ESE 503 - Stochastic Systems

Fall 1999

Solutions to Homework # 5

Problem 4.76: $E[X] = 2, E[Y] = 1, \text{Var}[X] = 1, \text{Var}[Y] = 1/4, \text{Cov}(X,Y) = -1/4$. The pdf given in this problem is "off" by a factor of $\sqrt{16/3}$.

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Problem 4.77:

$$K = \begin{bmatrix} 1 & \sqrt{1/2} & 0 \\ \sqrt{1/2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Problem 4.84:

$$K = \begin{bmatrix} \sqrt{1/2} & \sqrt{1/2} & 0 \\ -\sqrt{1/2} & \sqrt{1/2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Problem 6.3:

(a)

$$\begin{array}{rcccccccc} \omega = \text{Head} & \Longrightarrow & X_n & = & \ldots & 1 & -1 & 1 & \ldots \\ \omega = \text{Tail} & \Longrightarrow & X_n & = & \ldots & -1 & 1 & -1 & \ldots \\ & & & & & \Uparrow & & \Uparrow & \\ & & & & & n=0 & & n=2 & \end{array}$$

(b)

$$\begin{aligned} n\text{even} &\Longrightarrow P[X_n = 1] = P(\omega = \text{Head}) = 1/2 \\ n\text{odd} &\Longrightarrow P[X_n = 1] = P(\omega = \text{Tail}) = 1/2 \end{aligned}$$

(c) For k even:

$$\begin{aligned} P[X_n = 1, X_{n+k} = 1] &= 1/2 \\ P[X_n = 1, X_{n+k} = -1] &= 0 \\ P[X_n = -1, X_{n+k} = 1] &= 0 \\ P[X_n = -1, X_{n+k} = -1] &= 1/2 \end{aligned}$$

For k odd:

$$\begin{aligned} P[X_n = 1, X_{n+k} = 1] &= 0 \\ P[X_n = 1, X_{n+k} = -1] &= 1/2 \\ P[X_n = -1, X_{n+k} = 1] &= 1/2 \\ P[X_n = -1, X_{n+k} = -1] &= 0 \end{aligned}$$

(d) $E[X_n] = (1)(1/2) + (-1)(1/2) = 0$

$$R_X(k) = \begin{cases} 1 & \text{if} \quad k \text{ even}, \\ -1 & \text{if} \quad k \text{ odd}. \end{cases}$$

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Problem 6.4:

(a) The plot of X_n is a decreasing exponential with a random rate of decrease.

(b) For $0 \le x \le 1$, we have

$$F_{X_n}(x) = P[X_n \le x] = P[s^n \le x] = P[s \le x^{1/n}] = x^{1/n}.$$

(c) For $0 \le x_n, x_{n+1} \le 1$, we have

$$\begin{aligned} F_{X_n,X_{n+1}}(x_n, x_{n+1}) &= P[X_n \le x_n, X_{n+1} \le x_{n+1}] \\ &= P[s^n \le x_n, s^{n+1} \le x_{n+1}] \\ &= P[s \le x_n^{1/n}, s \le x_{n+1}^{1/(n+1)}] \\ &= \min(x_n^{1/n}, x_{n+1}^{1/(n+1)}). \end{aligned}$$

(d)

$$\begin{aligned} m_X(n) &= E[X_n] = E[s^n] = \int_0^1 s^n ds = 1/(n+1) \\ R_X(n_1, n_2) &= E[X_{n_1} X_{n_2}] = E[s^{n_1} s^{n_2}] = E[s^{n_1+n_2}] = 1/(n_1+n_2+1) \\ C_X(n_1, n_2) &= 1/(n_1+n_2+1) - 1/[(n_1+1)(n_2+1)] \end{aligned}$$

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Problem 6.5:

(a) Since $g(t) = 0$ outside the interval $[0, 1]$:

$$P[X(t) = 0] = 1 \quad \forall t \notin [0, 1].$$

For $t \in [0, 1]$, we have

$$P[X(t) = 1] = P[X(t) = -1] = 1/2.$$

(b) $m_X(t) = 0 \quad \forall t.$

(c) For $t \in [0, 1], t + d \in [0, 1]$:

$$\begin{aligned} P[X(t) = 1, X(t+d) = 1] &= 1/2 \\ P[X(t) = -1, X(t+d) = -1] &= 1/2 \end{aligned}$$

For $t \in [0, 1], t + d \notin [0, 1]$:

$$\begin{aligned} P[X(t) = 1, X(t+d) = 0] &= 1/2 \\ P[X(t) = -1, X(t+d) = 0] &= 1/2 \end{aligned}$$

For $t \notin [0, 1], t + d \notin [0, 1]$:

$$P[X(t) = 0, X(t+d) = 0] = 1$$

(d)

$$\begin{aligned}C_X(t,t+d) &= R_X(t,t+d) - m_X(t)m_X(t+d)\\ &= R_X(t,t+d)\\ &= \begin{cases}1 & \text{if} \quad t\in[0,1], t+d\in[0,1]\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

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Problem 6.9:

(a) The pdf of $Z(t)$ is the convolution of the pdf of At and B:

$$f_{Z(t)}(z) = \int_{-\infty}^{\infty} \frac{1}{|t|} f_A\left(\frac{z-b}{t}\right) f_B(b)db$$

(b)

$$\begin{aligned}m_X(t) &= E[A]t + E[B]\\ R_Z(t_1,t_2) &= E[A^2]t_1t_2 + E[AB](t_1+t_2) + E[B^2]\\ C_Z(t_1,t_2) &= R_Z(t_1,t_2) - m_X(t_1)m_X(t_2)\\ &= \text{Var}[A]t_1t_2 + \text{Var}[B]\end{aligned}$$

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Problem 6.13:

(a) $X(t)$ and $Y(t)$ are independent iff $X(t)$ and $Y(t)$ are uncorrelated.

(b) If at least one of the processes is zero mean, then independence is equivalent to orthogonality.

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Problem 6.15:

(a)

$$\begin{aligned}m_Z(t) &= m_Xt + m_Y\\ C_Z(t_1,t_2) &= \sigma_X^2 t_1t_2 + \sigma_X\sigma_Y\rho_{X,Y}(t_1+t_2) + \sigma_Y^2\end{aligned}$$

(b)

$$f_{Z(t)}(z) = \frac{\exp\left\{-\frac{(z-m_Xt-m_Y)^2}{2(\sigma_X^2t^2+2\sigma_X\sigma_Y\rho_{X,Y}t+\sigma_Y^2)}\right\}}{\sqrt{2\pi(\sigma_X^2t^2+2\sigma_X\sigma_Y\rho_{X,Y}t+\sigma_Y^2)}}$$

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Problem 6.18:

(a)

$$\begin{aligned}
m_Z(t) &= m_X(t)\cos(\omega t) + m_Y(t)\sin(\omega t) = 0 \\
C_Z(t_1,t_2) &= R_Z(t_1,t_2) \\
&= E\left[(X(t_1)\cos(\omega t_1) + Y(t_1)\sin(\omega t_1))(X(t_2)\cos(\omega t_2) + Y(t_2)\sin(\omega t_2))\right] \\
&= E[X(t_1)X(t_2)]\cos(\omega t_1)\cos(\omega t_2) \\
&\quad + \underbrace{E[X(t_1)Y(t_2)]}_{=0}\cos(\omega t_1)\sin(\omega t_2) + \underbrace{E[Y(t_1)X(t_2)]}_{=0}\sin(\omega t_1)\cos(\omega t_2) \\
&\quad + E[Y(t_1)Y(t_2)]\sin(\omega t_1)\sin(\omega t_2) \\
&= C_X(t_1,t_2)\cos(\omega t_1)\cos(\omega t_2) + C_Y(t_1,t_2)\sin(\omega t_1)\sin(\omega t_2) \\
&= C(t_1,t_2)[\cos(\omega t_1)\cos(\omega t_2) + \sin(\omega t_1)\sin(\omega t_2)] \\
&= C(t_1,t_2)\cos(\omega(t_1 - t_2))
\end{aligned}$$

(b)

$$f_{Z(t)}(z) = \frac{\exp\{-z^2/2C(t,t)\}}{\sqrt{2\pi C(t,t)}}$$

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Problem 6.23c: Assume that X_n is Bernoulli with parameter $p = P[X_n = 1]$.

$$\begin{aligned}
m_Y(n) &= E[Y_n] = (1/2)E[X_n] + (1/2)E[X_{n-1}] = (1/2)p + (1/2)p = p \\
C_Y(n,n+k) &= \begin{cases} \frac{1}{2}(p-p^2) & \text{if} \quad k = 0 \\ \frac{1}{4}(p-p^2) & \text{if} \quad k = \pm 1 \\ 0 & \text{if} \quad |k| > 1 \end{cases} \\
m_Z(n) &= E[Z_n] = (2/3)E[X_n] + (1/3)E[X_{n-1}] = (2/3)p + (1/3)p = p \\
C_Z(n,n+k) &= \begin{cases} \frac{5}{9}(p-p^2) & \text{if} \quad k = 0 \\ \frac{2}{9}(p-p^2) & \text{if} \quad k = \pm 1 \\ 0 & \text{if} \quad |k| > 1 \end{cases}
\end{aligned}$$

Both Y_n and Z_n are WSS.

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Problem 6.24:

(b) For $n \geq 1$:

$$\begin{aligned}
W_n &= 2W_{n-1} + X_n \\
&= 2(W_{n-2} + X_{n-1}) + X_n \\
&= 2^{n-1}X_1 + 2^{n-2}X_2 + \ldots + 2X_{n-1} + X_n \\
&= \sum_{k=0}^{n-1} 2^k X_{n-k} \\
E[W_n] &= \sum_{k=0}^{n-1} 2^k E[X_{n-k}] \\
&= \left(\sum_{k=0}^{n-1} 2^k\right)(1/2) \\
&= \frac{1-2^n}{1-2}(1/2) \\
&= 2^{n-1} - 1/2
\end{aligned}$$

For Z_n, we obtain:

$$\begin{aligned} Z_n &= \sum_{k=0}^{n-1}(1/2)^k X_{n-k} \\ E[Z_n] &= \frac{1-(1/2)^n}{4} \end{aligned}$$

(c) Neither W_n nor Z_n have independent increments.

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Problem 6.25:

$$M_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \ldots + X_n}{n}.$$

(a)

$$\begin{aligned} E[M_n] &= \frac{1}{n}E\left[\sum_{k=1}^{n} X_k\right] \stackrel{iid}{=} E[X_1] \\ C_M(n,k) &= E[(M_n - E[X_1])(M_k - E[X_1])] \\ &= E\left[\frac{1}{n}(S_n - nE[X_1])\frac{1}{k}(S_k - kE[X_1])\right] \\ &= \frac{1}{nk}C_S(n,k) \\ &\stackrel{(6.29)}{=} \frac{1}{nk}\min(n,k)\sigma_X^2 \\ \text{Var}[M_n] &= C_M(n,n) = \frac{\sigma_X^2}{n} \end{aligned}$$

(b) M_n does not have independent increments. It also does not have stationary increments.

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Problem 6.26: Y_n and Z_n are Gaussian random processes with mean:

$$\begin{aligned} E[Y_n] &= (1/2)E[X_n] + (1/2)E[X_{n-1}] = 0 \\ E[Z_n] &= (2/3)E[X_n] + (1/3)E[X_{n-1}] = 0 \end{aligned}$$

and variance:

$$\begin{aligned} E[Y_n^2] &= (1/4)E[X_n^2] + (1/4)E[X_{n-1}^2] = 1/2 \\ E[Z_n^2] &= (4/9)E[X_n^2] + (1/9)E[X_{n-1}^2] = 5/9 \end{aligned}$$

Thus:

$$f_{Y_n}(y) = \frac{e^{-y^2}}{\sqrt{\pi}} \quad f_{Z_n}(z) = \frac{e^{-9z^2/10}}{\sqrt{10\pi/9}}$$

Note that Y_n is identically distributed and Z_n is identically distributed. However, neither Y_n nor Z_n are independent.

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Problem 6.28:

(a) Using the characteristic function (the z-transform of the pmf):

$$G_{S_n}(z) = [G_{X_1}(z)]^n = e^{n\alpha(z-1)}$$

Thus S_n is Poisson with mean $n\alpha$:

$$P[S_n = k] = \frac{(n\alpha)^k}{k!}e^{-n\alpha}$$

(b) Assuming that $k > 0$:

$$\begin{aligned} P[S_n = i, S_{n+k} = j] &= P[S_n = i]P[S_k = j - i] \\ &= \begin{cases} \frac{(n\alpha)^i}{i!}e^{-n\alpha}\frac{(k\alpha)^{j-i}}{(j-i)!}e^{-k\alpha} & \text{if } \; j \geq i \geq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

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Problem 6.29:

(a) Since M_n is Gaussian with zero mean and variance $1/n$:

$$f_{M_n}(x) = \frac{e^{-nx^2/2}}{\sqrt{2\pi/n}}$$

(b) M_n and M_{n+k} are related to S_n and S_{n+k} by

$$\begin{aligned} M_n &= \frac{1}{n}S_n \\ M_{n+k} &= \frac{1}{n+k}S_{n+k} \\ J(S_n, S_{n+k}) &= \frac{1}{n(n+k)} \end{aligned}$$

Thus

$$\begin{aligned} f_{M_n,M_{n+k}}(x,y) &= n(n+k)f_{S_n,S_{n+k}}(nx,(n+k)y) \\ &= n(n+k)f_{S_n}(nx)f_{S_k}((n+k)y - nx) \\ &= n(n+k)\frac{e^{-(nx)^2/2n}}{\sqrt{2\pi n}}\frac{e^{-((n+k)y-nx)^2/2k}}{\sqrt{2\pi k}} \end{aligned}$$

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Problem 6.31: $\lambda = 10/60 = 1/6$ calls/minute. Let $N(t)$ be the total number of calls received up to minute t. Since the Poisson process has independent increments:

$$\begin{aligned} P[N(15) - N(0) = 0, N(60) - N(45) = 0] &= P[N(15) - N(0) = 0] \times P[N(60) - N(45) = 0] \\ &= \left(e^{-\lambda 15}\right) \times \left(e^{-\lambda 15}\right) \\ &= e^{-5}. \end{aligned}$$

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Problem 6.32: Let $C(t)$ be the number of customers who have arrived up to time t and $D(t)$ be the number of drinks dispensed up to time t.

$$\begin{aligned} P[D(t) = k] &= \sum_{n=k}^{\infty} P[D(t) = k | C(t) = n]P[C(t) = n] \\ &= \sum_{n=k}^{\infty} \left[\binom{n}{k} p^k(1-p)^{n-k}\right]\left[\frac{(\lambda t)^n}{n!}e^{-\lambda t}\right] \\ &= \left[\frac{e^{-\lambda t}}{k!}(p\lambda t)^k\right]\left[\sum_{n=k}^{\infty}\frac{(1-p)^{n-k}(\lambda t)^{n-k}}{(n-k)!}\right] \\ &= \left[\frac{e^{-\lambda t}}{k!}(p\lambda t)^k\right]\left[e^{+(1-p)\lambda t}\right] \\ &= \frac{(p\lambda t)^k e^{-p\lambda t}}{k!} \end{aligned}$$

Thus, $D(t)$ is a Possion process with arrival rate $p\lambda$.

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Problem 6.33:

(a)

$$P[N(t) = 0] = e^{-\lambda t}$$

(b)

$$P[N(t) \leq 1] = e^{-\lambda t} + \lambda t e^{-\lambda t}$$

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Problem 6.34: Let X_i=time of first arrival in line i, $i = 1, 2$.

(a)

$$\begin{aligned} P[X_1 < X_2] &= \int_0^\infty P[X_2 > x | X_1 = x] f_{X_1}(x) dx \\ &\overset{ind.}{=} \int_0^\infty P[X_2 > x] f_{X_1}(x) dx \\ &= \int_0^\infty \left(e^{-\lambda_2 x}\right) f_{X_1}(x) dx \\ &= \int_0^\infty \left(e^{-\lambda_2 x}\right) \left(\lambda_1 e^{-\lambda_1 x}\right) dx \\ &= \lambda_1 \int_0^\infty e^{-(\lambda_1 + \lambda_2) x} dx \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{aligned}$$

(b) Let $Z = \min(X_1, X_2)$= time till the first arrival.

$$\begin{aligned} P[Z > x] &= P[\min(X_1, X_2) > x] \\ &= P[X_1 > x, X_2 > x] \\ &= P[X_1 > x] P[X_2 > x] \\ &= e^{-\lambda_1 x} e^{-\lambda_2 x} = e^{-(\lambda_1 + \lambda_2) x} \end{aligned}$$

Thus, Z is exponential with mean $1/(\lambda_1 + \lambda_2)$.

(c) Let $N(t) = N_1(t) + N_2(t)$ be the total number of message arrived up to time t. Since $N(t)$ is a sum of to independent Poisson random variables, it is also a Poisson r.v. with rate $\lambda_1 + \lambda_2$.

(d) In the general case, $N(t)$ is a Poisson r.v. with rate $\lambda_1 + \lambda_2 + \ldots + \lambda_k$.

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Problem 6.37: Since the difference could take negative values, it could not be a Poisson process.

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Problem 6.46:

(a)

$$\begin{aligned} f_{Y(t)}(y) &= f_{X(t)}(y - \mu t) \\ &= \frac{e^{-(y-\mu t)^2/2\alpha t}}{\sqrt{2\pi\alpha t}} \end{aligned}$$

(b) Assuming that $s > 0$:

$$\begin{aligned}
f_{Y(t),Y(t+s)}(y_1, y_2) &= f_{X(t),X(t+s)}(y_1 - \mu t, y_2 - \mu(t+s)) \\
&= f_{X(t)}(y_1 - \mu t) f_{X(s)}(y_2 - y_1 - \mu s) \\
&= \frac{e^{-(y_1-\mu t)^2/2\alpha t}}{\sqrt{2\pi\alpha t}} \frac{e^{-(y_2-y_1-\mu s)^2/2\alpha s}}{\sqrt{2\pi\alpha s}}
\end{aligned}$$

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Problem 6.48:

(a) For fixed t and $s > 0$, $Z(t) = X(t) - aX(t-s)$ is a sum of two Gaussian r.v.s, thus it is Gaussian.

$$\begin{aligned}
m_Z(t) &= E[X(t) - aX(t-s)] = 0 \\
\text{Var}[Z(t)] &= E[(X(t) - aX(t-s))^2] \\
&= E[X^2(t)] - 2aE[X(t)X(t-s)] + a^2 E[X^2(t-s)] \\
&= \alpha t - 2a\alpha(t-s) + a^2\alpha(t-s) \\
&= \alpha t - 2a\alpha(t-s) + a^2\alpha(t-s)
\end{aligned}$$

Thus $Z(t) \sim \mathcal{N}(0, \alpha t - 2a\alpha(t-s) + a^2\alpha(t-s))$.

(b)

$$\begin{aligned}
m_Z(t) &= 0 \\
C_Z(t_1, t_2) &= E[(X(t_1) - aX(t_1 - s))(X(t_2) - aX(t_2 - s))] \\
&= C_X(t_1, t_2) - aC_X(t_1 - s, t_2) - aC_X(t_1, t_2 - s) + a^2 C_X(t_1 - s, t_2 - s) \\
&= \alpha \min(t_1, t_2) - a\alpha \min(t_1 - s, t_2) - a\alpha \min(t_1, t_2 - s) + a^2\alpha \min(t_1 - s, t_2 - s)
\end{aligned}$$

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Problem 6.49: Set $a = -1$ in prob. 6.48.

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