 2\gamma\omega_n s + \omega_n^2} = \frac{1}{LC} \cdot \frac{1}{(s-p)(s-p^*)}$$

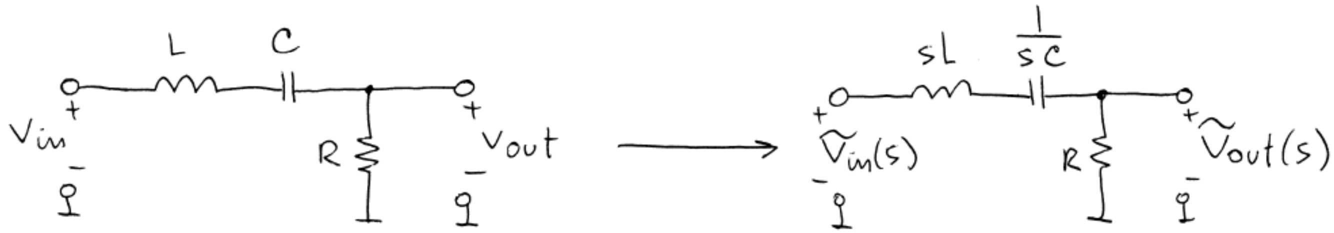
$$T(j\omega) = \frac{1}{LC} \cdot \frac{1}{(j\omega)^2 + 2\gamma\omega_n \cdot (j\omega) + \omega_n^2}$$

$$20 \log|T(j\omega)| = 20 \log \frac{1}{LC} - 20 \log \sqrt{(\omega_n^2 - \omega^2)^2 + (2\gamma\omega_n\omega)^2}$$



Underdamped case – the circuit can demonstrate apparent resonant behavior.

Series RLC resonance.



$$T(s) = \left.\frac{\tilde{V}_{out}(s)}{\tilde{V}_{in}(s)}\right|_{\text{ZERO IC}} = \frac{R}{sL + \frac{1}{sC} + R} = \frac{R}{L}\cdot\frac{s}{s^2 + \frac{R}{L}\cdot s + \frac{1}{LC}}$$

Zeros: $s = 0 = z$

Poles: $s_{1,2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = p_{1,2}$

Case 1: $\frac{R^2}{4L^2} > \frac{1}{LC}$, i.e. $R > 2\sqrt{\frac{L}{C}}$ - ***two different real poles***

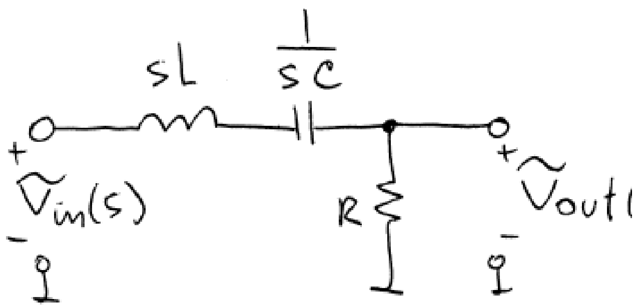
Case 2: $\frac{R^2}{4L^2} = \frac{1}{LC}$, i.e. $p_1 = p_2 = -\frac{R}{2L}$ - ***two identical real poles***

Case 3: $\frac{R^2}{4L^2} < \frac{1}{LC}$, i.e. $R < 2\sqrt{\frac{L}{C}}$ - ***complex poles***

$$s^2 + \frac{R}{L}\cdot s + \frac{1}{LC} = s^2 + 2\cdot\gamma\cdot\omega_n\cdot s + \omega_n^2$$

$$\omega_n = \frac{1}{\sqrt{LC}} \quad \& \quad \gamma = \frac{R}{2}\sqrt{\frac{C}{L}} < 1$$

Series RLC resonance: Case 1 – two different real poles.



$$R > 2\sqrt{L/C} \Rightarrow p_1 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

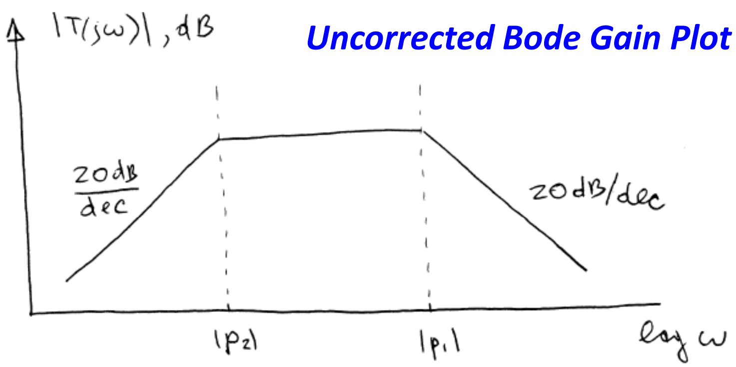
$$p_2 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$p_1 < p_2 < 0 \ \& \ |p_1| > |p_2|$$

$$T(s) = \frac{R}{L} \cdot \frac{s}{(s-p_1)(s-p_2)}$$

$$T(j\omega) = \frac{R}{L} \frac{j\omega}{(j\omega - p_1)(j\omega - p_2)}$$

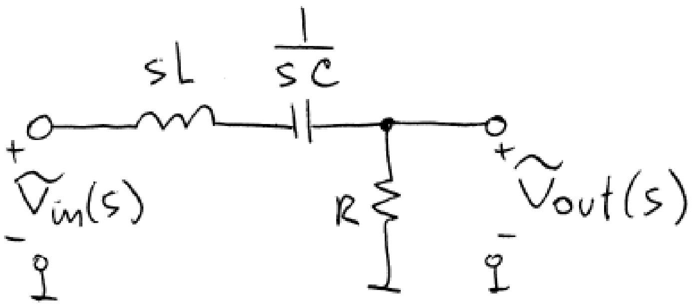
$$20\log|T(j\omega)| = 20\log\frac{R}{L} + 20\log\omega - 20\log\sqrt{\omega^2+p_1^2} - 20\log\sqrt{\omega^2+p_2^2}$$



Uncorrected Bode Gain Plot

Underdamped case - Resonance is not sharp.

Series RLC resonance: Case 2 – two identical real poles.



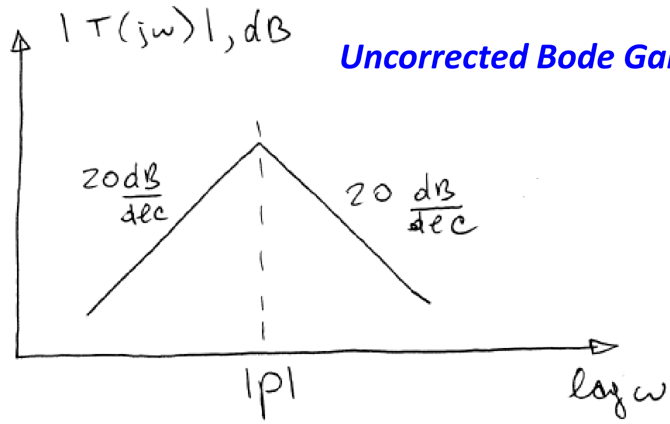
$$R = 2\sqrt{\frac{L}{C}}$$

$$p = p_1 = p_2 = -\frac{R}{2L} < 0$$

$$T(s) = \frac{R}{L} \cdot \frac{s}{(s-p)^2}$$

$$T(j\omega) = \frac{R}{L} \cdot \frac{j\omega}{(j\omega - p)^2}$$

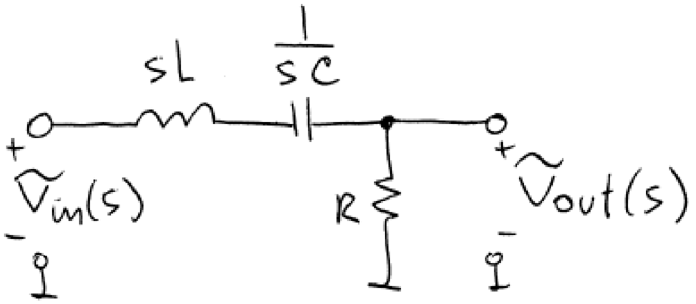
$$20 \log |T(j\omega)| = 20 \log \frac{R}{L} + 20 \log \omega - 40 \log \sqrt{\omega^2 + p^2}$$



Uncorrected Bode Gain Plot

Critically damped case - Resonance is apparent but still not sharp enough.

Series RLC resonance: Case 3 – complex poles.



$$R < 2\sqrt{\frac{L}{C}}$$

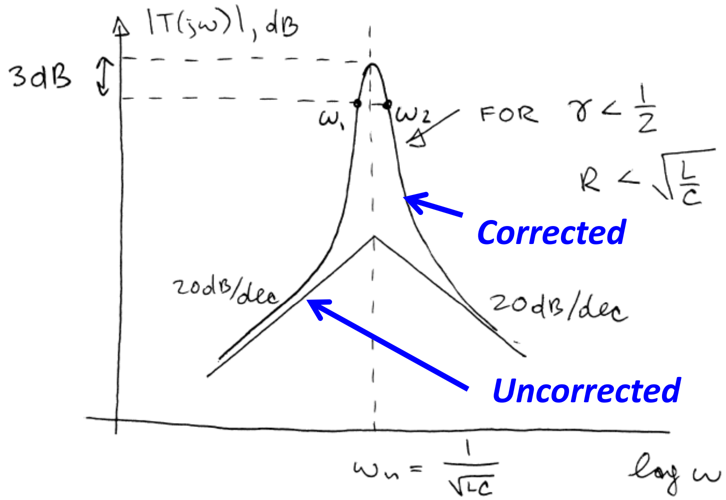
$$p_{1,2} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$T(s) = \frac{R}{L} \cdot \frac{s}{s^2 + 2\cdot\gamma\cdot\omega_n\cdot s + \omega_n^2}$$

$$\omega_n = \frac{1}{\sqrt{LC}} \qquad \gamma = \frac{R}{2}\sqrt{\frac{C}{L}} < 1$$

$$T(j\omega) = \frac{R}{L} \cdot \frac{j\omega}{(j\omega)^2 + 2\gamma\omega_n\cdot j\omega + \omega_n^2}$$

$$20\log|T(j\omega)| = 20\log\frac{R}{L} + 20\log\omega - 20\log\sqrt{(\omega_n^2-\omega^2)^2 + (2\gamma\omega_n\omega)^2}$$

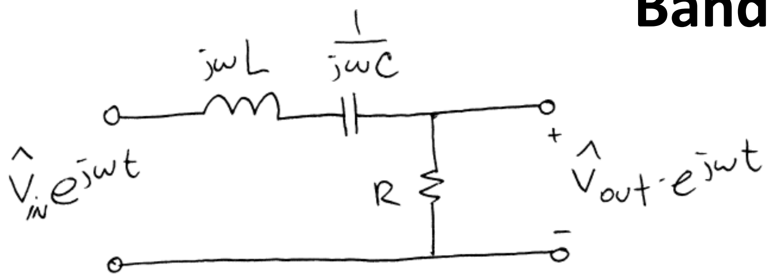


Half power bandwidth: $\left|T(j\omega)\right|_{\omega=\omega_1 \text{ or } \omega_2} = \frac{1}{\sqrt{2}}$

$$BW = R/L$$

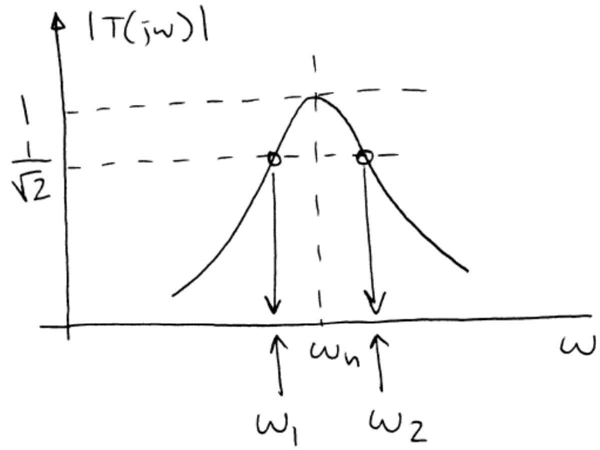
Quality factor: $Q = \frac{\omega_n}{BW} = \frac{1}{R}\sqrt{\frac{L}{C}}$

Another look at Series RLC resonance. Bandwidth.



$$T(j\omega) = \frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$|T(j\omega)| = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \; ; \; |T(j\omega_n)| = 1 \; ; \; \omega_n = \frac{1}{\sqrt{LC}}$$



$$|T(j\omega)| = \frac{1}{\sqrt{2}} \quad \text{when } R^2 = \left(\omega L - \frac{1}{\omega C}\right)^2$$

$$\left(\omega L - \frac{1}{\omega C}\right) = \pm R$$

$$\omega^2 \pm \frac{R}{L}\omega - \frac{1}{LC} = 0$$

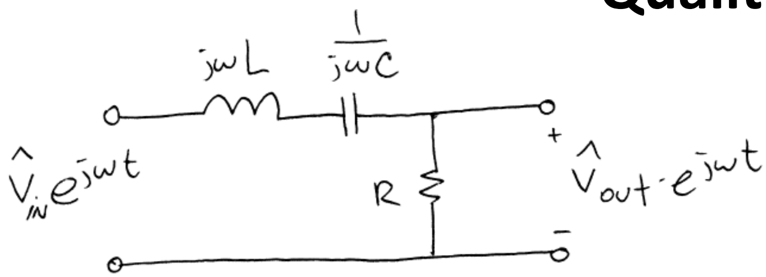
$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

Half power bandwidth:

$$BW = \omega_2 - \omega_1 = \frac{R}{L}$$

Another look at Series RLC resonance. Quality factor.

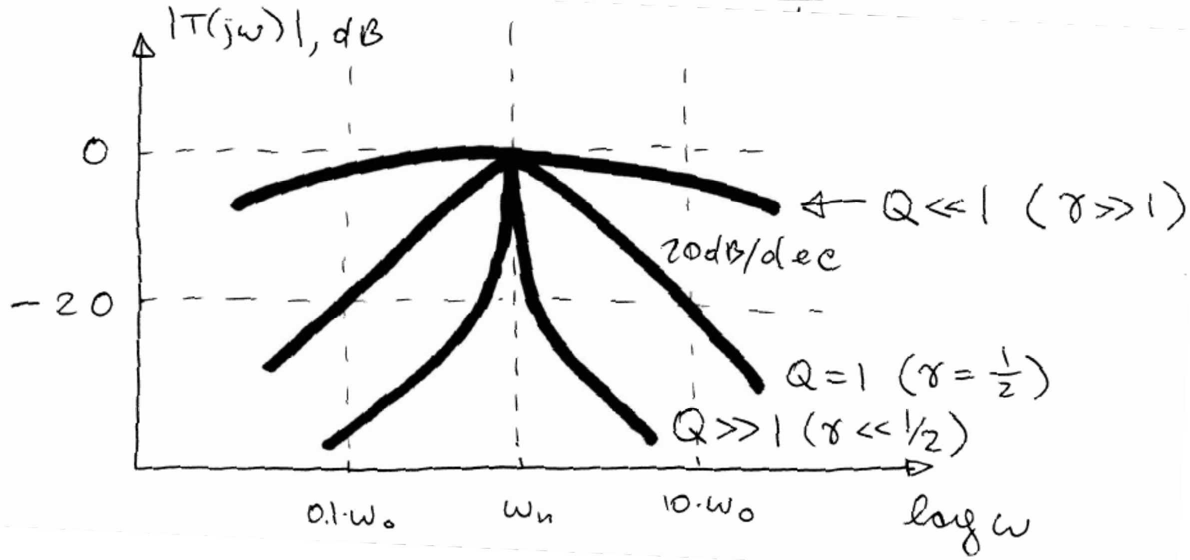


$$T(j\omega) = \frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$|T(j\omega)| = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \quad ; \quad |T(j\omega_n)| = 1 \quad ; \quad \omega_n = \frac{1}{\sqrt{LC}}$$

Quality factor:

$$Q = \frac{\omega_n}{BW} = \frac{\frac{1}{\sqrt{LC}}}{\frac{R}{L}} = \frac{1}{R} \cdot \sqrt{\frac{L}{C}} = \frac{1}{2 \cdot \gamma} \quad ; \quad \gamma = \frac{R}{2}\sqrt{\frac{C}{L}}$$



High values of Quality factor ensure sharp resonance peak.

Example 1.

Design series RLC circuit with resonant frequency of 1 kHz and quality factor of 100.

$$\omega_n = \frac{1}{\sqrt{LC}} \quad \& \quad Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

ASSUME $C = 1\mu F$

$$\Rightarrow \quad L = \frac{1}{(2\pi f_n)^2 \cdot C} = 25.3 \text{ mH}$$

$$\Rightarrow \quad R = \frac{1}{100} \cdot \sqrt{\frac{L}{C}} \approx 1.6\ \Omega$$

BUT : L is COIL OF WIRE $\Rightarrow$ $R_L \neq 0$!!!

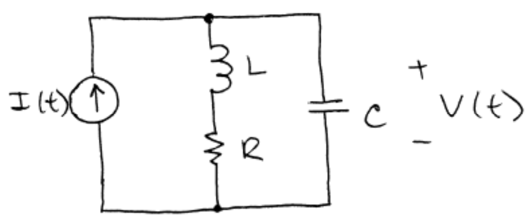
Inductors have series resistance that is often characterized by inductor Q-factor: $Q_L = \dfrac{\omega \cdot L}{r_s}$

Usual values of inductor Q-factor are 50-200.

Series resistance of the inductor has to be taken into account when calculating R.

Example.

Sketch the Bode Gain Plot of the following circuit:



$$T(s) = \frac{\tilde{V}(s)}{\tilde{I}(s)} = \frac{\frac{1}{sC}\cdot(sL+R)}{\frac{1}{sC}+sL+R}$$

ASSUME $L = 10\,H$; $R = 1\,\Omega$; $C = 0{,}1\,F$

$$T(s) = \frac{\frac{10}{s}(10s+1)}{\frac{10}{s}+10s+1} = 10\cdot\frac{\left(s+\frac{1}{10}\right)}{s^2+\frac{1}{10}s+1}$$

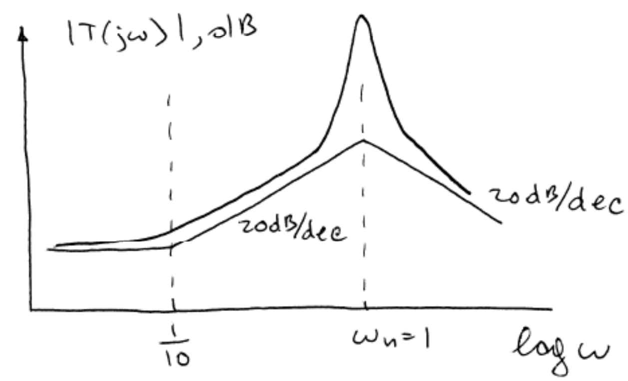
$$\begin{cases} \text{ZERO:} & z = -\frac{1}{10} \\ \text{poles:} & p_{1,2} = -\frac{1}{20} \pm \sqrt{\frac{1}{400}-1} = -\frac{1}{20} \pm j\sqrt{1-\frac{1}{400}} \end{cases}$$

– COMPLEX (UNDERDAMPED)

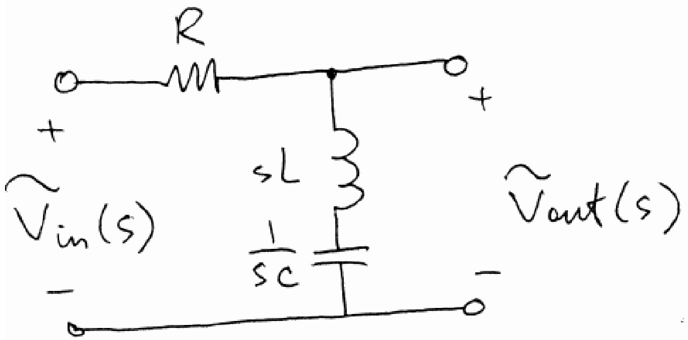
$$s^2 + \frac{1}{10}s + 1 = s^2 + 2\cdot\gamma\cdot\omega_n\cdot s + \omega_n^2$$

$\Rightarrow \omega_n = 1$ RAD/s

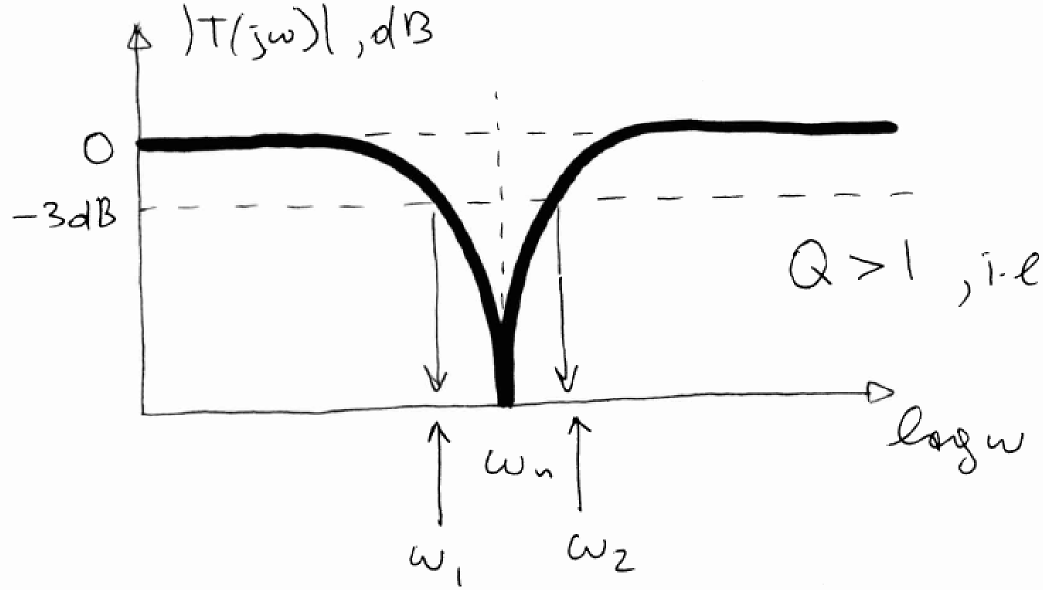
$\gamma = \frac{1}{20}$



RLC band stop circuit – notch filter.



$$T(s) = \frac{sL + \frac{1}{sC}}{R + sL + \frac{1}{sC}}$$



$$\omega_n = \frac{1}{\sqrt{LC}}$$

$$Q > 1 \text{, i.e. } \gamma = \frac{R}{2}\sqrt{\frac{C}{L}} < \frac{1}{2}$$

$$BW = \omega_2 - \omega_1 = R/L \quad ; \quad Q = \frac{\omega_n}{BW}$$