ESE 503 - Stochastic Systems

Fall 1999

Solutions to Homework # 6

Problem 6.50:

$$\begin{aligned} X(t,\omega) &= \omega\cos(2\pi t) \\ m_X(t) &= E[\omega]\cos(2\pi t) = 0 \\ C_X(t_1,t_2) &= \mathrm{Var}[\omega]\cos(2\pi t_1)\cos(2\pi t_2) \\ &= \frac{1}{3}\cos(2\pi t_1)\cos(2\pi t_2) \end{aligned}$$

Since $C_X(t_1,t_2)$ does not depend only on $t_1 - t_2$, the process is not WSS. Hence, it is also not SSS.

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Problem 6.51: $X(t) = \cos(\omega t + \Theta)$. From Example 6.7, $m_X(t) = 0$ and $C_X(t_1,t_2) = (1/2)\cos[\omega(t_1 - t_2)]$. Thus, $X(t)$ is WSS. It can also be checked from the n-th order joint pdf that $X(t)$ is SSS.

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Problem 6.53: $X(t) = A\cos(\omega t) + B\sin(\omega t)$.

(a)

$$\begin{aligned} E[X(t)] &= E[A\cos(\omega t) + B\sin(\omega t)] \\ &= E[A]\cos(\omega t) + E[B]\sin(\omega t) \\ &= 0 \\ C_X(t_1,t_2) &= E[(A\cos(\omega t_1) + B\sin(\omega t_1))(A\cos(\omega t_2) + B\sin(\omega t_2))] \\ &= E[A^2]\cos(\omega t_1)\cos(\omega t_2) + E[B^2]\sin(\omega t_1)\sin(\omega t_2) \\ &\quad + \underbrace{E[A]E[B]}_{=0}\cos(\omega t_1)\sin(\omega t_2) + \underbrace{E[A]E[B]}_{=0}\sin(\omega t_1)\cos(\omega t_2) \\ &= E[A^2]\cos(\omega t_1)\cos(\omega t_2) + E[B^2]\sin(\omega t_1)\sin(\omega t_2) \\ &= E[A^2](\cos(\omega t_1)\cos(\omega t_2) + \sin(\omega t_1)\sin(\omega t_2)) \\ &\quad (A \text{ and } B \text{ are iid, zero-mean r.v.}) \\ &= E[A^2](1/2)\cos(\omega(t_1 - t_2)) \end{aligned}$$

Thus $X(t)$ is WSS.

(b)

$$\begin{aligned} E[X^3(t)] &= E[(A\cos(\omega t) + B\sin(\omega t))^3] \\ &= E[A^3]\cos^3(\omega t) + 3E[A^2]E[B]\cos^2(\omega t)\sin(\omega t) \\ &\quad + 3E[A]E[B^2]\cos(\omega t)\sin^2(\omega t) + E[B^3]\sin^3(\omega t) \\ &= E[A^3]\cos^3(\omega t) + E[B^3]\sin^3(\omega t) \end{aligned}$$

If $\omega \neq 0$, the third moment of $X(t)$ varies with time. Thus, $X(t)$ is not SSS.

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Problem 6.55: $Z_n = (1/2)Z_{n-1} + X_n, \qquad Z_0 = 0$, where X_n is zero-mean iid.

(a) Solving the difference equation, we obtain

$$Z_n = \sum_{i=1}^{n} (1/2)^{n-i} X_i$$

Thus $E[Z_n] = 0$. Now assuming that $m \leq n$.

$$\begin{aligned} C_Z(m,n) &= E[Z_m Z_n] \\ &= E\left[\left(\sum_{i=1}^{m}(1/2)^{m-i}X_i\right)\left(\sum_{j=1}^{n}(1/2)^{n-j}X_j\right)\right] \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n}(1/2)^{m+n-i-j}E[X_iX_j] \\ &= \sum_{i=1}^{m}(1/2)^{m+n-2i}E[X_i^2] \\ &= E[X_1^2]\sum_{i=1}^{m}(1/2)^{m+n-2i}. \end{aligned}$$

Since

$$E[X_iX_j] = \begin{cases} 0 & \text{if } i \neq j \\ E[X_i^2] & \text{if } i = j \end{cases}.$$

From the above, we see that $C_Z(m,n)$ depends on both m and n. Thus the process is <u>not</u> WSS.

(b) From the above, we see that

$$\begin{aligned} C_Z(m,m+k) &= E[X_1^2]\sum_{i=1}^{m}(1/2)^{2m+k-2i} \\ &= E[X_1^2](1/4)^m(1/2)^k\sum_{i=1}^{m}(4)^i \\ &= E[X_1^2](1/4)^m(1/2)^k\left(\frac{4-4^{m+1}}{1-4}\right) \\ &= E[X_1^2](1/2)^k\left(\frac{4-(1/4)^{m-1}}{3}\right) \\ &= \frac{4}{3}E[X_1^2](1/2)^k \qquad \text{as } m \to \infty \end{aligned}$$

Thus, Z_n is asymptotically WSS.

(c) From the above, the pdf of Z_n approaches $\mathcal{N}(0, 4/3)$

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Problem 6.57: $Z(t) = aX(t) + bY(t)$, where $X(t)$ and $Y(t)$ are independent, zero-mean WSS processes with the same covariance function $C_X(\tau)$.

(a)

$$\begin{aligned}
E[Z(t)] &= E[aX(t)+bY(t)] \\
&= aE[X(t)]+bE[Y(t)] \\
&= 0 \\
C_Z(t_1,t_2)] &= E[(aX(t_1)+bY(t_1))(aX(t_2)+bY(t_2))] \\
&= a^2E[X(t_1)X(t_2)]+b^2E[Y(t_1)Y(t_2)] \\
&\quad +abE[X(t_1)Y(t_2)]+abE[Y(t_1)X(t_2)] \\
&= a^2C_X(t_1,t_2)+b^2C_Y(t_1,t_2) \\
&= (a^2+b^2)C_X(t_1,t_2) \\
&= (a^2+b^2)C_X(t_1-t_2)
\end{aligned}$$

Thus, $Z(t)$ is also WSS.

(b) $Z(t) \sim \mathcal{N}(0, a^2+b^2)$

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Problem 6.70:

(a) $R_X(\tau) = e^{-2\alpha|\tau|}$ is continuous at $\tau = 0$. Thus, $X(t)$ is mean-square continuous at every t.

(b)

$$\frac{\partial}{\partial\tau}R_X(\tau) = -2\alpha\text{sign}(\tau)e^{-2\alpha|\tau|}$$

$$\frac{\partial^2}{\partial^2\tau}R_X(\tau) = 4\alpha^2e^{-2\alpha|\tau|}+4\alpha\delta(\tau)$$

The discontinuity in the sign function gives rise to the delta function. Thus, $X(t)$ is not mean-square differentiable.

(c) We assume the integral starts from 0:

$$Y(t) = \int_0^t X(\tau)d\tau$$

Consider the double integral:

$$\begin{aligned}
R_Y(t_1,t_2) &= \int_0^{t_1}\int_0^{t_2} R_X(u,v)dudv \\
&= \int_0^{t_1}\int_0^{t_2} R_X(u-v)dudv \\
&= \int_0^{t_1}\int_0^{t_2} e^{-2\alpha|u-v|}dudv \\
&= \frac{\min(t_1,t_2)}{\alpha}\left(1-e^{-2\alpha(t_1-t_2)}\right)+\frac{1}{4\alpha^2}\left(e^{-2\alpha t_1}+e^{-2\alpha t_2}-1-e^{-2\alpha(t_1-t_2)}\right)
\end{aligned}$$

Since the above integral exists for $t_1 = t_2$, $X(t)$ is mean-square integrable. The mean of $Y(t)$ is zero and the covariance is given above.

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Problem 6.71: $X(t)$ is WSS with autocorrelation function

$$R_X(\tau) = \sigma^2e^{-\alpha\tau^2}$$

(a) Since $R_X(\tau)$ is continuous at $\tau = 0$, $X(t)$ is mean-square continuous at every t.

(b) Since $d^2R_X(\tau)/d\tau^2$ exists at $\tau = 0$, $X(t)$ is mean-square differentiable at every t. Let $X'(t) = dX(t)/dt$. Since $E[X(t)]$=constant,

$$E[X'(t)] = \frac{d}{dt}E[X(t)] = 0$$

Also,

$$R_{X'}(\tau) = -\frac{d^2}{d\tau^2}R_X(\tau) = 2\alpha\sigma^2 e^{-\alpha\tau^2}(1 - 2\alpha\tau^2)$$

(c) Since $X(t)$ is mean-square continuous, its mean-square integral exists. Consider

$$Y(t) = \int_0^t X(t')dt'$$

$$\begin{aligned} E[Y(t)] &= \int_0^t E[X(t)]dt \\ &= mt \end{aligned}$$

Assume $t_1 < t_2$:

$$\begin{aligned} R_Y(t_1, t_2) &= \int_0^{t_1}\int_0^{t_2} R_X(u - v)dudv \\ &= \sigma^2\left[t_1 \int_{-t_1}^{t_2-t_1} e^{-\alpha\tau^2}d\tau + t_2\int_{t_2-t_1}^{t_2} e^{-\alpha\tau^2}d\tau + \frac{1}{2\alpha}\left(-1 + e^{-\alpha t_1^2} + e^{-\alpha t_2^2} - e^{-\alpha(t_1-t_2)^2}\right)\right] \end{aligned}$$

If $t_1 \geq t_2$:

$$R_Y(t_1, t_2) = \sigma^2\left[t_2 \int_{-t_2}^{t_1-t_2} e^{-\alpha\tau^2}d\tau + t_1\int_{t_1-t_2}^{t_1} e^{-\alpha\tau^2}d\tau + \frac{1}{2\alpha}\left(-1 + e^{-\alpha t_1^2} + e^{-\alpha t_2^2} - e^{-\alpha(t_1-t_2)^2}\right)\right]$$

(d) $X(t)$ is not necessarily Gaussian even though its autocorrelation function looks like a Gaussian pdf.

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Problem 6.78: As discussed in class, $R_X(\tau) = 1$ and

$$\begin{aligned} \text{Var}[< X(t) >_T] &= \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|\tau|}{2T}\right)C_X(\tau)d\tau \\ &= \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|\tau|}{2T}\right)d\tau \\ &= (1/2T)(1/2)(4T)(1) \\ &= 1. \end{aligned}$$

Thus $X(t)$ is not mean-ergodic.

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Problem 6.79: Since $R_X(\tau) \to 0$ as $\tau \to \infty$, $X(t)$ is mean-ergodic.

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Problem 6.80: $X(t) = A\cos(\omega t)$, where A has mean m and variance σ^2.

(a)

$$\begin{aligned} < X(t) >_T &= \frac{1}{2T}\int_{-T}^{T} X(t)dt \\ &= \frac{1}{2T}\int_{-T}^{T} A\cos(\omega t)dt \\ &= \frac{1}{2T}\frac{2A\sin(\omega T)}{\omega} \\ &= A\text{sinc}(\omega T) \to 0 \quad \text{as } T \to \infty \end{aligned}$$

$E[X(t)] = E[A]\cos(\omega t) = m\cos(\omega t)$. Thus, $X(t)$ is mean-ergodic iff $m = E[A] = 0$.

(b)

$$\begin{aligned} R_X(t,t+\tau) &= E[A^2]\cos(\omega t)\cos(\omega(t+\tau)) \\ &= (\sigma^2+m^2)\cos(\omega t)\cos(\omega(t+\tau)) \\ < X(t)X(t+\tau >_T &= \frac{1}{2T}\int_{-T}^{T} A^2\cos(\omega t)\cos(\omega(t+\tau))dt \\ &= \frac{1}{2T}\int_{-T}^{T} \frac{A^2}{2}\left[\cos(\omega\tau)+\cos(\omega(2t+\tau))\right]dt \\ &= \frac{1}{2T}\frac{A^2}{2}\cos(\omega\tau)(2T) + \frac{A^2}{2}\frac{\sin(\omega(2T+\tau))-\sin(\omega(-2T+\tau))}{4\omega T} \\ &= \frac{A^2}{2}\cos(\omega\tau) \qquad \text{as } T\to\infty \end{aligned}$$

Thus

$$< X(t)X(t+\tau >_T \not\longrightarrow R_X(t,t+\tau)$$

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Problem 6.81: $X(t) = A\cos(\omega t+\Theta)$, where A has mean m and variance σ^2 and Θ is uniform on $[0,2\pi]$.

(a)

$$\begin{aligned} E[X(t)] &= E[A]E[\cos(\omega t+\Theta)] = m\cdot 0 = 0 = m_X \\ < X(t) >_T &= \frac{1}{2T}\int_{-T}^{T} A\cos(\omega t+\Theta)dt \\ &= \frac{A\sin(\omega T+\Theta)}{\omega T} \\ &\to 0 = m_X \end{aligned}$$

(b)

$$\begin{aligned} E[X(t)X(t+\tau] &= E[A^2]E[\cos(\omega t+\Theta)\cos(\omega(t+\tau)+\Theta)] \\ &= E[A^2]E[(1/2)\cos(\omega\tau)+(1/2)\cos(\omega(2t+\tau)+2\Theta)] \\ &= \frac{(\sigma^2+m^2)\cos(\omega\tau)}{2} = R_X(\tau) \\ < X(t)X(t+\tau) >_T &= \frac{1}{2T}\int_{-T}^{T} A^2\cos(\omega t+\Theta)\cos(\omega(t+\tau)+\Theta)dt \\ &= \frac{1}{2T}\int_{-T}^{T} A^2[(1/2)\cos(\omega\tau)+(1/2)\cos(\omega(2t+\tau)+2\Theta)]dt \\ &\to \frac{A^2\cos(\omega\tau)}{2} \neq R_X(\tau) \end{aligned}$$

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