ESE 503 - Stochastic Systems
Fall 1999

Solutions to Homework # 4

Problem 5.1:

(a) $E[X+Y+Z] = E[X] + E[Y] + E[Z] = 0$. From Eqn 5.3, we have

$$\begin{aligned}\text{Var}[X+Y+Z] &= \text{Var}[X] + \text{Var}[Y] + \text{Var}[Z] \\ &\quad +2\text{Cov}(X,Y) + 2\text{Cov}(X,Z) + 2\text{Cov}(Y,Z) \\ &= 1+1+1+2(1/4)+2(0)+2(-1/4) = 3.\end{aligned}$$

(b) From Eqn 5.3, we have

$$\begin{aligned}\text{Var}[X+Y+Z] &= \text{Var}[X] + \text{Var}[Y] + \text{Var}[Z] \\ &= 1+1+1 = 3.\end{aligned}$$

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Problem 5.2: The mean of $S_n = \sum_{i=1}^{n} X_i$ is

$$E[S_n] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \mu = n\mu.$$

From Eqn 5.3, we have

$$\begin{aligned}\text{Var}[S_n] &= \sum_{i=1}^{n} \text{Var}[X_i] + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \text{Cov}(X_i, X_j) \\ &= n\sigma^2 + 2(n-1)\rho\sigma^2.\end{aligned}$$

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Problem 5.7:

(a) From Table 3.1, we have

$$\Phi_Z(\omega) = \Phi_X(\omega)\Phi_Y(\omega) = \left(\frac{\alpha}{\alpha - j\omega}\right)\left(\frac{\beta}{\beta - j\omega}\right)$$

(b) By partial fraction, we have

$$\Phi_Z(\omega) = \left(\frac{a}{\alpha - j\omega}\right) + \left(\frac{b}{\beta - j\omega}\right),$$

where $a = (\alpha\beta)/(\beta - \alpha)$ and $b = (\alpha\beta)/(\alpha - \beta)$. Thus

$$\Phi_Z(\omega) = (a/\alpha)\left(\frac{\alpha}{\alpha - j\omega}\right) + (b/\beta)\left(\frac{\beta}{\beta - j\omega}\right).$$

Taking the inverse Fourier Transform, we get

$$\begin{aligned}f_Z(z) &= (a/\alpha)\alpha e^{-\alpha z} + (b/\beta)\beta e^{-\beta z} \\ &= ae^{-\alpha z} + be^{-\beta z} \\ &= (\alpha\beta)/(\beta - \alpha)e^{-\alpha z} + (\alpha\beta)/(\alpha - \beta)e^{-\beta z}.\end{aligned}$$

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Problem 5.16: From Eqn. 5.20, we have

$$P[|f_A(n) - p| < \epsilon] \geq 1 - \frac{p(1-p)}{n\epsilon^2} = 0.95.$$

Setting $p = 0.1, \epsilon = 0.02$, we get $n = 4500$.

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Problem 5.17:

$$\begin{aligned} M_{100} &= \frac{1}{100}(X_1 + X_2 + \ldots, X_{100}) = \frac{S_{100}}{100}, \\ \mu &= E[X_1] = \frac{1+2+3+4+5+6}{6} = 3.5, \\ \sigma^2 &= E[X_1^2] - \mu^2 = \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) - 3.5^2 = 2.91667. \end{aligned}$$

$$\begin{aligned} P[300 < S_{100} < 400] &= P[3 < \frac{S_{100}}{100} < 4] \\ &= P[-0.5 < M_{100} - 3.5 < 0.5] \\ &= P[|M_{100} - 3.5| < 0.5] \\ &\geq 1 - \frac{2.92}{100(0.5)^2} = 0.8832. \end{aligned}$$

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Problem 5.19:

$$\begin{aligned} P[|\frac{S_n}{n} - \mu| \geq \epsilon] &\leq \frac{\text{Var}[\frac{S_n}{n}]}{\epsilon^2} \\ &= \frac{\text{Var}[S_n]}{n^2\epsilon^2} \\ &= \frac{n\sigma^2 + 2(n-1)\rho\sigma^2}{n^2\epsilon^2} \to 0 \qquad \text{as} \quad n \to \infty \end{aligned}$$

Thus, the WLLN holds.

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Problem 5.21:

(a)

$$\begin{aligned} LHS &= \sum_{j=1}^{n}(X_j^2 - 2\mu X_j + \mu^2) = \left[\sum_{j=1}^{n} X_j^2\right] - 2\mu(nM_n) + n\mu^2 \\ RHS &= \sum_{j=1}^{n}(X_j^2 - 2M_nX_j + M_n^2) + n(M_n - \mu)^2 \\ &= \left[\sum_{j=1}^{n} X_j^2\right] - 2M_n(nM_n) + nM_n^2 + nM_n^2 - 2n\mu M_n + n\mu^2 \\ &= \left[\sum_{j=1}^{n} X_j^2\right] - 2\mu(nM_n) + n\mu^2 = LHS. \end{aligned}$$

(b) From part (a), we have

$$\begin{aligned} E\left[k\sum_{j=1}^{n}(X_j-M_n)^2\right] &= kE\left[\sum_{j=1}^{n}(X_j-\mu)^2 - n(M_n-\mu)^2\right] \\ &= k\left[\sum_{j=1}^{n}E[(X_j-\mu)^2]\right] - knE[(M_n-\mu)^2] \\ &= kn\sigma^2 - kn(\sigma^2/n) \\ &= k(n-1)\sigma^2 \end{aligned}$$

(c) If $k = 1/(n-1)$, then $E[V_n^2] = \sigma^2$. Thus, V_n is an unbiased variance estimator.

(c) If $k = 1/n$, then

$$E\left[\frac{1}{n}\sum_{j=1}^{n}(X_j-M_n)^2\right] = \frac{n-1}{n}\sigma^2 = \sigma^2 - \sigma^2/n.$$

This is thus a biased estimator.

We note in this case (as well as in the general case) that the best unbiased estimator is not necessarily better than the best biased estimator (in term of minimizing the mean square error).

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Problem 5.24: $S_{100} = X_1 + X_2 + \ldots + X_{100}$, where $E[X_1] = 3.5$ and $\text{Var}[X_1] = 2.92$. Thus, $E[S_{100}] = 100 \times E[X_1] = 350$ and $\text{Var}[S_{100}] = 100 \times \text{Var}[X] = 292$.

$$\begin{aligned} P[300 < S_{100} < 400] &= P\left[\frac{300-350}{\sqrt{292}} < \frac{S_{100}-350}{\sqrt{292}} < \frac{400-350}{\sqrt{292}}\right] \\ &\approx 1 - 2Q(2.926) = 0.998. \end{aligned}$$

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Problem 5.25: $S_{16} = X_1 + X_2 + \ldots + X_{16}$, where $E[X_1] = (1/\lambda) = 36$ and $\text{Var}[X_1] = (1/\lambda^2) = 36^2$.
Thus, $E[S_{16}] = 16 \times E[X_1] = (16)(36)$ and $\text{Var}[S_{16}] = 16 \times \text{Var}[X] = (16)(36)^2$.

$$\begin{aligned} P[S_{16} < 600] &= P\left[\frac{S_{16}-(16)(36)}{\sqrt{(16)(36)^2}} < \frac{600-(16)(36)}{\sqrt{(16)(36)^2}}\right] \\ &\approx 1 - Q(1/6) = 0.5692. \end{aligned}$$

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Problem 5.41:

(a) $X_n(\omega) = \omega^n$. This sequence converges to 0 for all $\omega \in [0,1)$. It converges to 1 if $\omega = 1$. Let $X(\omega) = 0\ \forall \omega \in [0,1]$. Since $P[\omega \in [0,1)] = 1$, X_n converges to X almost surely. By the theorem discussed in class, it also converges in probability and in distribution. Obviously, it does not converge surely. Now

$$E[(X_n - X)^2] = \int_0^1 (\omega^n - 0)^2(1)d\omega$$

$$
\begin{aligned}
&= \int_0^1 \omega^{2n} d\omega \\
&= \left. \frac{\omega^{2n+1}}{(2n+1)} \right|_0^1 \\
&= 1/(2n+1) \to 0.
\end{aligned}
$$

Thus, it converges in mean square. In summary,

$$
\begin{aligned}
X_n &\stackrel{s}{\not\longrightarrow} X \text{ (not sure convergence)} \ , \\
X_n &\stackrel{a.s.}{\longrightarrow} X \text{ (almost-sure convergence)} \ , \\
X_n &\stackrel{m.s.}{\longrightarrow} X \text{ (mean square convergence)} \ , \\
X_n &\stackrel{p}{\longrightarrow} X \text{ (convergence in probability)} \ , \\
X_n &\stackrel{d}{\longrightarrow} X \text{ (convergence in distribution)} \ .
\end{aligned}
$$

(b) $Y_n(\omega) = \cos^2(2\pi\omega)$. Since this is independent of n, we let $Y(\omega) = \cos^2(2\pi\omega)\ \forall \omega \in [0,1]$. Then

$$
\begin{aligned}
Y_n &\stackrel{s}{\longrightarrow} Y, \\
Y_n &\stackrel{a.s.}{\longrightarrow} Y, \\
Y_n &\stackrel{m.s.}{\longrightarrow} Y, \\
Y_n &\stackrel{p}{\longrightarrow} Y, \\
Y_n &\stackrel{d}{\longrightarrow} Y.
\end{aligned}
$$

[Note: It seems that there was a typo in this problem. The author probably meant to say that $Y_n(\omega) = \cos^2(2\pi n\omega)$.]

(c) $Z_n(\omega) = \cos^n(2\pi\omega)$. This sequence converges to 0 for all $\omega \in [0,1]$ — except $\omega = 0, 1/2, 1$. In these three cases, $Z_n(\omega)$ converges to ± 1. Let $Z(\omega) = 0\ \forall \omega \in [0,1]$. Since $P[\omega \in \{0, 1/2, 1\}] = 0$, we have almost-sure convergence which implies convergence in probability which in turn implies convergence in distribution. Now,

$$
\begin{aligned}
E[(Z_n - Z)^2] &= \int_0^1 (\cos^n(2\pi\omega) - 0)^2 (1) d\omega \\
&= \int_0^1 \cos^{2n}(2\pi\omega) d\omega \\
&= \frac{1}{2\pi} \int_0^{2\pi} \cos^{2n}(x) dx \qquad (x = 2\pi\omega) \\
&= \frac{1}{2\pi} \left[\frac{1}{2n-1} \cos^{2n-1}(x) \sin(x) \right]_0^{2\pi} + \frac{2n-1}{2n} \int_0^{2\pi} \cos^{2n-2}(x) dx \\
&= \frac{1}{2\pi} \frac{2n-1}{2n} \int_0^{2\pi} \cos^{2n-2}(x) dx \\
&\ \ \vdots \qquad \vdots \\
&= \frac{1}{2\pi} \prod_{k=1}^{n} \left(\frac{2k-1}{2k} \right) \int_0^{2\pi} \cos^0(x) dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\prod_{k=1}^{n}\left(\frac{2k-1}{2k}\right)\int_0^{2\pi} dx \\
&= \frac{1}{2\pi}\prod_{k=1}^{n}\left(\frac{2k-1}{2k}\right)(2\pi) \\
&= \prod_{k=1}^{n}\left(\frac{2k-1}{2k}\right) = a_n
\end{aligned}
$$

The sequence $\{a_n\}$ is monotonically decreasing and is bounded below by 0. Hence, it converges. To show that it converges to 0, we use the following argument.

$$
\begin{aligned}
\ln(a_n) &= \ln\left[\prod_{k=1}^{n}\left(\frac{2k-1}{2k}\right)\right] \\
&= \sum_{k=1}^{n}\ln\left(\frac{2k-1}{2k}\right).
\end{aligned}
$$

Using the fact that $\ln(x) \leq x - 1$, we get

$$
\begin{aligned}
\ln(a_n) &\leq \sum_{k=1}^{n}\left(\frac{2k-1}{2k} - 1\right) \\
&= \sum_{k=1}^{n}\frac{-1}{2k} \to -\infty
\end{aligned}
$$

Thus $\ln(a_n) \to -\infty$, implying that $a_n \to 0$. Thus, Z_n converges in mean square to 0. In summary,

$$
\begin{aligned}
Z_n &\stackrel{s}{\not\longrightarrow} Z, \\
Z_n &\stackrel{a.s.}{\longrightarrow} Z, \\
Z_n &\stackrel{m.s.}{\longrightarrow} Z, \\
Z_n &\stackrel{p}{\longrightarrow} Z, \\
Z_n &\stackrel{d}{\longrightarrow} Z.
\end{aligned}
$$

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Problem 5.43:

(a)

$$
Y_n = 2^n X_1 X_2 \ldots X_n = \begin{cases} 2^n & \text{if } X_1 = X_2 = \ldots = X_n = 1 \\ 0 & \text{otherwise.} \end{cases}
$$

Since

$$
P(\{\omega : X_i(\omega) = 1 \quad \forall i\}) = \lim_{n\to\infty}(1/2)^n = 0,
$$

$X_n \stackrel{a.s.}{\longrightarrow} 0.$

(b)

$$
E[(Y_n - 0)^2] = E[Y_n^2] = (2^n)^2(0.5)^n + (0)^2(1 - 0.5^n) = 2^n \to \infty.
$$

Thus $Y_n \stackrel{m.s.}{\not\longrightarrow} 0.$

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Problem 5.45: We are given that $X_n \stackrel{m.s.}{\longrightarrow} X$ and $Y_n \stackrel{m.s.}{\longrightarrow} Y$. Consider

$$\begin{aligned} E\left[((X_n + Y_n) - (X+Y))^2\right] &= E\left[((X_n - X) + (Y_n - Y))^2\right] \\ &= E\left[(X_n - X)^2\right] + E\left[(Y_n - Y)^2\right] \\ &\quad +2E\left[(X_n - X)(Y_n - Y)\right]. \end{aligned}$$

The first two terms converge to 0 by assumption. To show that the third term also converges to 0, we use Scharwz inequality:

$$E[ZW] \leq \sqrt{E[Z^2]E[W^2]}.$$

Using this inequality, we have all three terms converging to 0. Thus

$$X_n + Y_n \stackrel{m.s.}{\longrightarrow} X + Y.$$

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Problem 5.46:

(a) No, X_n does not converge in mean square to any random variable.

(b) Yes, $X_n \stackrel{d}{\longrightarrow} \mathcal{N}(0, 1/2)$.

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Problem Assigned in Class: Given $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}[0,1], P_X)$, where P_X is induced by the uniform random variable. Let

$$Y_n(\omega) = \begin{cases} n & \text{if } 0 \leq \omega < 1/n \\ 0 & \text{if } 1/n \leq \omega \leq 1 \end{cases}$$

and $Y(\omega) = 0 \quad \forall \omega \in [0,1]$.

(a) $Y_n \stackrel{s}{\not\longrightarrow} Y$. This is true because the sequence does not converge for $\omega = 0$.

(b) $Y_n \stackrel{a.s.}{\longrightarrow} Y$. This is true because the sequence converges for every $\omega \in (0,1]$.

(c) $Y_n \stackrel{p}{\longrightarrow} Y$. This is true because given any $\epsilon > 0$:

$$P(\{\omega : |Y_n(\omega) - 0| < \epsilon\}) = 1 - \frac{1}{n} \to 1.$$

(d) $Y_n \stackrel{d}{\longrightarrow} Y$. This is true because:

$$F_{Y_n}(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 - \frac{1}{n} & \text{if } 0 \leq y < n \\ 1 & \text{if } n \leq y \end{cases}$$

For each $y \in (-\infty, \infty)$, $F_{Y_n}(y) \to F_Y(y)$.

(e) $Y_n \stackrel{m.s.}{\not\longrightarrow} Y$. This is true because:

$$\begin{aligned} E[Y_n^2] &= \int_0^1 Y_n(\omega)^2 (1) d\omega \\ &= \int_0^{1/n} n^2 d\omega + \int_{1/n}^1 0^2 d\omega \\ &= n^2(1/n) = n \to \infty. \end{aligned}$$

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