HW # 1 Solution

Problem 3.1

Let us denote by r_n (b_n) the event of drawing a red (black) ball with number n. Then

$$\begin{aligned}
1.\ E_1 &= \{r_2, r_4, b_2\} \\
2.\ E_2 &= \{r_2, r_3, r_4\} \\
3.\ E_3 &= \{r_1, r_2, b_1, b_2\} \\
4.\ E_4 &= \{r_1, r_2, r_4, b_1, b_2\} \\
5.\ E_5 &= \{r_2, r_4, b_2\} \cup [\{r_2, r_3, r_4\} \cap \{r_1, r_2, b_1, b_2\}] \\
&= \{r_2, r_4, b_2\} \cup \{r_2\} = \{r_2, r_4, b_2\}
\end{aligned}$$

Problem 3.2

Solution:

Since the seven balls equally likely to be drawn, the probability of each event E_i is proportional to its cardinality.

$$P(E_1) = \frac{3}{7}, \quad P(E_2) = \frac{3}{7}, \quad P(E_3) = \frac{4}{7}, \quad P(E_4) = \frac{5}{7}, \quad P(E_5) = \frac{3}{7}$$

Problem 3.5

Let us denote by nS the event that n was produced by the source and sent over the channel, and by nC the event that n was observed at the output of the channel. Then

1)

$$\begin{aligned}
P(1C) &= P(1C|1S)P(1S) + P(1C|0S)P(0S) \\
&= .8 \cdot .7 + .2 \cdot .3 = .62
\end{aligned}$$

where we have used the fact that $P(1S) = .7$, $P(0S) = .3$, $P(1C|0S) = .2$ and $P(1C|1S) = 1 - .2 = .8$

2)

$$P(1S|1C) = \frac{P(1C, 1S)}{P(1C)} = \frac{P(1C|1S)P(1S)}{P(1C)} = \frac{.8 \cdot .7}{.62} = .9032$$

Problem 3.6

1) X can take four different values. 0, if no head shows up, 1, if only one head shows up in the four flips of the coin, 2, for two heads and 3 if the oucome of each flip is head.

2) X follows the binomial distribution with $n = 3$. Thus

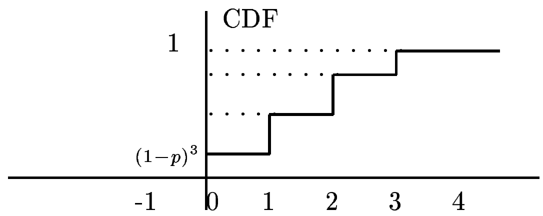
$$P(X = k) = \begin{cases} \binom{3}{k} p^k (1-p)^{3-k} & \text{for } 0 \le k \le 3 \\ 0 & \text{otherwise} \end{cases}$$

3)

$$F_X(k) = \sum_{m=0}^{k} \binom{3}{m} p^m (1-p)^{3-m}$$

Hence

$$F_X(k) = \begin{cases} 0 & k < 0 \\ (1-p)^3 & k = 0 \\ (1-p)^3 + 3p(1-p)^2 & k = 1 \\ (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p) & k = 2 \\ (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p) + p^3 = 1 & k = 3 \\ 1 & k > 3 \end{cases}$$



4)

$$P(X > 1) = \sum_{k=2}^{3} \binom{3}{k} p^k (1-p)^{3-k} = 3p^2(1-p) + p^3$$

Problem 3.7

1) The random variables X and Y follow the binomial distribution with $n = 4$ and $p = 1/4$ and $1/2$ respectively. Thus

$$p(X = 0) = \binom{4}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^4 = \frac{3^4}{2^8} \qquad p(Y = 0) = \binom{4}{0}\left(\frac{1}{2}\right)^4 = \frac{1}{2^4}$$

$$p(X = 1) = \binom{4}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^3 = \frac{3^3 4}{2^8} \qquad p(Y = 1) = \binom{4}{1}\left(\frac{1}{2}\right)^4 = \frac{4}{2^4}$$

$$p(X = 2) = \binom{4}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2 = \frac{3^3 2}{2^8} \qquad p(Y = 2) = \binom{4}{2}\left(\frac{1}{2}\right)^4 = \frac{6}{2^4}$$

$$p(X = 3) = \binom{4}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^1 = \frac{3 \cdot 4}{2^8} \qquad p(Y = 3) = \binom{4}{3}\left(\frac{1}{2}\right)^4 = \frac{4}{2^4}$$

$$p(X = 4) = \binom{4}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^0 = \frac{1}{2^8} \qquad p(Y = 4) = \binom{4}{4}\left(\frac{1}{2}\right)^4 = \frac{1}{2^4}$$

Since X and Y are independent we have

$$p(X = Y = 2) = p(X = 2)p(Y = 2) = \frac{3^3 2}{2^8}\frac{6}{2^4} = \frac{81}{1024}$$

2)

$$\begin{aligned} p(X = Y) &= p(X = 0)p(Y = 0) + p(X = 1)p(Y = 1) + p(X = 2)p(Y = 2) \\ &\quad + p(X = 3)p(Y = 3) + p(X = 4)p(Y = 4) \\ &= \frac{3^4}{2^12} + \frac{3^3 \cdot 4^2}{2^12} + \frac{3^4 \cdot 2^2}{2^12} + \frac{3 \cdot 4^2}{2^12} + \frac{1}{2^12} = \frac{886}{4096} \end{aligned}$$

3)

$$\begin{aligned} p(X > Y) &= p(Y = 0)\left[p(X = 1) + p(X = 2) + p(X = 3) + p(X = 4)\right] + \\ &\quad p(Y = 1)\left[p(X = 2) + p(X = 3) + p(X = 4)\right] + \\ &\quad p(Y = 2)\left[p(X = 3) + p(X = 4)\right] + \\ &\quad p(Y = 3)\left[p(X = 4)\right] \\ &= \frac{535}{4096} \end{aligned}$$

4) In general $p(X + Y \leq 5) = \sum_{l=0}^{5} \sum_{m=0}^{l} p(X = l - m)p(Y = m)$. However it is easier to find $p(X + Y \leq 5)$ through $p(X + Y \leq 5) = 1 - p(X + Y > 5)$ because fewer terms are involved in the

calculation of the probability $p(X+Y>5)$. Note also that $p(X+Y>5|X=0) = p(X+Y>5|X=1) = 0$.

$$\begin{aligned} p(X+Y>5) &= p(X=2)p(Y=4) + p(X=3)[p(Y=3)+p(Y=4)] + \\ &\quad p(X=4)[p(Y=2)+p(Y=3)+p(Y=4)] \\ &= \frac{125}{4096} \end{aligned}$$

Hence, $p(X+Y \leq 5) = 1 - p(X+Y>5) = 1 - \frac{125}{4096}$

Problem 3.8

1) Since $\lim_{x\to\infty} F_X(x) = 1$ and $F_X(x) = 1$ for all $x \geq 1$ we obtain $K = 1$.

2) The random variable is of the mixed-type since there is a discontinuity at $x = 1$. $\lim_{\epsilon\to 0} F_X(1-\epsilon) = 1/2$ whereas $\lim_{\epsilon\to 0} F_X(1+\epsilon) = 1$

3)

$$P(\frac{1}{2} < X \leq 1) = F_X(1) - F_X(\frac{1}{2}) = 1 - \frac{1}{4} = \frac{3}{4}$$

4)

$$P(\frac{1}{2} < X < 1) = F_X(1^-) - F_X(\frac{1}{2}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

5)

$$P(X > 2) = 1 - P(X \leq 2) = 1 - F_X(2) = 1 - 1 = 0$$

Problem 3.10

1) The random variable X is Gaussian with zero mean and variance $\sigma^2 = 10^{-8}$. Thus $p(X > x) = Q(\frac{x}{\sigma})$ and

$$\begin{aligned} p(X > 10^{-4}) &= Q\left(\frac{10^{-4}}{10^{-4}}\right) = Q(1) = .159 \\ p(X > 4\times 10^{-4}) &= Q\left(\frac{4\times 10^{-4}}{10^{-4}}\right) = Q(4) = 3.17\times 10^{-5} \\ p(-2\times 10^{-4} < X \leq 10^{-4}) &= 1 - Q(1) - Q(2) = .8182 \end{aligned}$$

2)

$$p(X > 10^{-4}|X>0) = \frac{p(X>10^{-4},\ X>0)}{p(X>0)} = \frac{p(X>10^{-4})}{p(X>0)} = \frac{.159}{.5} = .318$$

3) $y = g(x) = xu(x)$. Clearly $f_Y(y) = 0$ and $F_Y(y) = 0$ for $y < 0$. If $y > 0$, then the equation $y = xu(x)$ has a unique solution $x_1 = y$. Hence, $F_Y(y) = F_X(y)$ and $f_Y(y) = f_X(y)$ for $y > 0$. $F_Y(y)$ is discontinuous at $y = 0$ and the jump of the discontinuity equals $F_X(0)$.

$$F_Y(0^+) - F_Y(0^-) = F_X(0) = \frac{1}{2}$$

In summary the PDF $f_Y(y)$ equals

$$f_Y(y) = f_X(y)u(y) + \frac{1}{2}\delta(y)$$

The general expression for finding $f_Y(y)$ can not be used because $g(x)$ is constant for some interval so that there is an uncountable number of solutions for x in this interval.

4)

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= \int_{-\infty}^{\infty} y \left[f_X(y) u(y) + \frac{1}{2}\delta(y) \right] dy \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy = \frac{\sigma}{\sqrt{2\pi}} \end{aligned}$$

5) $y = g(x) = |x|$. For a given $y > 0$ there are two solutions to the equation $y = g(x) = |x|$, that is $x_{1,2} = \pm y$. Hence for $y > 0$

$$\begin{aligned} f_Y(y) &= \frac{f_X(x_1)}{|\text{sgn}(x_1)|} + \frac{f_X(x_2)}{|\text{sgn}(x_2)|} = f_X(y) + f_X(-y) \\ &= \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} \end{aligned}$$

For $y < 0$ there are no solutions to the equation $y = |x|$ and $f_Y(y) = 0$.

$$E[Y] = \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy = \frac{2\sigma}{\sqrt{2\pi}}$$

Problem 3.13

1)

$$\begin{aligned} E[Y] &= \int_0^{\infty} y f_Y(y) dy \geq \int_{\alpha}^{\infty} y f_Y(y) dy \\ &\geq \alpha \int_{\alpha}^{\infty} f_Y(y) dy = \alpha p(Y \geq \alpha) \end{aligned}$$

Thus $p(Y \geq \alpha) \leq E[Y]/\alpha$.

2) Clearly $p(|X - E[X]| > \epsilon) = p((X - E[X])^2 > \epsilon^2)$. Thus using the results of the previous question we obtain

$$p(|X - E[X]| > \epsilon) = p((X - E[X])^2 > \epsilon^2) \leq \frac{E[(X - E[X])^2]}{\epsilon^2} = \frac{\sigma^2}{\epsilon^2}$$

Problem 3.20

Let $Z = X + Y$. Then,

$$F_Z(z) = p(X + Y \leq z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{X,Y}(x, y) dx dy$$

Differentiating with respect to z we obtain

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} \frac{d}{dz} \int_{-\infty}^{z-y} f_{X,Y}(x, y) dx dy \\ &= \int_{-\infty}^{\infty} f_{X,Y}(z - y, y) \frac{d}{dz}(z - y) dy \\ &= \int_{-\infty}^{\infty} f_{X,Y}(z - y, y) dy \\ &= \int_{-\infty}^{\infty} f_X(z - y) f_Y(y) dy \end{aligned}$$

where the last line follows from the independence of X and Y. Thus $f_Z(z)$ is the convolution of $f_X(x)$ and $f_Y(y)$. With $f_X(x) = \alpha e^{-\alpha x} u(x)$ and $f_Y(y) = \beta e^{-\beta x} u(x)$ we obtain

$$f_Z(z) = \int_0^z \alpha e^{-\alpha v} \beta e^{-\beta(z-v)} dv$$

If $\alpha = \beta$ then

$$f_Z(z) = \int_0^z \alpha^2 e^{-\alpha z} dv = \alpha^2 z e^{-\alpha z} u_{-1}(z)$$

If $\alpha \neq \beta$ then

$$f_Z(z) = \alpha\beta e^{-\beta z} \int_0^z e^{(\beta-\alpha)v} dv = \frac{\alpha\beta}{\beta - \alpha} \left[e^{-\alpha z} - e^{-\beta z}\right] u_{-1}(z)$$