ESE 503 - Stochastic Systems

Fall 1999

Solutions to Homework # 3

Problem 4.3:

(a) By independence, we have

$$\begin{aligned} P[|X| < 5, Y > 2, Z^2 \geq 2] &= P[|X| < 5]P[Y > 2]P[Z^2 \geq 2] \\ &= P[-5 < X < 5]\,(1 - P[Y \leq 2]) \\ &\quad \left(P[Z \leq -\sqrt{2}] + P[Z \geq \sqrt{2}]\right) \\ &= (F_X(5^-) - F_X(-5))\,(1 - F_Y(2)) \\ &\quad \left(F_Z(-\sqrt{2}^-) + 1 - F_Z(\sqrt{2})\right). \end{aligned}$$

(b)

$$\begin{aligned} P[X > 5, Y < 0, Z = 1] &= P[X > 5]P[Y < 0]P[Z = 1] \\ &= (1 - F_X(5))F_Y(0^-)(F_Z(1) - F_Z(1^-)). \end{aligned}$$

(c)

$$\begin{aligned} P[\min(X, Y, Z) > 2] &= P[X > 2, Y > 2, Z > 2] \\ &= P[X > 2]P[Y > 2]P[Z > 2] \\ &= (1 - F_Z(2))(1 - F_Y(2))(1 - F_Z(2)). \end{aligned}$$

(d)

$$\begin{aligned} P[\max(X, Y, Z) < 6] &= P[X < 6, Y < 6, Z < 6] \\ &= P[X < 6]P[Y < 6]P[Z < 6] \\ &= F_X(6^-)F_Y(6^-)F_Z(6^-). \end{aligned}$$

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Problem 4.11:

(a) $f_{XY}(x, y) = k$ for all points (x, y) in the regions shown in Fig. P4.1.

(i) Since

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\; dx\; dy = 1,$$

we get that

$$k(\text{area of the circle}) = 1.$$

Thus $k\pi = 1$, and $k = 1/\pi$.

(ii) Similarly $k(2) = 1$. Thus $k = 1/2$.

(ii) Similarly $k(1/2) = 1$. Thus $k = 2$.

(b) The marginal pdf's of X and Y are obtained as follows:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\ dy,$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\ dx.$$

(i) For $-1 < x < 1$,

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\ dy = \frac{2}{\pi}\sqrt{1-x^2}.$$

Thus

$$f_X(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$f_Y(y) = \begin{cases} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\ dx = \frac{2}{\pi}\sqrt{1-y^2} & \text{if } -1 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Similarly,

$$f_X(x) = \begin{cases} \int_{-x-1}^{x+1} (1/2)\ dy = x+1 & \text{if } -1 < x < 0 \\ \int_{x-1}^{-x+1} (1/2)\ dy = -x+1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases},$$

and

$$f_Y(y) = \begin{cases} \int_{-y-1}^{y+1} (1/2)\ dx = y+1 & \text{if } -1 < y < 0 \\ \int_{y-1}^{-y+1} (1/2)\ dx = -y+1 & \text{if } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Similarly,

$$f_X(x) = \begin{cases} \int_0^{-x+1} 2\ dy = 2(1-x) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases},$$

and

$$f_Y(y) = \begin{cases} \int_0^{-y+1} 2\ dx = 2(1-y) & \text{if } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

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Problem 4.15:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\ dy \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right) + \left(\frac{y-m_2}{\sigma_2}\right)^2\right]}\ dy. \end{aligned}$$

Completing the square of the argument in the exponent yields

$$\begin{aligned} &\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right) + \left(\frac{y-m_2}{\sigma_2}\right)^2 + \rho^2\left(\frac{x-m_1}{\sigma_1}\right)^2 - \rho^2\left(\frac{x-m_1}{\sigma_1}\right)^2 \\ &= \left[\left(\frac{y-m_2}{\sigma_2}\right) - \rho\left(\frac{x-m_1}{\sigma_1}\right)\right]^2 + (1-\rho^2)\left(\frac{x-m_1}{\sigma_1}\right)^2 \\ &= \left[\frac{1}{\sigma_2}\left(y - \left(m_2 + \rho\sigma_2\frac{x-m_1}{\sigma_1}\right)\right)\right]^2 + (1-\rho^2)\left(\frac{x-m_1}{\sigma_1}\right)^2. \end{aligned}$$

Thus for any $-\infty < x < \infty$,

$$\begin{aligned} f_X(x) &= \frac{e^{\frac{-(x-m_1)^2}{2\sigma_1^2}}}{\sqrt{2\pi}\sigma_1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} e^{\frac{-\left[y-\left(m_2+\rho\sigma_2\frac{x-m_1}{\sigma_1}\right)\right]^2}{2\sigma_2^2(1-\rho^2)}} \, dy \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} e^{\frac{-(x-m_1)^2}{2\sigma_1^2}}, \end{aligned}$$

where the above integral is equal to 1 since it is integral of Gaussian pdf with mean $m_2 + \rho\sigma_2(x - m_1)/\sigma_1$ and variance $\sigma_2^2(1-\rho^2)$. Hence $X \sim N(m_1, \sigma_1^2)$.
In an analogous way, it can be shown that $Y \sim N(m_2, \sigma_2^2)$.

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Problem 4.17:

(a) If $k = 1$, then

$$\begin{aligned} P[X=1, Y \le y] &= P[Y \le y | X = 1] P[X = 1] \\ &= P[N + X \le y | X = 1] P[X = 1] \\ &= P[N + 1 \le y](1/2) \\ &= \frac{1}{2} \int_{-\infty}^{y-1} f_N(t) \, dt \\ &= \begin{cases} \frac{1}{2} \int_{-\infty}^{y-1} \frac{\alpha}{2} e^{\alpha t} \, dt & \text{if } y < 1 \\ \frac{1}{2} \int_{-\infty}^{0} \frac{\alpha}{2} e^{\alpha t} \, dt + \frac{1}{2} \int_0^{y-1} \frac{\alpha}{2} e^{-\alpha t} \, dt & \text{if } y \ge 1 \end{cases} \\ &= \begin{cases} \frac{1}{4} e^{\alpha(y-1)} & \text{if } y < 1 \\ \frac{1}{4}(2 - e^{-\alpha(y-1)}) & \text{if } y \ge 1. \end{cases} \end{aligned}$$

Similarly if $k = -1$, we get

$$\begin{aligned} P[X=-1, Y \le y] &= P[Y \le y | X = -1] P[X = -1] \\ &= P[N + X \le y | X = -1] P[X = -1] \\ &= P[N - 1 \le y](1/2) \\ &= \frac{1}{2} \int_{-\infty}^{y+1} f_N(t) \, dt \\ &= \begin{cases} \frac{1}{4} e^{\alpha(y+1)} & \text{if } y < -1 \\ \frac{1}{4}(2 - e^{-\alpha(y+1)}) & \text{if } y \ge -1. \end{cases} \end{aligned}$$

(b) By the law of total probability,

$$\begin{aligned} F_Y(y) &= P[Y \le y] \\ &= P[Y \le y | X = 1] P[X = 1] + [Y \le y | X = -1] P[X = -1] \\ &= P[N + 1 \le y](1/2) + P[N - 1 \le y](1/2) \\ &= \frac{1}{2} F_N(y-1) + \frac{1}{2} F_N(y+1). \end{aligned}$$

Thus

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{1}{2} \frac{d}{dy} F_N(y-1) + \frac{1}{2} \frac{d}{dy} F_N(y+1) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} f_N(y-1) + \frac{1}{2} f_N(y+1) \\
&= \frac{\alpha}{4}\left[e^{-\alpha|y-1|} + e^{-\alpha|y+1|}\right],
\end{aligned}
$$

where $-\infty < y < \infty$.

(c) We need to compare $P[X=1|Y>0]$ versus $P[X=-1|Y>0]$.

$$
\begin{aligned}
P[X=1|Y>0] &= \frac{P[Y>0|X=1]P[X=1]}{P[Y>0]} \\
&= \frac{P[N>-1](1/2)}{P[Y>0]} \\
&= \frac{(1/2)\int_{-1}^{\infty} f_N(t)\ dt}{P[Y>0]} \\
&= \frac{1-\frac{1}{2}e^{-\alpha}}{2P[Y>0]}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
P[X=-1|Y>0] &= \frac{P[Y>0|X=-1]P[X=-1]}{P[Y>0]} \\
&= \frac{P[N>1](1/2)}{P[Y>0]} \\
&= \frac{(1/2)\int_{1}^{\infty} f_N(t)\ dt}{P[Y>0]} \\
&= \frac{\frac{1}{2}e^{-\alpha}}{2P[Y>0]}.
\end{aligned}
$$

Thus

$$
P[X=1|Y>0] - P[X=-1|Y>0] = \frac{1-e^{-\alpha}}{2P[Y>0]} > 0.
$$

Therefore, given that $Y>0$, $X=1$ is more likely.

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Problem 4.18: N is a Binomial(n,p) RV; thus its pmf is

$$
P[N=k] = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k \in \{0, 1, \cdots, n\}.
$$

Furthermore, given $N=k$, then T is an exponential RV with rate $k\alpha$; thus

$$
P[T \le t|N=k] = 1 - e^{-k\alpha t}, \qquad t > 0.
$$

Hence

$$
\begin{aligned}
P[N=k, T \le t] &= P[T \le t|N=k]P[N=k] \\
&= \left(1-e^{-k\alpha t}\right)\binom{n}{k} p^k (1-p)^{n-k}.
\end{aligned}
$$

Also, by the law of total probability,

$$\begin{aligned} P[T \le t] &= \sum_{k=0}^{n} P[T \le t | N = k] P[N = k] \\ &= \sum_{k=0}^{n} \left(1 - e^{-k\alpha t}\right) \binom{n}{k} p^k (1-p)^{n-k} \\ &= 1 - \sum_{k=0}^{n} \binom{n}{k} (pe^{-\alpha t})^k (1-p)^{n-k} \\ &= 1 - \left[pe^{-\alpha t} + (1-p)\right]^n, \end{aligned}$$

where the last equality follows from the Binomial formula. As $t \to \infty$,

$$P[T \le t] \to 1 - (1-p)^n.$$

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Problem 4.22: $f_X(x)$ and $f_Y(y)$ are functions of x and y, respectively, but $f_{X,Y}(x,y)$ =constant$\neq$ $f_X(x)f_Y(y)$. Therefore, X and Y are *not* independent in all three cases.

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Problem 4.24: X and Y are independent RV's with

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 1 & \text{if } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Thus

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1, \end{cases}$$

and

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ y & \text{if } 0 \le y < 1 \\ 1 & \text{if } y \ge 1. \end{cases}$$

(a)

$$\begin{aligned} P[X^2 < 1/2, |Y-1| < 1/2] &= P[X^2 < 1/2] P[|Y-1| < 1/2] \quad \text{by independence} \\ &= P[X < 1/\sqrt{2}] P[1/2 < Y < 3/2] \\ &= F_X(1/\sqrt{2}) \left(F_Y(3/2) - F_Y(1/2)\right) \\ &= (1/\sqrt{2})(1 - 1/2) = 1/(2\sqrt{2}) = 0.3535. \end{aligned}$$

(b)

$$\begin{aligned} P[X/2 < 1, Y > 0] &= P[X < 2] P[Y > 0] \\ &= F_X(2)(1 - F_Y(0)) = (1)(1-0) = 1. \end{aligned}$$

(c)

$$P[XY < 1/2] = 1 - \int_{1/2}^{1} \int_{1/(2y)}^{1} f_{XY}(x, y)\ dx\ dy.$$

But since X and Y are independent, $f_{XY}(x, y) = f_X(x) f_Y(y)$. Thus

$$\begin{aligned} P[XY < 1/2] &= 1 - \int_{1/2}^{1} \int_{1/(2y)}^{1} dx\ dy \\ &= (1/2) + (1/2) \ln 2 = 0.8466. \end{aligned}$$

(d)

$$\begin{aligned} P[\min(X, Y) > 1/3] &= P[X > 1/3, Y > 1/3] \\ &= P[X > 1/3] P[Y > 1/3] \\ &= (1 - F_X(1/3))(1 - F_Y(1/3)) \\ &= (1 - 1/3)^2 = 4/9. \end{aligned}$$

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Problem 4.26:

(a) N and M are independent since the first n trials are independent of the remaining m trials.

(b) N is a Binomial(n,p) RV and M is a Binomial(m,p) RV. Thus

$$P[N = i] = \binom{n}{i} p^i (1-p)^{n-i}, \qquad i \in \{0, 1, \cdots, n\},$$

$$P[M = j] = \binom{m}{j} p^j (1-p)^{m-j}, \qquad j \in \{0, 1, \cdots, m\},$$

and

$$P[N = i, M = j] = P[N = i] P[M = j], \quad (i, j) \in \{0, 1, \cdots, n\} \times \{0, 1, \cdots, m\}.$$

(c) Let Z denote the total number of successes in $n + m$ trials. Then Z is a Binomial(n+m,p) RV. Hence

$$P[Z = k] = \binom{n+m}{k} p^k (1-p)^{n+m-k}, \qquad k \in \{0, 1, \cdots, n+m\}.$$

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Problem 4.37:

(a) The pmf of N is as follows. For any $k = 0, 1, \cdots$,

$$\begin{aligned} P[N = k] &= \int_0^\infty P[N = k | R = r]\ f_R(r)\ dr \\ &= \int_0^\infty \frac{r^k}{k!} e^{-r}\ \frac{\lambda (\lambda r)^{\alpha - 1} e^{-\lambda r}}{\Gamma(\alpha)}\ dr \end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda^\alpha}{k!\,\Gamma(\alpha)} \int_0^\infty r^{k+\alpha-1} e^{-(1+\lambda)r}\, dr \qquad \text{let } t = (1+\lambda)r \\
&= \frac{\lambda^\alpha}{k!\,\Gamma(\alpha)} \frac{1}{(1+\lambda)^{k+\alpha}} \underbrace{\int_0^\infty t^{k+\alpha-1} e^{-t}\, dt}_{\Gamma(k+\alpha)} \\
&= \frac{\Gamma(k+\alpha)}{k!\,\Gamma(\alpha)} \left(\frac{\lambda}{1+\lambda}\right)^\alpha \left(\frac{1}{1+\lambda}\right)^k.
\end{aligned}
$$

N is called a *generalized Binomial RV*.

(b) By the law of total expectation, we have

$$
\begin{aligned}
E[N] &= \int_0^\infty E[N|R=r]\; f_R(r)\; dr \\
&= \int_0^\infty r\; f_R(r)\; dr \\
&= E[R] = \frac{\alpha}{\lambda}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E[N^2] &= \int_0^\infty E[N^2|R=r]\; f_R(r)\; dr \\
&= \int_0^\infty (r+r^2)\; f_R(r)\; dr \\
&= E[R] + E[R^2].
\end{aligned}
$$

Thus

$$
\begin{aligned}
VAR[N] &= E[N^2] - E[N]^2 \\
&= E[R] + E[R^2] - E[R]^2 \\
&= E[R] + VAR[R] \\
&= \frac{\alpha}{\lambda} + \frac{\alpha}{\lambda^2}.
\end{aligned}
$$

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Problem 4.39:

(a)

$$
\begin{aligned}
1 &= \int_0^1 \int_0^1 \int_0^1 k(x+y+z)dxdydz \\
&= k\int_0^1 \int_0^1 (1/2+y+z)dydz \\
&= k\int_0^1 (1/2+1/2+z)dz \\
&= k(1/2+1/2+1/2) \\
&= 3k/2.
\end{aligned}
$$

Thus, $k = 2/3$.

(b) For $0 \le x \le 1$, $0 \le y \le 1$,

$$f_{X,Y}(x,y) = (2/3)\int_0^1 (x+y+z)dz = (2/3)[x+y+1/2].$$

So

$$f_{z|X,Y}(z|x,y) = \frac{f_{X,Y,Z}(x,y,z)}{f_{X,Y}(x,y)} = \begin{cases} (x+y+z)/(x+y+1/2) & \text{if } 0 \le z \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

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Problem 4.43:

(a) From Problem 4.41:

$$\begin{aligned} f_{X_1,X_2,X_3}(x_1,x_2,x_3) &= f_{X_3|X_2,X_1}(x_3|x_2,x_1)f_{X_2|X_1}(x_2|x_1)f_{X_1}(x_1) \\ &= \begin{cases} 1 \cdot \frac{1}{x_1} \cdot \frac{1}{x_2} & \text{if } 0 \le x_3 \le x_2 \le x_1 \le 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(b) Clearly, X_1 is uniformly distributed on [0,1]. Thus

$$f_{X_1}(x_1) = \begin{cases} 1 & \text{if } 0 \le x_1 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since,

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} \frac{1}{x_1} & \text{if } 0 \le x_2 \le x_1 \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

the marginal pdf of X_2 can be obtained by integrating the above over x_1:

$$\begin{aligned} f_{X_2}(x_2) &= \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)dx_1 \\ &= \int_{x_2}^{1} \frac{1}{x_1}dx_1 \\ &= -\ln x_2 \qquad \text{for } 0 \le x_2 \le 1. \end{aligned}$$

Finally, the pdf of X_3 can be obtained as follows:

$$\begin{aligned} f_{X_3}(x_3) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(x_1,x_2,x_3)dx_1dx_2 \\ &= \int_{x_3}^{1}\left[\int_{x_2}^{1} \frac{1}{x_1x_2}dx_1\right]dx_2 \\ &= \int_{x_3}^{1} \frac{-\ln x_2}{x_2}dx_2 \\ &= \frac{(\ln x_3)^2}{2}. \end{aligned}$$

(c) For $0 \le x_3 \le x_1 \le 1$, we have

$$\begin{aligned} f_{X_3|X_1}(x_3|x_1) &= \int_{-\infty}^{\infty} f_{X_2,X_3|X_1}(x_2,x_3|x_1)dx_2 \\ &= \int_{-\infty}^{\infty} \frac{f_{X_1,X_2,X_3}(x_1,x_2,x_3)}{f_{X_1}(x_1)}dx_2 \\ &= \int_{x_3}^{x_1} \frac{1}{x_1x_2}dx_2 \\ &= \frac{1}{x_1}[\ln x_1 - \ln x_3]. \end{aligned}$$

(d) Generally, we have

$$f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) = \begin{cases} \left(\prod_{i=1}^{n-1} x_i\right)^{-1} & \text{if } 0 \le x_n \le x_{n-1} \le \ldots \le x_1 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, it can be shown using induction that $\forall n \ge 1$ and for $0 \le x_n \le 1$

$$f_{X_n}(x_n) = \frac{(-1)^{n-1}(\ln x_n)^{n-1}}{(n-1)!}.$$

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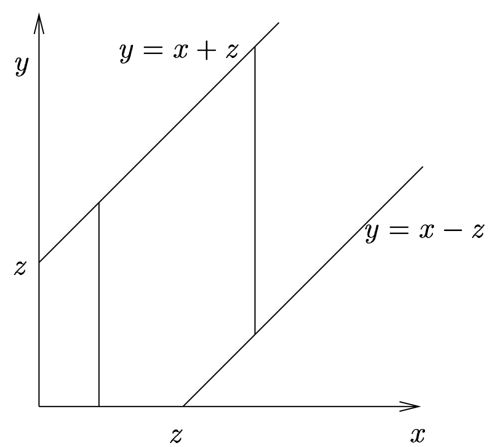
Problem 4.48: Assume that X and Y have means $1/\lambda_1$ and $1/\lambda_2$, respectively. For $z \ge 0$, we need to integrate over the following area:



$$\begin{aligned} F_Z(z) &= P(|X-Y| \le z) \\ &= \int_0^z f_X(x)\left[\int_0^{x+z} f_Y(y)dy\right]dx + \int_z^\infty f_X(x)\left[\int_{x-z}^{x+z} f_Y(y)dy\right]dx \\ &= \int_0^z \lambda_1 e^{-\lambda_1 x}\left[1-e^{-\lambda_2(x+z)}\right]dx + \int_z^\infty \lambda_1 e^{-\lambda_1 x}\left[e^{-\lambda_2(x-z)} - e^{-\lambda_2(x+z)}\right]dx \\ &= \lambda_1\left[\int_0^z e^{-\lambda_1 x}dx - e^{-\lambda_2 z}\int_0^z e^{-(\lambda_1+\lambda_2)x}dx \right. \\ &\quad \left. + e^{\lambda_2 z}\int_z^\infty e^{-(\lambda_1+\lambda_2)x}dx - e^{-\lambda_2 z}\int_z^\infty e^{-(\lambda_1+\lambda_2)x}dx\right]. \end{aligned}$$

Solving the first integral and combining the second and fourth integrals, we get

$$\begin{aligned} F_Z(z) &= \lambda_1\left[\frac{(1-e^{-\lambda_1 z})}{\lambda_1} - e^{-\lambda_2 z}\int_0^\infty e^{-(\lambda_1+\lambda_2)x}dx + e^{\lambda_2 z}\int_z^\infty e^{-(\lambda_1+\lambda_2)x}dx\right] \\ &= \left(1-e^{-\lambda_1 z}\right) - \frac{\lambda_1 e^{-\lambda_2 z}}{\lambda_1+\lambda_2} + \frac{\lambda_1 e^{\lambda_2 z} e^{-(\lambda_1+\lambda_2)z}}{\lambda_1+\lambda_2} \\ &= 1 - \frac{\lambda_1 e^{-\lambda_2 z}}{\lambda_1+\lambda_2} - \frac{\lambda_2 e^{-\lambda_1 z}}{\lambda_1+\lambda_2}. \end{aligned}$$

Taking the derivative w.r.t. z we get

$$f_Z(z) = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\left(e^{-\lambda_1 z} + e^{-\lambda_2 z}\right) \quad \text{for } z > 0.$$

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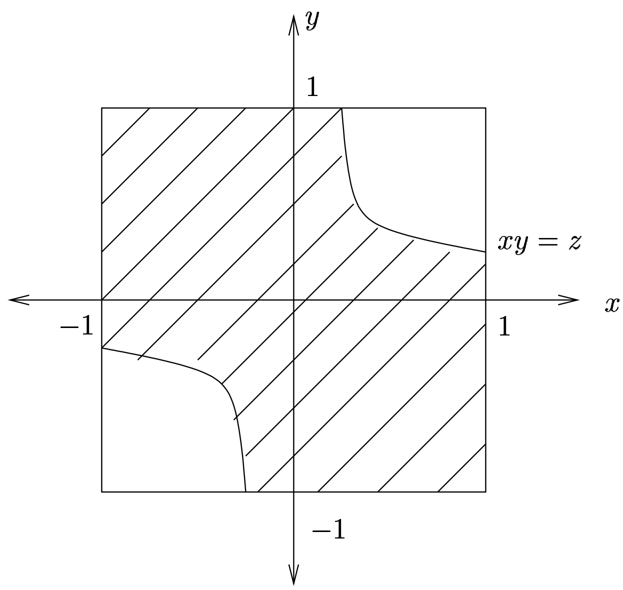
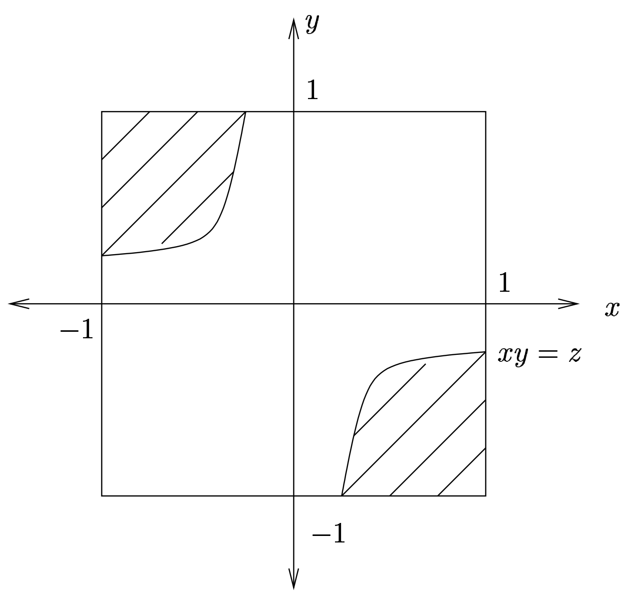
Problem 4.49: Note that $f_{X,Y}(x, y) = 1/4$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. For $0 < z \leq 1$, we need to integrate the shaded area:



$$
\begin{aligned}
F_Z(z) &= P(XY \leq z) \\
&= (1/4) \times \text{shaded area} \\
&= (1/4) \left[2 + 2 \left(z + \int_z^1 \int_0^{z/x} dy dx \right) \right] \\
&= (1/2) \left[1 + z - z \ln z \right].
\end{aligned}
$$

For $1 \leq z < 0$, we need to integrate the shaded area:



$$
\begin{aligned}
F_Z(z) &= P(XY \le z) \\
&= (1/4) \times \text{shaded area} \\
&= (1/4)2\left[1 - \left(-z + \int_{-z}^{1}\int_{0}^{z/x} dydx\right)\right] \\
&= (1/2)\left[1 + z - z\ln(-z)\right].
\end{aligned}
$$

Taking the derivative, we get

$$
f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} -(1/2)\ln z & \text{if } 0 < z \le 1 \\ -(1/2)\ln(-z) & \text{if } -1 \le z < 0 \\ 0 & \text{otherwise.} \end{cases}
$$

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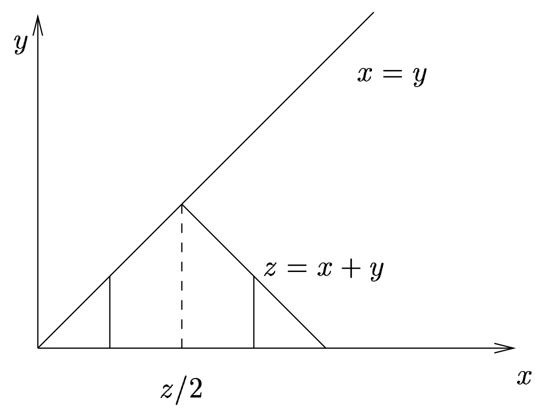
Problem 4.51:



For $z \ge 0$:

$$
\begin{aligned}
F_Z(z) &= P(X + Y \le z) \\
&= \int_0^{z/2}\int_0^{x} 2e^{-(x+y)}dydx + \int_{z/2}^{z}\int_0^{z-x} 2e^{-(x+y)}dydx \\
&= (1 - 2e^{-z/2} - e^{-z}) + (2e^{-z/2} - 2e^{-z} - ze^{-z}) \\
&= 1 - e^{-z} - ze^{-z}.
\end{aligned}
$$

Taking the derivative, we get

$$
f_Z(z) = ze^{-z}.
$$

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Problem 4.59:

(a)

$$
\begin{aligned}
E[(X+Y)^2] &= E[X^2 + 2XY + Y^2] \\
&= E[X^2] + 2E[XY] + E[Y^2].
\end{aligned}
$$

(b)

$$\begin{aligned} VAR[X+Y] &= E[(X+Y)^2] - E[X+Y]^2 \\ &= E[X^2] + 2E[XY] + E[Y^2] - E[X]^2 - E[Y]^2 - 2E[X]E[Y] \\ &= VAR[X] + VAR[Y] + 2COV(X,Y). \end{aligned}$$

(c) We have that $VAR[X+Y] = VAR[X] + VAR[Y]$ if

$$COV(X,Y) = 0,$$

or

$$E[XY] = E[X]E[Y],$$

that is if X and Y are uncorrelated.

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Problem 4.61: Let X and Y be two independent RV's, where $X \sim N(0,1)$ and Y is uniform over $[-1,3]$. Since X and Y are independent, we have that

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

for all functions $g(\cdot)$ and $h(\cdot)$. Thus

$$E[X^2Y] = E[X^2]E[Y] = (1)(1) = 1.$$

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Problem 4.67:

Let $Y = aX + b$. Then the correlation coefficient is

$$\begin{aligned} \rho &= \frac{COV(X,Y)}{\sqrt{VAR[X]\ VAR[Y]}} \\ &= \frac{E[XY] - E[X]E[Y]}{\sqrt{VAR[X]\ VAR[Y]}} \\ &= \frac{E[X(aX+b)] - E[X]E[aX+b]}{\sqrt{VAR[X]\ VAR[aX+b]}} \\ &= \frac{aE[X^2] + bE[X] - aE[X]^2 - bE[X]}{\sqrt{VAR[X]\ a^2VAR[X]}} \\ &= \frac{aVAR[X]}{VAR[X]\sqrt{a^2}} \\ &= \frac{a}{|a|} \\ &= \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0. \end{cases} \end{aligned}$$