ESE 503 - Stochastic Systems

Fall 1999

Solutions to Homework # 1

Problem 2.3:

(a) Sample space $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

(b) A: "total number of dots showing is even."

$$A = \{2, 4, 6, 8, 10, 12\}.$$

(c) The event "sum of dots = 2" corresponds to the elementary event $(1, 1)$ in # 2.2; thus the elementary event $\{\text{sum of dots} = 2\}$ can be written as

$$\{\text{sum of dots} = 2\} = \{(1, 1)\}.$$

Also, since the event "the sum of dots = 3" corresponds to $(1, 2)$ or $(2, 1)$, we have

$$\{\text{sum of dots} = 3\} = \{(1, 2)\} \cup \{(2, 1)\} = \{(1, 2), (2, 1)\}.$$

Similarly,

$$\{\text{sum of dots} = 4\} = \{(1, 3), (2, 2,), (3, 1)\},$$

and

$$\{\text{sum of dots} = 5\} = \{(1, 4), (2, 3), (3, 2), (4, 1)\}.$$

In general, for $1 < k \leq 12$,

$$\{\text{sum of dots} = k\} = \bigcup_{i=\max(1,k-6)}^{\min(6,k-1)} \{(i, k - i)\}.$$

Problem 2.6:

(a) The sample space Ω has nine elements and is given by:

$$\Omega = \{(F, F), (F, R), (F, K), (R, F), (R, R), (R, K), (K, F), (K, R), (K, K)\}.$$

It is the set of ordered pairs (s_1, s_2) where s_1 specifies the state of C_1 and s_2 specifies the state of C_2.

(b) Let A represent the event "none of the components is kaput;" then

$$A = \{(F, F), (F, R), (R, F), (R, R)\}.$$

Problem 2.7:

(a) Sample space Ω is

$$\Omega = \{(1, 2, 3), (1, 3, 2), (2, 3, 1), (2, 1, 3), (3, 1, 2), (3, 2, 1)\}.$$

(b) If A_k denotes the event "ball number k is selected in the kth draw," then

$$\begin{aligned} A_1 &= \{(1,2,3),(1,3,2)\}, \\ A_2 &= \{(1,2,3),(3,2,1)\}, \\ A_3 &= \{(1,2,3),(2,1,3)\}. \end{aligned}$$

(c) $A_1 \cap A_2 \cap A_3 = \{(1,2,3)\}$. This event can be described as: "the number of each ball corresponds to the number of the draw."

(d) $A_1 \cup A_2 \cup A_3 = \{(1,2,3),(1,3,2),(3,2,1),(2,1,3)\}$. This event can be described as: "the number of at least one ball corresponds to the number of the draw."

(e) $(A_1 \cup A_2 \cup A_3)^c = \{(2,3,1),(3,1,2)\}$. This event can be described as: "none of the balls has a number corresponding to the number of the draw."

Problem 2.17:

The sample space has 36 elements and is given by the following set

$$\begin{aligned} \Omega \;=\; & \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ & (3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ & (5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}. \end{aligned}$$

Since the outcomes are equally likely, we get that the probability of each elementary event $\{(i,j)\}$ is

$$P[\{(i,j)\}] = \frac{1}{36}, \quad i = 1,\cdots,6, \quad j = 1,\cdots,6.$$

(a) Let A_k denote the event "the sum of the two outcomes is k," for $k = 2,3,\cdots,12$. Therefore, using the fact that each event A_k can be written as the disjoint union of elementary events of Ω (cf # 2.3(c)), we get

$$P[A_2] = P[\{(1,1)\}] = \frac{1}{36},$$

$$P[A_3] = P[\{(1,2),(2,1)\}] = P[\{(1,2)\}] + P[\{(2,1)\}] = \frac{1}{36} + \frac{1}{36} = \frac{2}{36},$$

$$P[A_4] = P[\{(1,3),(2,2),(3,1)\}] = P[\{(1,3)\}] + P[\{(2,2)\}] + P[\{(3,1)\}] = \frac{3}{36}.$$

Proceeding similarly, we obtain:

$$P[A_5] = \frac{4}{36}, \quad P[A_6] = \frac{5}{36}, \quad P[A_7] = \frac{6}{36}, \quad P[A_8] = \frac{5}{36},$$

$$P[A_9] = \frac{4}{36}, \quad P[A_{10}] = \frac{3}{36}, \quad P[A_{11}] = \frac{2}{36}, \quad P[A_{12}] = \frac{1}{36}.$$

Note that $P[A_2] + P[A_3] + \cdots + P[A_{12}] = 1$, as expected.

(b) Let B denote the event "the outcomes of the two tosses are different." Then, B^c represents the complimentary event which is:"the outcomes of the two tosses are the same." Thus,

$$B^c = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\},$$

and

$$P[B^c] = \frac{6}{36}.$$

Hence,

$$P[B] = 1 - P[B^c] = 1 - \frac{6}{36} = \frac{30}{36} = \frac{5}{6}.$$

Problem 2.23:

Let A_i denotes the event "the i'th toss results in heads," $i = 1, 2, 3, 4$. The sample space Ω has 16 elements; it consists of the set

$$\begin{aligned}\Omega &= \{(H,H,H,H),(H,H,H,T),(H,H,T,H),(H,H,T,T),\\ &\quad (H,T,H,H),(H,T,H,T),(H,T,T,H),(H,T,T,T),\\ &\quad (T,H,H,H),(T,H,H,T),(T,H,T,H),(T,H,T,T),\\ &\quad (T,T,H,H),(T,T,H,T),(T,T,T,H),(T,T,T,T)\},\end{aligned}$$

where H denotes heads and T denotes tails. Since the coin is fair, we obtain that the probability of each elementary event in Ω is $1/16$. We thus obtain:

$$\begin{aligned}P[A_2] &= P\left[\{(H,H,H,H),(H,H,H,T),(H,H,T,H),(H,H,T,T),\right.\\ &\quad \left.(T,H,H,H),(T,H,H,T),(T,H,T,H),(T,H,T,T)\}\right]\\ &= \frac{8}{16} = \frac{1}{2},\end{aligned}$$

$$\begin{aligned}P[A_1 \cap A_3] &= P[\text{1st toss and 3rd toss are heads}]\\ &= P\left[\{(H,H,H,H),(H,H,H,T),(H,T,H,H),(H,T,H,T)\}\right]\\ &= \frac{4}{16} = \frac{1}{4},\end{aligned}$$

$$\begin{aligned}P[A_1 \cap A_2 \cap A_3 \cap A_4] &= P[\text{all four tosses are heads}]\\ &= P\left[\{(H,H,H,H)\}\right] = \frac{1}{16},\end{aligned}$$

and

$$\begin{aligned}P[A_1 \cup A_2 \cup A_3 \cup A_4] &= P[\text{at least one of the four tosses is heads}]\\ &= 1 - P[\text{all four tosses are tails}]\\ &= 1 - P\left[\{(T,T,T,T)\}\right] = 1 - \frac{1}{16} = \frac{15}{16}.\end{aligned}$$

Problem 2.27:

(a) Since the number is selected at random, we assume a uniform probability law. Hence,

$$P[B] = \frac{\text{length of interval } B}{\text{length of } [-1,1]} = \frac{\text{length of } (-0.5,1)}{\text{length of } [-1,1]} = \frac{1.5}{2} = \frac{3}{4} = 0.75,$$

$$P[A \cap B] = P[\{x \in (-0.5,0)\}] = \frac{\text{length of } (-0.5,0)}{\text{length of } [-1,1]} = \frac{0.5}{2} = \frac{1}{4} = 0.25,$$

and

$$P[A \cap C] = P[\emptyset] = 0.$$

(b)

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = \frac{1}{2} + \frac{3}{4} - \frac{1}{4} = 1.$$

It can also be directly verified that $A \cup B = [-1,1]$; thus $P[A \cup B] = 1$.

Also,

$$P[A \cup C] = P[A] + P[C] - P[A \cap C] = \frac{1}{} + \frac{1}{} - 0 = \frac{5}{},$$

$P[A \cup C]$ can also be directly computed.

Finally,

$$\begin{aligned} P[A \cup B \cup C] &= P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] \\ &\quad - P[B \cap C] + P[A \cap B \cap C] \\ &= \frac{1}{2} + \frac{3}{4} + \frac{1}{8} - \frac{1}{4} - 0 - \frac{1}{8} + 0 = 1. \end{aligned}$$

This result can also be computed by finding the set $A \cup B \cup C$ and calculating its probability directly.

Problem 2.30:

(a) Since $(-\infty, r] \subset (-\infty, s]$ for $r < s$, we must have by Corollary 7 that

$$P[(-\infty, r]] \leq P[(-\infty, s]].$$

(b) First note that we can write $(-\infty, s]$ as the disjoint union of $(r, s]$ and $(-\infty, r]$:

$$(-\infty, s] = (-\infty, r] \cup (r, s]$$

where the two intervals on the right hand side are disjoint. We hence get (by Axiom III) that

$$P(-\infty, s]] = P[(-\infty, r]] + P[(r, s]].$$

Thus

$$P[(r, s]] = P(-\infty, s]] - P[(-\infty, r]].$$

Problem 2.48:

(a) We know that if $P[B] > 0$, then

$$P[A|B] = \frac{P[A \cap B]}{P[B]}.$$

- If $A \cap B = \emptyset$, then $P[A \cap B] = 0$ and $P[A|B] = 0$.
- If $A \subset B$, then $A \cap B = A$ and

$$P[A|B] = \frac{P[A]}{P[B]}.$$

- If $B \subset A$, then $A \cap B = B$ and

$$P[A|B] = \frac{P[B]}{P[B]} = 1.$$

(b) If $P[A|B] > P[A]$, then

$$\frac{P[A \cap B]}{P[B]} > P[A],$$

and thus

$$P[A \cap B] > P[A]P[B].$$

Hence,

$$P[B|A] = \frac{P[A \cap B]}{P[A]} > \frac{P[A]P[B]}{P[A]} = P[B].$$

□

This leads us to conclude that A and B tend to occur jointl

Problem 2.51:

Assuming that the die is fair, let A denote the event "total number of dots is even," and let B denote the event "both tosses are even." Then

$$\begin{aligned} A &= \{(1,1),(1,3),(1,5),(2,2),(2,4),(2,6),(3,1),(3,3),(3,5) \\ &\quad (4,2),(4,4),(4,6),(5,1),(5,3),(5,5),(6,2),(6,4),(6,6)\}, \end{aligned}$$

and

$$B = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}.$$

Clearly, $B \subset A$; then $A \cap B = B$. Hence

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[B]}{P[B]} = 1,$$

and

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[B]}{P[A]} = \frac{\#\text{ of elements in } B}{\#\text{ of elements in } A} = \frac{9}{18} = \frac{1}{2}.$$

Problem 2.56:

$$P[\text{arrival in next minute}|\text{there was no arrival by 8:30}] = \frac{(1/60)}{(30/60)} = \frac{1}{30},$$

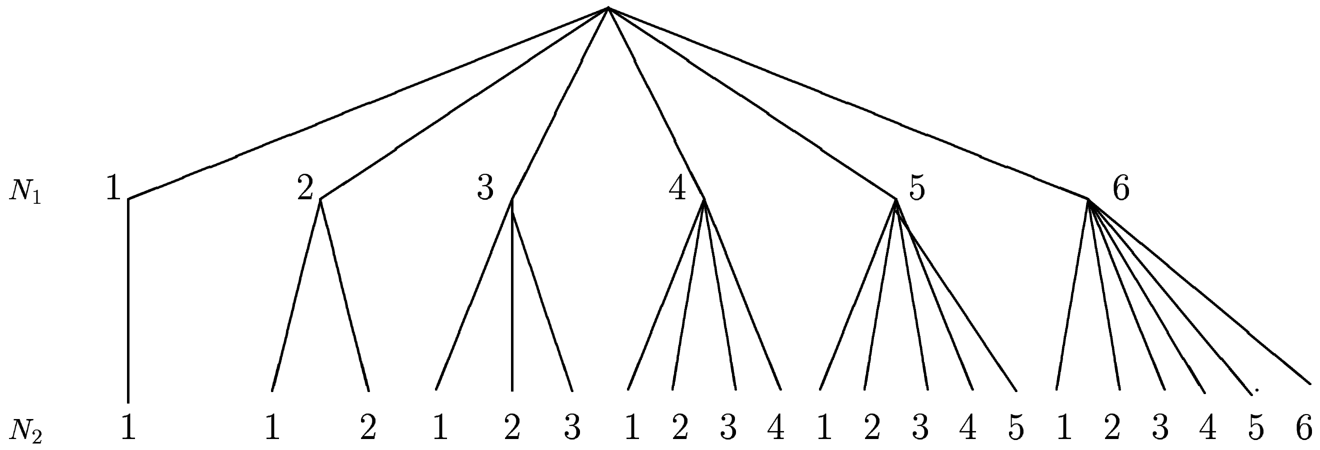
and

$$P[\text{arrival in next minute}|\text{there was no arrival by 8:50}] = \frac{(1/60)}{(10/60)} = \frac{1}{10}.$$

The knowledge that the professor will arrive at 9 A.M. at the latest implies that the probability of arrival within a minute approaches 1 as times draws closer to 9 A.M.

Problem 2.58:

(a) The tree diagram is as follows.



(b) By the law of total probability, we get

$$\begin{aligned} P[N_2 = 3] &= \sum_{i=1}^{6} P[N_2 = 3|N_1 = i]P[N_1 = i] \\ &= (0)(1/6) + (0)(1/6) + (1/3)(1/6) \\ &\quad +(1/4)(1/6) + (1/5)(1/6) + (1/6)(1/6) \\ &= \frac{57}{(60)(6)} = \frac{19}{120} = 0.15833. \end{aligned}$$

(c) By Bayes' rule, we obtain

$$\begin{aligned} P[N_1 = 4 | N_2 = 3] &= \frac{P[\{N_1 = 4\} \cap \{N_2 = 3\}]}{P[N_2 = 3]} \\ &= \frac{P[N_1 = 4, N_2 = 3]}{P[N_2 = 3]} \\ &= \frac{P[N_2 = 3 | N_1 = 4] P[N_1 = 4]}{P[N_2 = 3]} \\ &= \frac{(1/4)(1/6)}{(19/120)} = \frac{5}{19}. \end{aligned}$$

(d) By Bayes' rule, we obtain

$$P[N_1 = 4 | N_2 = 5] = \frac{P[N_2 = 5 | N_1 = 4] P[N_1 = 4]}{\sum_{i=1}^{6} P[N_2 = 5 | N_1 = i] P[N_1 = i]} = 0.$$

Problem 2.60:

(a) Let X denote the input and Y the output of the ternary communication channel. By the law of total probability, we have

$$P[Y = j] = \sum_{i=0}^{2} P[Y = j | X = i] P[X = i], \quad \text{for } j = 0, 1, 2.$$

This yields:

$$P[Y = 0] = (1 - \epsilon)(1/2) + (\epsilon)(1/4) = \frac{1}{2} - \frac{\epsilon}{4},$$

$$P[Y = 1] = (\epsilon)(1/2) + ((1 - \epsilon)(1/4) = \frac{1}{4} + \frac{\epsilon}{4},$$

and

$$P[Y = 2] = (\epsilon)(1/4) + ((1 - \epsilon)(1/4) = \frac{1}{4}.$$

Note that $P[Y = 0] + P[Y = 1] + P[Y = 2] = 1$, as expected.

(b) By Bayes' rule, we get

$$P[X = 0 | Y = 1] = \frac{P[Y = 1 | X = 0] P[X = 0]}{P[Y = 1]} = \frac{(\epsilon)(1/2)}{(1/4) + (\epsilon/4)} = \frac{2\epsilon}{1 + \epsilon},$$

$$P[X = 1 | Y = 1] = \frac{P[Y = 1 | X = 1] P[X = 1]}{P[Y = 1]} = \frac{(1 - \epsilon)(1/4)}{(1/4) + (\epsilon/4)} = \frac{1 - \epsilon}{1 + \epsilon},$$

and

$$P[X = 2 | Y = 1] = \frac{P[Y = 1 | X = 2] P[X = 2]}{P[Y = 1]} = 0.$$

Problem 2.64:

(a) If A and B are independent, then $P[A \cap B] = P[A]P[B]$. Hence

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = P[A] + P[B] - P[A]P[B].$$

(b) If A and B are mutually exclusive (i.e. disjoint), $A \cap B = \emptyset$. So $P[A \cap B] = 0$ and

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = P[A] + P[B]$$

Problem 2.66:

Assume that events A, B and C are independent.

(a) The event "exactly one of the three events occur" is the event D described by

$$D = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C).$$

Since D is a disjoint union, we get (by Axiom III) that

$$\begin{aligned} P[D] &= P[A \cap B^c \cap C^c] + P[A^c \cap B \cap C^c] + P[A^c \cap B^c \cap C] \\ &= P[A]P[B^c]P[C^c] + P[A^c]P[B]P[C^c] + P[A^c]P[B^c]P[C] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{(by independence)} \\ &= P[A](1 - P[B])(1 - P[C]) + (1 - P[A])P[B](1 - P[C]) \\ &\quad +(1 - P[A])(1 - P[B])P[C]. \end{aligned}$$

(b) The event "exactly two of the events occur" is the event D given by

$$D = (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C).$$

Thus

$$\begin{aligned} P[D] &= P[A \cap B \cap C^c] + P[A \cap B^c \cap C] + P[A^c \cap B \cap C] \\ &= P[A]P[B]P[C^c] + P[A]P[B^c]P[C] + P[A^c]P[B]P[C] \\ &= P[A]P[B](1 - P[C]) + P[A](1 - P[B])P[C] + (1 - P[A])P[B]P[C]. \end{aligned}$$

(c) The event "one or more of the three events occur" is the event $A \cup B \cup C$. Noting that $A \cup B \cup C = (A^c \cap B^c \cap C^c)^c$ (De Morgan's law), we get

$$\begin{aligned} P[A \cup B \cup C] &= 1 - P[A^c \cap B^c \cap C^c] \\ &= 1 - P[A^c]P[B^c]P[C^c] \quad \text{(by independence)} \\ &= 1 - (1 - P[A])(1 - P[B])(1 - P[C]). \end{aligned}$$

(d) The event "two or more of the events occur" is the event D given by

$$D = (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B \cap C).$$

Hence

$$\begin{aligned} P[D] &= P[A \cap B \cap C^c] + P[A \cap B^c \cap C] \\ &\quad +P[A^c \cap B \cap C] + P[A \cap B \cap C] \\ &= P[A]P[B]P[C^c] + P[A]P[B^c]P[C] \\ &\quad +P[A^c]P[B]P[C] + P[A]P[B]P[C] \\ &= P[A]P[B](1 - P[C]) + P[A](1 - P[B])P[C] \\ &\quad +(1 - P[A])P[B]P[C] + P[A]P[B]P[C]. \end{aligned}$$

(e) The event "none of the events occur" is $A^c \cap B^c \cap C^c$. Hence

$$P[A^c \cap B^c \cap C^c] = P[A^c]P[B^c]P[C^c] = (1 - P[A])(1 - P[B])(1 - P[C]).$$

Problem 2.70:

For $i = 0, 1$, let A_i be the event "input is i", and let B_i denote the event "output is i. Note that $A_0 = A_1^c$ and $B_0 = B_1^c$. If the input and output are independent, then $P[A_0 \cap B_0] = P[A_0]P[B_0]$. But

$$P[A_0 \cap B_0] = (1 - \epsilon)(1 - p),$$

$$P[A_0] = 1 - p,$$

and

$$P[B_0] = (1 - \epsilon)(1 - p) + p\epsilon = 1 - p - \epsilon + 2p\epsilon.$$

Setting $P[A_0 \cap B_0] = P[A_0]P[B_0]$ yields

$$\epsilon = 1/2.$$

Thus each input has an equal chance of being received perfectly or in error; so the channel cannot transmit any useful information since the receiver cannot guess from the observation which input was sent (since $P[A_0|B_i] = P[A_1|B_i] = 1/2$).