ESE 503 - Stochastic Systems

Fall 1999

Solutions to Homework # 2

Problem 3.1:

(a) The underlying space has 100 elements, with each element w_i corresponding to a bill. Thus $S = \{w_1, w_2, \cdots, w_{100}\}$. Since all bills are equiprobable,

$$P[\{w_i\}] = \frac{1}{100}, \quad \forall i = 1, \cdots, 100.$$

(b) The random variable X can have three possible values: 1, 5 or 50. So

$$S_X = \{1, 5, 50\},$$

and

$$P[X = 1] = P[\{w \in S : X(w) = 1\}] = \frac{90}{100} = 0.9,$$
$$P[X = 5] = P[\{w \in S : X(w) = 5\}] = \frac{9}{100} = 0.09,$$
$$P[X = 50] = P[\{w \in S : X(w) = 50\}] = \frac{1}{100} = 0.01.$$

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Problem 3.2: The sample space of random variable Y is

$$S_Y = \{1, 2, 3, 4\}.$$

The probabilities for Y are:

$$P[Y = 1] = P[\{w \in S : Y(w) = 1\}] = P[\{a\}] = 1/2,$$
$$P[Y = 2] = P[\{w \in S : Y(w) = 2\}] = P[\{b\}] = 1/4,$$
$$P[Y = 3] = P[\{w \in S : Y(w) = 3\}] = P[\{c\}] = 1/8,$$

and

$$P[Y = 4] = P[\{w \in S : Y(w) = 4\}] = P[\{d, e\}] = 1/16 + 1/16 = 1/8.$$

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Problem 3.4:

(a) The sample space of random variable Y is $S_Y = [0, 1)$, the interval from 0 to 1.

(b) $\{Y \leq y\} = \{\text{point inside the circle of radius } y\}$.

(c) Assuming a uniform probability assignment, we get that

$$P[Y \leq y] = \frac{\text{area of circle of radius } y}{\text{area of unit circle}} = \frac{\pi y^2}{\pi (1)^2} = y^2,$$

where $0 \leq y < 1$.

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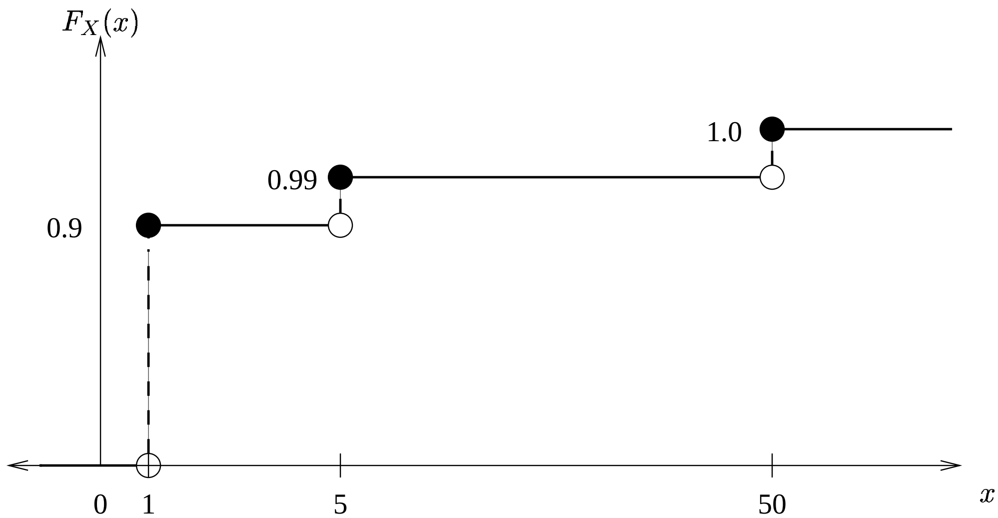
Problem 3.6: The random variable X is discrete with cdf:



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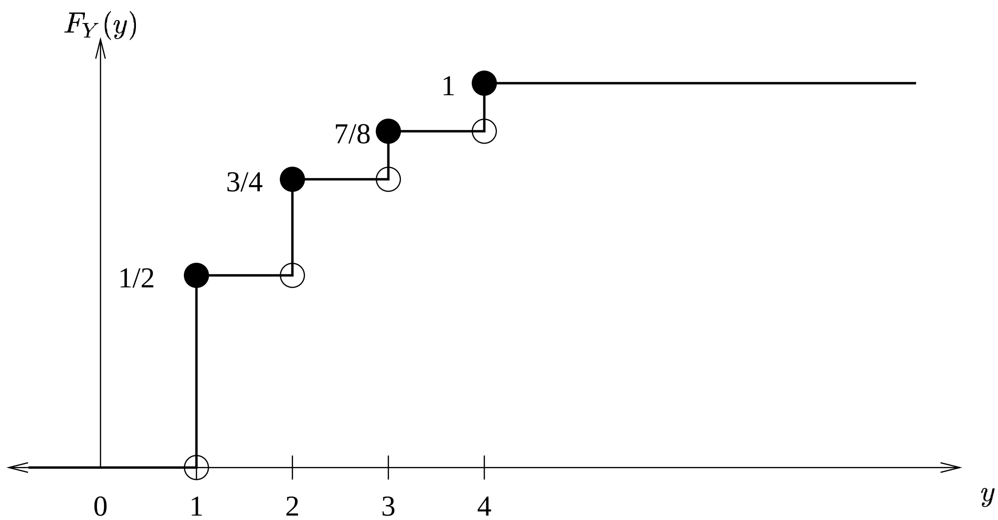
Problem 3.7: The random variable Y is discrete with cdf:



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Problem 3.12: U is a uniform continuous RV over the interval $[-1, 1]$; so

$$F_U(u) = P[U \le u] = \frac{\text{length of } \{U \le u\}}{\text{length of } [-1,1]} = \begin{cases} 0 & \text{if } \ u < -1, \\ \frac{u+1}{2} & \text{if } -1 \le u < 1, \\ 1 & \text{if } \ u \ge 1. \end{cases}$$

We thus get:

$$\begin{aligned} P[U > 0] &= 1 - P[U \le 0] = 1 - F_U(0) = 1/2, \\ P[U < 5] &= F_U(5^-) = 1, \\ P[|U| < 1/3] &= P[-1/3 < U < 1/3] = F_U((1/3)^-) - F_U(-1/3) = 1/3, \\ P[1/3 < U < 1/2] &= F_U((1/2)^-) - F_U(1/3) = 3/4 - 2/3 = 1/12, \\ P[|U| \ge 3/4] &= 1 - P[|U| < 3/4] = 1 - 6/8 = 2/8 = 1/4. \end{aligned}$$

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Problem 3.13: Consider RV X with the following cdf

$$F_X(x) = \begin{cases} 0 & \text{if } x < -1, \\ 1/3 + (2/3)(x+1)^2 & \text{if } -1 \le x \le 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Then

$$P[A] = P[X > 1/3] = 1 - P[X \le 1/3] = 1 - F_X(1/3) = 1 - 1 = 0,$$
$$P[B] = P[|X| \ge 1] = 1 - P[|X| < 1] = 1 - (F_X(1^-) - F_X(-1)) = 1/3,$$

$$\begin{aligned} P[C] &= P[|X - 1/3| < 1] = P[-2/3 < X < 4/3] \\ &= F_X((4/3)^-) - F_X(-2/3) = 1 - 11/27 = 16/27, \end{aligned}$$

and

$$P[D] = P[X < 0] = F_X(0^-) = 1.$$

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Problem 3.19: Consider RV X with the following pdf

$$f_X(x) = \begin{cases} cx(1-x) & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) To find c, set $\int_{-\infty}^{\infty} f_X(x)dx = 1$. Thus

$$\int_0^1 cx(1-x)dx = 1,$$

or

$$c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1.$$

Hence

$$c = 6.$$

(b)

$$P[1/2 \le X \le 3/4] = \int_{1/2}^{3/4} 6x(1-x)dx = \left[3x^2 - 2x^3\right]_{1/2}^{3/4} = 0.34375.$$

(c) The cdf is

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt.$$

Thus

$$F_X(x) = \begin{cases} 0 & \text{if } x \le 0, \\ \int_0^x 6t(1-t)dt = 3x^2 - 2x^3 & \text{if } 0 \le x < 1, \\ \int_0^1 f_X(t)dt = 1 & \text{if } x \ge 1. \end{cases}$$

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Problem 3.23: The cdf of a Cauchy RV X is

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_{-\infty}^{x} \frac{\alpha/\pi}{t^2 + \alpha^2}dt.$$

Make a change a variable by letting

$$t = \alpha \tan\theta,$$

so

$$dt = \frac{\alpha}{\cos^2\theta} d\theta.$$

Hence

$$F_X(x) = \int_{-\pi/2}^{\tan^{-1}(x/\alpha)} \frac{1}{\pi}\, d\theta = \frac{1}{\pi}\tan^{-1}\left(\frac{x}{\alpha}\right) + \frac{1}{2},$$

where $-\infty < x < \infty$.

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Problem 3.27: Consider an exponential RV X with parameter λ; so its pdf is

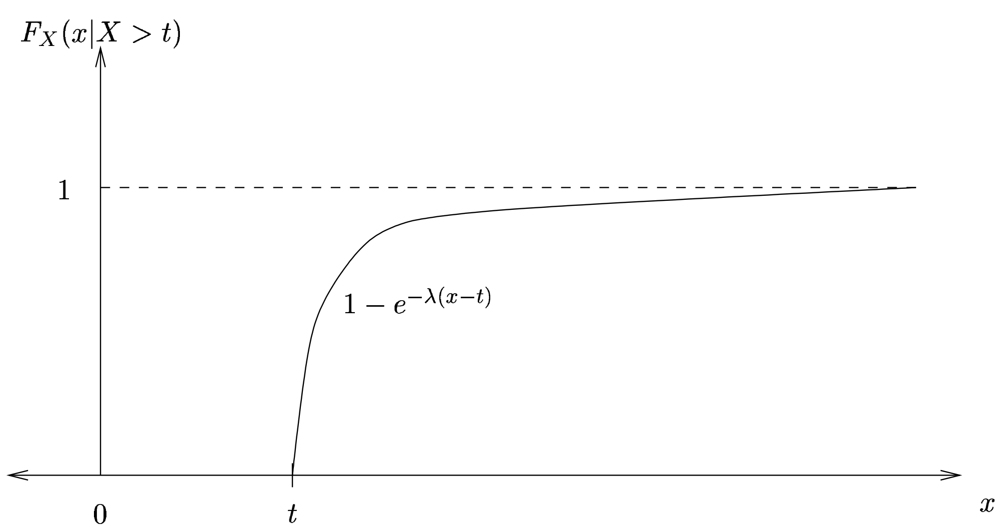
$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and its cdf is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - e^{-\lambda x} & \text{if } x \geq 0. \end{cases}$$

(a) Assume that $t > 0$. Then

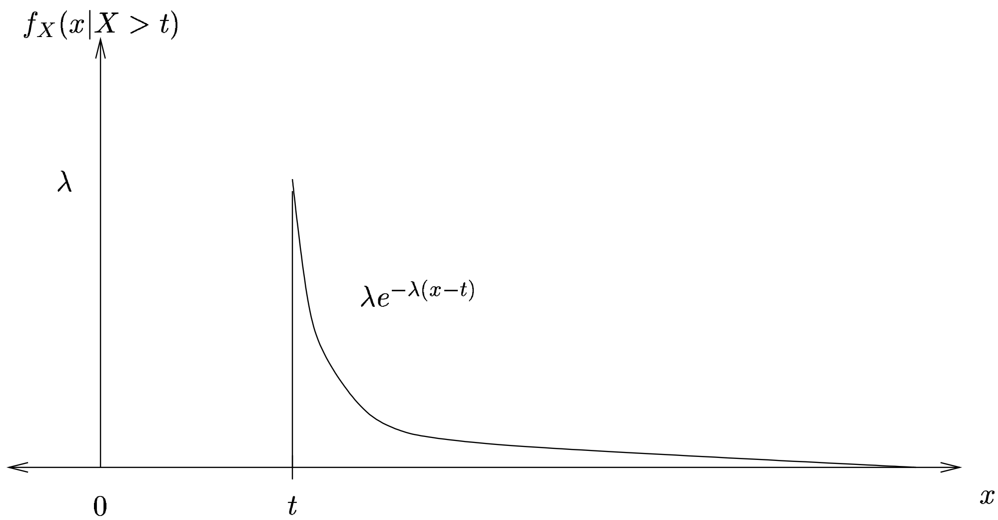
$$\begin{aligned} F_X(x|X>t) &= P[X \leq x \mid X > t] = \frac{P[\{X \leq x\} \cap \{X > t\}]}{P[X > t]} \\ &= \begin{cases} 0 & \text{if } x < t, \\ \frac{F_X(x) - F_X(t)}{1 - F_X(t)} & \text{if } x \geq t, \end{cases} \\ &= \begin{cases} 0 & \text{if } x < t, \\ \frac{e^{-\lambda t} - e^{-\lambda x}}{e^{-\lambda t}} & \text{if } x \geq t, \end{cases} \\ &= \begin{cases} 0 & \text{if } x < t, \\ 1 - e^{-\lambda(x-t)} & \text{if } x \geq t. \end{cases} \end{aligned}$$



Note that $F_X(x|X>t) = F_X(x-t)$; thus $F_X(x|X>t)$ is a delayed version of $F_X(x)$.

(b)

$$\begin{aligned} f_X(x|X>t) &= \frac{d}{dt}F_X(x|X>t) = \begin{cases} 0 & \text{if } x<t, \\ \frac{f_X(x)}{1-F_X(t)} & \text{if } x \geq t, \end{cases} \\ &= \begin{cases} 0 & \text{if } x<t, \\ \lambda e^{-\lambda(x-t)} & \text{if } x \geq t. \end{cases} \end{aligned}$$



Note also that $f_X(x|X>t)$ is a delayed version of $f_X(x)$.

(c) Let us show the *memoryless property.*

$$\begin{aligned} P[X>t+x|X>t] &= \frac{P[\{X>t+x\} \cap \{X>t\}]}{P[X>t]} \\ &= \begin{cases} \frac{P[X>t]}{P[X>t]} & \text{if } x<0, \\ \frac{P[X>x+t]}{P[X>t]} & \text{if } x \geq 0, \end{cases} \\ &= \begin{cases} 1 & \text{if } x<0, \\ \frac{1-F_X(x+t)}{1-F_X(t)} & \text{if } x \geq 0, \end{cases} \\ &= \begin{cases} 1 & \text{if } x<0, \\ \frac{1-1+e^{-\lambda(x+t)}}{1-1+e^{-\lambda t}} & \text{if } x \geq 0, \end{cases} \\ &= \begin{cases} 1 & \text{if } x<0, \\ e^{-\lambda x} & \text{if } x \geq 0, \end{cases} \\ &= P[X>x]. \end{aligned}$$

This is called the *memoryless property* because the probability of waiting additional x seconds is independent of the previous waiting time t; it is the same as if we just started to wait at time 0.

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Problem 3.34: N is a geometric RV with $S_N = \{0, 1, \cdots\}$. Thus, its pmf is

$$P[N=k] = p(1-p)^k, \qquad k \in S_N.$$

(a)

$$\begin{aligned} P[N>k] &= \sum_{i=k+1}^{\infty} P[N=i] = \sum_{i=k+1}^{\infty} p(1-p)^i \\ &= \sum_{i=0}^{\infty} p(1-p)^i - \sum_{i=0}^{k} p(1-p)^i \\ &= 1 - p\frac{1-(1-p)^{k+1}}{1-(1-p)} \\ &= 1 - 1 + (1-p)^{k+1} = (1-p)^{k+1}. \end{aligned}$$

(b) The cdf of N is

$$\begin{aligned} F_N(x) &= P[N \le x] \\ &= \begin{cases} 0 & \text{if } x < 0 \\ \sum_{i=0}^{\lfloor x \rfloor} P[N=i] & \text{if } x \ge 0 \end{cases} \\ &= \begin{cases} 0 & \text{if } x < 0 \\ p\sum_{i=0}^{\lfloor x \rfloor} (1-p)^i & \text{if } x \ge 0 \end{cases} \\ &= \begin{cases} 0 & \text{if } x < 0 \\ p\frac{1-(1-p)^{\lfloor x \rfloor + 1}}{1-(1-p)} & \text{if } x \ge 0 \end{cases} \\ &= \begin{cases} 0 & \text{if } x < 0 \\ 1-(1-p)^{\lfloor x \rfloor + 1} & \text{if } x \ge 0. \end{cases} \end{aligned}$$

(c)

$$\begin{aligned} P[\text{N even}] &= \sum_{i=0}^{\infty} p(1-p)^{2i} = p\frac{1}{1-(1-p)^2} \\ &= \frac{1}{2-p}. \end{aligned}$$

(d) Assuming that m and k are non-negative integers, we get

$$\begin{aligned} P[N=k|N\le m] &= \frac{P[\{N=k\}\cap\{N\le m\}]}{P[N\le m]} \\ &= \begin{cases} 0 & \text{if } m < k \\ \frac{P[N=k]}{P[N\le m]} & \text{if } m \ge k \end{cases} \\ &= \begin{cases} 0 & \text{if } m < k \\ \frac{p(1-p)^k}{1-(1-p)^{m+1}} & \text{if } m \ge k. \end{cases} \end{aligned}$$

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Problem 3.35: Consider a geometric RV X with sample space $S_X = \{1, 2, \cdots\}$ and pmf

$$P[X=k] = p(1-p)^{k-1}, \qquad k \in S_X.$$

Let $k, j > 0$. Then

$$P[X \ge k+j|X > j] = \frac{P[X \ge k+j \text{ and } X > j]}{P[X > j]}$$

$$
\begin{aligned}
&= \frac{P[X \geq k+j]}{P[X > j]} \\
&= \frac{\sum_{i=k+j}^{\infty} p(1-p)^{i-1}}{\sum_{i=j+1}^{\infty} p(1-p)^{i-1}} \\
&= \frac{1 - p\frac{1-(1-p)^{k+j}}{1-(1-p)}}{1 - p\frac{1-(1-p)^{j+1}}{1-(1-p)}} \\
&= \frac{(1-p)^{k+j}}{(1-p)^{j+1}} = (1-p)^{k-1}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
P[X \geq k] &= \sum_{i=k}^{\infty} p(1-p)^{i-1} = 1 - \sum_{i=1}^{k-1} p(1-p)^{i-1} \\
&= 1 - p\sum_{j=0}^{k-2}(1-p)^j = 1 - p\frac{1-(1-p)^{k-1}}{1-(1-p)} \\
&= (1-p)^{k-1}.
\end{aligned}
$$

Thus,

$$P[X \geq k+j | X > j] = P[X \geq k].$$

$\square$

X is memoryless in the sense that the probability of k additional trials until the first success is independent of how many failures have already occurred.

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Problem 3.40: Consider a Poisson RV X with pmf

$$P[X = k] = \frac{\alpha^k}{k!}e^{-\alpha}, \qquad k = 0, 1, 2, \cdots,$$

where $\alpha = \lambda/(n\mu)$.
Given that $\lambda = 3$ and $\mu = 1$, we need to find n such that

$$P[X > 4] < 0.9,$$

or

$$P[X \leq 4] > 0.1,$$

where

$$P[X \leq 4] = \sum_{k=0}^{4} \frac{\alpha^k}{k!}e^{-\alpha} = \sum_{k=0}^{4} \frac{(3/n)^k}{k!}e^{-3/n}.$$

Try different values of n ($n \geq 1$) until $P[X \leq 4] > 0.1$. If $n = 1$, we get $P[X \leq 4] = 0.815 > 0.1$! Therefore *one* employee is sufficient ($n = 1$).

$$P[\text{no orders waiting}] = P[X = 0] = e^{-\alpha} = e^{-3} = 0.05.$$

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Problem 3.43:

$$\begin{aligned}
Q(-x) &= \frac{1}{\sqrt{2\pi}}\int_{-x}^{\infty} e^{-t^2/2}dt \\
&= 1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x} e^{-t^2/2}dt \\
&= 1-\frac{1}{\sqrt{2\pi}}\int_{\infty}^{x} e^{-u^2/2}(-du) \qquad \text{where } u=-t \\
&= 1-\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-u^2/2}du \\
&= 1-Q(x).
\end{aligned}$$

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Problem 3.44: Let $X \sim N(m, \sigma^2)$. Then

$$P[X < m] = \Phi\left(\frac{m-m}{\sigma}\right) = \Phi(0) = 1/2.$$

$$\begin{aligned}
P[|X-m| > k\sigma] &= 1 - P[-k\sigma + m \le X \le m + k\sigma] \\
&= 1 - \left[\Phi\left(\frac{m+k\sigma-m}{\sigma}\right) - \Phi\left(\frac{-k\sigma+m-m}{\sigma}\right)\right] \\
&= 1 - \Phi(k) + \Phi(-k) \\
&= 1 - (1-Q(k)) + (1-Q(-k)) \\
&= Q(k) + Q(k) = 2Q(k).
\end{aligned}$$

Using Table 3.3, we get the following results.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$2Q(k)$	0.318	4.56×10^{-2}	2.7×10^{-3}	6.34×10^{-5}	5.74×10^{-7}

$$P[X > m + k\sigma] = 1 - P[X \le m + k\sigma] = 1 - \Phi\left(\frac{m+k\sigma-m}{\sigma}\right) = 1 - \Phi(k) = Q(k).$$

Using Table 3.4, we get the following.

	$k=1.28$	$k=3.09$	$k=4.26$	$k=5.20$
$Q(k)$	$\approx 10^{-1}$	$\approx 10^{-3}$	$\approx 10^{-5}$	$\approx 10^{-7}$

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Problem 3.45: Let $Y = v + N$ where $N \sim N(0,1)$.

$$\begin{aligned}
P[\text{error}|v=-1] &= P[Y \ge 0|v=-1] = P[-1+N \ge 0] \\
&= P[N \ge 1] = Q(1) = 0.159, \qquad \text{(from Table 3.3)}
\end{aligned}$$

and

$$\begin{aligned}
P[\text{error}|v=1] &= P[Y < 0|v=1] = P[1+Y < 0] \\
&= P[Y < -1] = \Phi(-1) = 1 - Q(-1) = Q(1) = 0.159.
\end{aligned}$$

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Problem 3.53: Consider RV $X \sim N(m, \sigma^2)$. Let $Y = aX + b$ such that $Y \sim N(m', \sigma'^2)$. Then, for $-\infty < y < \infty$,

$$\begin{aligned} F_Y(y) &= P[Y \le y] = P[aX + b \le y] \\ &= P\left[X \le \frac{y-b}{a}\right] \qquad \text{(assuming that } a > 0) \\ &= F_X\left(\frac{y-b}{a}\right). \end{aligned}$$

Thus

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \\ &= \frac{1}{\sqrt{2\pi} a\sigma} e^{\frac{-(y-(am+b)^2}{a^2\sigma^2}}. \end{aligned}$$

But $Y \sim N(m', \sigma'^2)$. Thus

$$m' = am + b,$$

and

$$\sigma' = a\sigma.$$

Hence

$$a = \frac{\sigma'}{\sigma},$$

and

$$b = m' - \frac{\sigma'}{\sigma} m.$$

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Problem 3.54:

(a) $Y = |X|$ where X is assumed to be a continuous random variable (input voltage). Let us first find the cdf of Y, $F_Y(y) = P[Y \le y]$.

- If $y < 0$, then $F_Y(y) = P[|X| \le y] = 0$.
- If $y \ge 0$, then

$$F_Y(y) = P[-y \le X \le y] = F_X(y) - F_X(-y).$$

Thus

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ F_X(y) - F_X(-y) & \text{if } y \ge 0. \end{cases}$$

The pdf of Y is hence

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ f_X(y) + f_X(-y) & \text{if } y \ge 0. \end{cases}$$

(b) If $y < 0$, then $P[y < Y \le y + dy] = 0$, since $Y = |X|$ is non-negative.

If $y \ge 0$, then

$$P[y < Y \le y + dy] = P[y < X \le y + dy] + P[-y - dy < X \le -y].$$

Thus

$$f_Y(y)dy = f_X(y)dy + f_X(-y)dy,$$

and so

$$f_Y(y) = f_X(y) + f_X(-y).$$

Hence

$$f_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ f_X(y) + f_X(-y) & \text{if } y \geq 0, \end{cases}$$

and we get the same answer as in part a (as expected).

(c) If $f_X(\cdot)$ is an even function, then $f_X(x) = f_X(-x)$ for all x. Thus

$$f_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ 2f_X(y) & \text{if } y \geq 0. \end{cases}$$

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Problem 3.58:

(a) Given RV X, define RV Y by $Y = h(X)$ where

$$h(x) = \begin{cases} x + a & \text{if } x < -a \\ 0 & \text{if } -a \leq x \leq a \\ x - a & \text{If } x > a. \end{cases}$$

Let us first find the cdf of Y, $F_Y(y) = P[Y \leq y]$.

- If $y < 0$, $F_Y(y) = P[X + a \leq y] = P[X \leq y - a] = F_X(y - a)$.
- If $y \geq 0$, then $F_Y(y) = P[X - a \leq y] = P[X \leq y + a] = F_X(y + a)$.

Thus

$$F_Y(y) = \begin{cases} F_X(y - a) & \text{if } y < 0 \\ F_X(y + a) & \text{if } y \geq 0. \end{cases}$$

or

$$F_Y(y) = F_X(y - a)[1 - u(y)] + F_X(y + a)u(y),$$

where $u(\cdot)$ is the unit-step function. Note that $F_Y(y)$ has a discontinuity (jump) at $y = 0$; thus Y is a mixed RV. The (generalized) pdf of Y is

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) \\ &= f_X(y - a)[1 - u(y)] + f_X(y + a)u(y) + [F_X(a) - F_X(-a^-)]\delta(y). \end{aligned}$$

(b) If X is a Laplacian RV, then its pdf is $f_X(x) = \frac{\alpha}{2}e^{-\alpha|x|}$, for $x \in \mathcal{R}$, and its cdf is

$$F_X(x) = \begin{cases} \frac{1}{2}e^{\alpha x} & \text{if } x < 0, \\ 1 - \frac{1}{2}e^{-\alpha x} & \text{if } x \geq 0. \end{cases}$$

Then

$$F_Y(y) = \begin{cases} \frac{1}{2}e^{\alpha(y-a)} & \text{if } y < 0, \\ 1 - \frac{1}{2}e^{-\alpha(y+a)} & \text{if } y \geq 0, \end{cases}$$

and

$$f_Y(y) = \frac{\alpha}{2}e^{\alpha(y-a)}[1 - u(y)] + \frac{\alpha}{2}e^{-\alpha(y+a)}u(y) + (1 - e^{-\alpha a})\delta(y).$$

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Problem 3.60: Let X be a RV with pdf

$$f_X(x) = \begin{cases} 6x(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Let Y be the area covered by a disc of radius X; thus $Y = \pi X^2$. Then, for $0 \le y \le \pi$,

$$\begin{aligned} f_Y(y) &= \left[\frac{f_X(x)}{|dy/dx|}\right]_{x=x_1} \qquad (x_1 \text{ is the positive root of } y = \pi x^2) \\ &= \left[\frac{6x(1-x)}{|2\pi x|}\right]_{x=\sqrt{y/\pi}} \\ &= 6\sqrt{y/\pi}(1-\sqrt{y/\pi})\frac{1}{2\sqrt{\pi y}} \\ &= \frac{3}{\pi}(1-\sqrt{y/\pi}). \end{aligned}$$

Thus

$$f_Y(y) = \begin{cases} \frac{3}{\pi}(1-\sqrt{y/\pi}) & \text{if } 0 \le y \le \pi, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Let Y be the volume of the sphere of radius X; then $Y = (4/3)\pi X^3$. Hence, for $0 \le y \le (4/3)\pi$,

$$\begin{aligned} f_Y(y) &= \left[\frac{f_X(x)}{|dy/dx|}\right]_{x=x_1} \qquad \left(\text{where } x_1 = ((3/4\pi)y)^{1/3}\right) \\ &= 6\left(\frac{3}{4\pi}y\right)^{1/3}\left(1-\left(\frac{3}{4\pi}y\right)^{1/3}\right)\frac{1}{4\pi(\frac{3}{4\pi}y)^{2/3}} \\ &= 2\left(\frac{3}{4\pi}\right)^{2/3} y^{-1/3}\left(1-\left(\frac{3}{4\pi}y\right)^{1/3}\right). \end{aligned}$$

Thus

$$f_Y(y) = \begin{cases} 2\left(\frac{3}{4\pi}\right)^{2/3} y^{-1/3}\left(1-\left(\frac{3}{4\pi}y\right)^{1/3}\right) & \text{if } 0 \le y \le (4/3)\pi, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Let $Y = X^n$. Then, for $0 \le y \le 1$,

$$\begin{aligned} f_Y(y) &= \left[\frac{f_X(x)}{|dy/dx|}\right]_{x=y^{1/n}} \\ &= 6y^{1/n}(1-y^{1/n})\frac{1}{ny^{\frac{n-1}{n}}} \\ &= \frac{6}{n}y^{\frac{2-n}{n}}(1-y^{1/n}). \end{aligned}$$

Thus

$$f_Y(y) = \begin{cases} \frac{6}{n}y^{\frac{2-n}{n}}(1-y^{1/n}) & \text{if } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

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Problem 3.65: Let X be a discrete uniform RV with $S_X = \{1, 2, \cdots, n\}$ and pmf $P[X = k] = 1/n$, for all $k \in S_X$. Then

$$\begin{aligned} E[X] &= \sum_{k \in S_x} kP[X = k] = \sum_{k=1}^{n} k \, \frac{1}{n} \\ &= \frac{1}{n} \sum_{k=1}^{n} k = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}. \end{aligned}$$

Furthermore,

$$\begin{aligned} E[X^2] &= \sum_{k \in S_x} k^2 P[X = k] = \sum_{k=1}^{n} k^2 \, \frac{1}{n} \\ &= \frac{1}{n} \sum_{k=1}^{n} k^2 = \frac{1}{n} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}. \end{aligned}$$

Thus

$$\begin{aligned} VAR[X] &= E[X^2] - E[X]^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} \\ &= \frac{n^2 - 1}{12}. \end{aligned}$$

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Problem 3.70: We first prove equation (3.59). Let X be a continuous non-negative RV. Then

$$\begin{aligned} E[X] &= \int_0^\infty x f_X(x) \, dx = \int_0^\infty f_X(x) \int_0^x \, dt \, dx \\ &= \int_0^\infty \int_0^x f_X(x) \, dt \, dx \\ &= \int_0^\infty \int_t^\infty f_X(x) \, dx \, dt \qquad \text{(by changing the order of integration)} \\ &= \int_0^\infty (1 - F_X(t) \, dt. \end{aligned}$$

□

We next show equation (3.60). Let X be a non-negative integer-valued RV. Then

$$\begin{aligned} \sum_{k=0}^{\infty} P[X > k] &= \sum_{k=0}^{\infty} \sum_{i=k+1}^{\infty} P[X = i] \\ &= (P[X = 1] + P[X = 2] + \cdots) \\ &\quad + (P[X = 2] + P[X = 3] + \cdots) \\ &\quad + (P[X = 3] + \cdots) + \cdots \\ &= P[X = 1] + 2P[X = 2] + 3P[X = 3] + \cdots \\ &= \sum_{k=0}^{\infty} k \; P[X = k] \\ &= E[X]. \end{aligned}$$

□

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Problem 3.75:

(a) If X is the number of heads in n tosses of a coin, then X is a Binomial(n, p) RV, where p is the probability of heads in a single coin toss. We already know from Problem # 3.66 that

$$E[X] = np \qquad \text{and} \qquad VAR[X] = np(1-p).$$

Now let Y denote the net reward in n tosses, then

$$Y = aX^2 + bX - nd.$$

So

$$\begin{aligned} E[Y] &= E[aX^2 + bX - nd] \\ &= aE[X^2] + bE[X] - nd \\ &= a\left(VAR[X] + E[X]^2\right) + bE[X] - nd \\ &= a[np(1-p) + n^2p^2] + bnp - nd. \end{aligned}$$

(b) Let $Y = a^X$, where $a > 0$ is a constant. Then

$$\begin{aligned} E[Y] &= E[a^X] \\ &= \sum_{k=0}^{n} a^k P[X = k] \\ &= \sum_{k=0}^{n} a^k \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} (ap)^k (1-p)^{n-k} \\ &= [ap + (1-p)]^n, \end{aligned}$$

where the last equality follows from the Binomial theorem:

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n.$$

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Problem 3.79: Let X be a discrete RV that is uniformly distributed on the set $S = \{1, 2, \cdots, n\}$. It can be easily verified that

$$E[X] = \frac{n+1}{2} \qquad \text{and} \qquad VAR[X] = \frac{n^2-1}{12}.$$

Let $Y = K + LX$, where K and L are fixed integers. Then

$$E[Y] = E[K + LX] = K + LE[X] = K + \frac{L(n+1)}{2},$$

and

$$VAR[Y] = VAR[K + LX] = L^2 VAR[X] = L^2 \frac{(n^2-1)}{12}.$$

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Problem 3.82: Let X be the number of successes in n Bernoulli trials where the probability of success is p. Then X is Binomial(n,p) RV with

$$P[X=k] = \binom{n}{k} p^k(1-p)^{n-k}, \qquad k = 0, 1, \cdots, n,$$

$E[X] = np$ and $VAR[X] = np(1-p)$. Let $Y = \frac{X}{n}$, so

$$E[Y] = \frac{1}{n}E[X] = \frac{np}{n} = p.$$

By Chebyshev,

$$P[|Y - E[Y]| > a] \le \frac{VAR[Y]}{a^2},$$

where $a > 0$ is fixed. But

$$VAR[Y] = VAR[X/n] = \frac{1}{n^2}VAR[X] = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}.$$

Therefore

$$P[|Y-p| > a] \le \frac{p(1-p)}{a^2 n} \to 0, \qquad \text{as } n \to \infty.$$

Thus as $n \to \infty$,

$$P[|Y-p| > a] \to 0,$$

and in this case we say that "Y converges to p in probability."

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Problem 3.89: Let X be a Laplacian RV with parameter $\alpha > 0$, then its pdf is

$$f_X(x) = \frac{\alpha}{2}e^{-\alpha|x|}, \qquad -\infty < x < \infty.$$

Then its characteristic function is

$$\begin{aligned}
\Phi_X(w) &= E[e^{jwX}] = \int_{-\infty}^{\infty} f_X(x)e^{jwx}\,dx \\
&= \int_{-\infty}^{0} \frac{\alpha}{2}e^{\alpha x}e^{jwx}\,dx \;+\; \int_{0}^{\infty} \frac{\alpha}{2}e^{-\alpha x}e^{jwx}\,dx \\
&= \frac{\alpha}{2}\frac{1}{\alpha + jw} + \frac{\alpha}{2}\frac{1}{\alpha - jw} \\
&= \frac{\alpha^2}{\alpha^2 + w^2}.
\end{aligned}$$

The Moment Theorem states that

$$E[X^n] = \frac{1}{j^n}\frac{d^n}{dw^n}\Phi_X(w)\bigg|_{w=0}.$$

Thus

$$E[X] = \frac{1}{j}\frac{d\Phi_X(w)}{dw}\bigg|_{w=0} = \frac{1}{j}\frac{-\alpha^2 2w}{(\alpha^2+w^2)^2}\bigg|_{w=0} = 0.$$

Furthermore,

$$\begin{aligned}
E[X^2] &= \frac{1}{j^2}\frac{d^2}{dw^2}\Phi_X(w)\bigg|_{w=0} \\
&= -\frac{\alpha^2}{j^2}\frac{2(\alpha^2+w^2)^2 - (2w)(4w)(\alpha^2+w^2)}{(\alpha^2+w^2)^4}\bigg|_{w=0} \\
&= \frac{2}{\alpha^2}.
\end{aligned}$$

Thus

$$VAR[X] = E[X^2] - E[X]^2 = \frac{2}{\alpha^2}.$$

Problem 3.91: Let X be a Cauchy RV with pdf given by

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \qquad -\infty < x < \infty.$$

We will show that $\Phi_X(w) = e^{-|w|}$ by verifying that the inverse Fourier transform of $e^{-|w|}$ yields the pdf of X.

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|w|}e^{-jwx}\,dw &= \frac{1}{2\pi}\int_{-\infty}^{0} e^{w}e^{-jwx}\,dw + \frac{1}{2\pi}\int_{0}^{\infty} e^{-w}e^{-jwx}\,dw \\
&= \frac{1}{2\pi}\left[\frac{1}{1-jx} + \frac{1}{1+jx}\right] \\
&= \frac{1}{\pi(1+x^2)} = f_X(x).
\end{aligned}$$

Thus

$$\Phi_X(w) = e^{-|w|}.$$

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Problem 3.100:

(a) Let $Y = aX + b$ where a and b are constants. Then

$$\begin{aligned}
\Phi_Y(w) &= E[e^{jwY}] = E[e^{jw(aX+b)}] \\
&= E[e^{jwaX}e^{jwb}] \\
&= e^{jwb}E[e^{jwaX}] \\
&= e^{jwb}\Phi_X(aw).
\end{aligned}$$

(b) If X is uniform over $[0,1]$, then

$$\Phi_X(w) = \frac{e^{jw}-1}{jw}.$$

Hence

$$\Phi_Y(w) = e^{jwb}\Phi_X(aw) = e^{jwb}\left(\frac{e^{jaw}-1}{jaw}\right).$$

(c) If $X \sim N(0,1)$, then

$$\Phi_X(w) = e^{-w^2/2}.$$

Thus

$$\Phi_Y(w) = e^{jwb}\Phi_X(aw) = e^{jwb}e^{-a^2w^2/2}.$$

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Problem 3.112: First note that, for this to be a valid pdf, $c = 1/a$. The cdf is found as

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(t)dt, \\
&= \begin{cases} 0 & \text{if } x < -a \\ (a+x)^2/2a^2 & \text{if } -a \le x < 0 \\ 1-(a-x)^2/2a^2 & \text{if } 0 \le x < a \\ 1 & \text{if } a \le x \end{cases}.
\end{aligned}$$

Let Y be a r.v. uniformly distributed on $[0,1]$. To generate X, we use the transformation:

$$\begin{aligned} X &= F_X^{-1}(Y), \\ &= \begin{cases} \sqrt{2a^2Y} - a & \text{if } 0 \le Y < 0.5 \\ a - \sqrt{2a^2(1-Y)} & \text{if } 0.5 \le Y \le 1 \end{cases}. \end{aligned}$$

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Problem 3.113: The Laplacian pdf is given by (see Table 3.2):

$$f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}.$$

Its cdf is found as

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(t)dt, \\ &= \begin{cases} \frac{1}{2}e^{\alpha x} & \text{if } x < 0 \\ 1 - \frac{1}{2}e^{-\alpha x} & \text{if } x \ge 0 \end{cases}. \end{aligned}$$

Let Y be a r.v. uniformly distributed on $[0,1]$. To generate X, we use the transformation:

$$\begin{aligned} X &= F_X^{-1}(Y), \\ &= \begin{cases} \frac{1}{\alpha}\ln(2Y) & \text{if } 0 \le Y < 0.5 \\ -\frac{1}{\alpha}\ln(2-2Y) & \text{if } 0.5 \le Y \le 1 \end{cases}. \end{aligned}$$

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Problem 3.151: Let X denote the service time. Then

$$X = \begin{cases} X_1 & \text{with probability } 1/2 \\ X_2 & \text{with probability } 1/8 \\ 2 \ \text{(a constant)} & \text{with probability } 3/8, \end{cases}$$

where X_1 is an exponential RV with parameter $\lambda_1 = 1/E[X_1] = 1$ and X_2 is an exponential RV with parameter $\lambda_2 = 1/E[X_2] = 1/10$. Recall that for an exponential RV Z with parameter λ, its cdf is given by

$$F_Z(z) = P[Z \le z] = 1 - P[Z > z] = \begin{cases} 1 - e^{-\lambda z} & \text{if } z > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, by the law of total probability, we obtain

$$\begin{aligned} P[X > 15] &= P[X > 15|X = X_1]P[X = X_1] \\ &\quad + P[X > 15|X = X_2]P[X = X_2] + P[X > 15|X = 2]P[X = 2] \\ &= P[X_1 > 15] \cdot \frac{1}{2} + P[X_2 > 15] \cdot \frac{1}{8} + (0) \cdot \frac{3}{8} \\ &= \frac{1}{2}e^{-15} + \frac{1}{8}e^{-1.5} \\ &= 0.028. \end{aligned}$$

Similarly, by the law of total probability, we have

$$\begin{aligned}
E[X] &= E[X|X = X_1]P[X = X_1] \\
&\quad +E[X|X = X_2]P[X = X_2] + E[X|X = 2]P[X = 2] \\
&= E[X_1]P[X = X_1] + E[X_2]P[X = X_2] + E[2]P[X = 2] \\
&= (1)(1/2) + (10)(1/8) + (2)(3/8) \\
&= 5/2 = 2.5.
\end{aligned}$$

Thus, the bound provided by Markov's inequality is

$$P[X \geq 15] \leq \frac{E[X]}{15} = \frac{2.5}{15} = \frac{1}{6} = 0.167.$$

Observe that the bound is rather loose.

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