
ESE 532
HW #12 Solutions

Problem 7.3

a) To show that the waveforms $\psi_n(t)$, $n = 1, \dots, 3$ are orthogonal we have to prove that

$$\int_{-\infty}^{\infty} \psi_m(t)\psi_n(t)dt = 0, \quad m \neq n$$

Clearly,

$$\begin{aligned} c_{12} &= \int_{-\infty}^{\infty} \psi_1(t)\psi_2(t)dt = \int_0^4 \psi_1(t)\psi_2(t)dt \\ &= \int_0^2 \psi_1(t)\psi_2(t)dt + \int_2^4 \psi_1(t)\psi_2(t)dt \\ &= \frac{1}{4} \int_0^2 dt - \frac{1}{4} \int_2^4 dt = \frac{1}{4} \times 2 - \frac{1}{4} \times (4 - 2) \\ &= 0 \end{aligned}$$

Similarly,

$$\begin{aligned} c_{13} &= \int_{-\infty}^{\infty} \psi_1(t)\psi_3(t)dt = \int_0^4 \psi_1(t)\psi_3(t)dt \\ &= \frac{1}{4} \int_0^1 dt - \frac{1}{4} \int_1^2 dt - \frac{1}{4} \int_2^3 dt + \frac{1}{4} \int_3^4 dt \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} c_{23} &= \int_{-\infty}^{\infty} \psi_2(t)\psi_3(t)dt = \int_0^4 \psi_2(t)\psi_3(t)dt \\ &= \frac{1}{4} \int_0^1 dt - \frac{1}{4} \int_1^2 dt + \frac{1}{4} \int_2^3 dt - \frac{1}{4} \int_3^4 dt \\ &= 0 \end{aligned}$$

Thus, the signals $\psi_n(t)$ are orthogonal.

b) We first determine the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t)\psi_n(t)dt, \quad n = 1, 2, 3$$

$$\begin{aligned} x_1 &= \int_0^4 x(t)\psi_1(t)dt = -\frac{1}{2} \int_0^1 dt + \frac{1}{2} \int_1^2 dt - \frac{1}{2} \int_2^3 dt + \frac{1}{2} \int_3^4 dt = 0 \\ x_2 &= \int_0^4 x(t)\psi_2(t)dt = \frac{1}{2} \int_0^4 x(t)dt = 0 \\ x_3 &= \int_0^4 x(t)\psi_3(t)dt = -\frac{1}{2} \int_0^1 dt - \frac{1}{2} \int_1^2 dt + \frac{1}{2} \int_2^3 dt + \frac{1}{2} \int_3^4 dt = 0 \end{aligned}$$

As it is observed, $x(t)$ is orthogonal to the signal waveforms $\psi_n(t)$, $n = 1, 2, 3$ and thus it can not be represented as a linear combination of these functions.

Problem 7.4

a) The expansion coefficients $\{c_n\}$, that minimize the mean square error, satisfy (see Problem 2.33)

$$c_n = \int_{-\infty}^{\infty} x(t)\psi_n(t)dt = \int_0^4 \sin \frac{\pi t}{4} \psi_n(t)dt$$

Hence,

$$\begin{aligned} c_1 &= \int_0^4 \sin \frac{\pi t}{4} \psi_1(t)dt = \frac{1}{2} \int_0^2 \sin \frac{\pi t}{4} dt - \frac{1}{2} \int_2^4 \sin \frac{\pi t}{4} dt \\ &= -\frac{2}{\pi} \cos \frac{\pi t}{4} \Big|_0^2 + \frac{2}{\pi} \cos \frac{\pi t}{4} \Big|_2^4 \\ &= -\frac{2}{\pi}(0-1) + \frac{2}{\pi}(-1-0) = 0 \end{aligned}$$

Similarly,

$$\begin{aligned} c_2 &= \int_0^4 \sin \frac{\pi t}{4} \psi_2(t)dt = \frac{1}{2} \int_0^4 \sin \frac{\pi t}{4} dt \\ &= -\frac{2}{\pi} \cos \frac{\pi t}{4} \Big|_0^4 = -\frac{2}{\pi}(-1-1) = \frac{4}{\pi} \end{aligned}$$

and

$$\begin{aligned} c_3 &= \int_0^4 \sin \frac{\pi t}{4} \psi_3(t)dt \\ &= \frac{1}{2} \int_0^1 \sin \frac{\pi t}{4} dt - \frac{1}{2} \int_1^2 \sin \frac{\pi t}{4} dt + \frac{1}{2} \int_2^3 \sin \frac{\pi t}{4} dt - \frac{1}{2} \int_3^4 \sin \frac{\pi t}{4} dt \\ &= 0 \end{aligned}$$

Note that c_1, c_2 can be found by inspection since $\sin \frac{\pi t}{4}$ is even with respect to the $x = 2$ axis and $\psi_1(t), \psi_3(t)$ are odd with respect to the same axis.

b) The residual mean square error E_{\min} can be found from (see Problem 2.33)

$$E_{\min} = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^3 |c_i|^2$$

Thus,

$$\begin{aligned} E_{\min} &= \int_0^4 \left(\sin \frac{\pi t}{4} \right)^2 dt - \left(\frac{4}{\pi} \right)^2 = \frac{1}{2} \int_0^4 \left(1 - \cos \frac{\pi t}{2} \right) dt - \frac{16}{\pi^2} \\ &= 2 - \frac{1}{\pi} \sin \frac{\pi t}{2} \Big|_0^4 - \frac{16}{\pi^2} = 2 - \frac{16}{\pi^2} \end{aligned}$$

Problem 7.5

a) As an orthonormal set of basis functions we consider the set

$$\begin{aligned} \psi_1(t) &= \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{o.w} \end{cases} & \psi_2(t) &= \begin{cases} 1 & 1 \leq t < 2 \\ 0 & \text{o.w} \end{cases} \\ \psi_3(t) &= \begin{cases} 1 & 2 \leq t < 3 \\ 0 & \text{o.w} \end{cases} & \psi_4(t) &= \begin{cases} 1 & 3 \leq t < 4 \\ 0 & \text{o.w} \end{cases} \end{aligned}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \\ \psi_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

b) The representation vectors are

$$\begin{aligned}\mathbf{s}_1 &= [2 \quad -1 \quad -1 \quad -1] \\ \mathbf{s}_2 &= [-2 \quad 1 \quad 1 \quad 0] \\ \mathbf{s}_3 &= [1 \quad -1 \quad 1 \quad -1] \\ \mathbf{s}_4 &= [1 \quad -2 \quad -2 \quad 2]\end{aligned}$$

c) The distance between the first and the second vector is

$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{|[4 \quad -2 \quad -2 \quad -1]|^2} = \sqrt{25}$$

Similarly we find that

$$\begin{aligned}d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{|[1 \quad 0 \quad -2 \quad 0]|^2} = \sqrt{5} \\ d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{|[1 \quad 1 \quad 1 \quad -3]|^2} = \sqrt{12} \\ d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{|[-3 \quad 2 \quad 0 \quad 1]|^2} = \sqrt{14} \\ d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{|[-3 \quad 3 \quad 3 \quad -2]|^2} = \sqrt{31} \\ d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{|[0 \quad 1 \quad 3 \quad -3]|^2} = \sqrt{19}\end{aligned}$$

Thus, the minimum distance between any pair of vectors is $d_{\min} = \sqrt{5}$.

Problem 7.6

As a set of orthonormal functions we consider the waveforms

$$\psi_1(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{o.w.} \end{cases} \quad \psi_2(t) = \begin{cases} 1 & 1 \leq t < 2 \\ 0 & \text{o.w.} \end{cases} \quad \psi_3(t) = \begin{cases} 1 & 2 \leq t < 3 \\ 0 & \text{o.w.} \end{cases}$$

The vector representation of the signals is

$$\begin{aligned}\mathbf{s}_1 &= [2 \quad 2 \quad 2] \\ \mathbf{s}_2 &= [2 \quad 0 \quad 0] \\ \mathbf{s}_3 &= [0 \quad -2 \quad -2] \\ \mathbf{s}_4 &= [2 \quad 2 \quad 0]\end{aligned}$$

Note that $s_3(t) = s_2(t) - s_1(t)$ and that the dimensionality of the waveforms is 3.

Problem 7.18

a) The received signal may be expressed as

$$r(t) = \begin{cases} n(t) & \text{if } s_0(t) \text{ was transmitted} \\ A + n(t) & \text{if } s_1(t) \text{ was transmitted} \end{cases}$$

Assuming that $s(t)$ has unit energy, then the sampled outputs of the crosscorrelators are

$$r = s_m + n, \quad m = 0, 1$$

where $s_0 = 0$, $s_1 = A\sqrt{T}$ and the noise term n is a zero-mean Gaussian random variable with variance

$$\begin{aligned}\sigma_n^2 &= E \left[\frac{1}{\sqrt{T}} \int_0^T n(t) dt \frac{1}{\sqrt{T}} \int_0^T n(\tau) d\tau \right] \\ &= \frac{1}{T} \int_0^T \int_0^T E [n(t)n(\tau)] dt d\tau \\ &= \frac{N_0}{2T} \int_0^T \int_0^T \delta(t - \tau) dt d\tau = \frac{N_0}{2}\end{aligned}$$

The probability density function for the sampled output is

$$f(r|s_0) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}}$$

$$f(r|s_1) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}}$$

Since the signals are equally probable, the optimal detector decides in favor of s_0 if

$$\text{PM}(\mathbf{r}, \mathbf{s}_0) = f(r|s_0) > f(r|s_1) = \text{PM}(\mathbf{r}, \mathbf{s}_1)$$

otherwise it decides in favor of s_1 . The decision rule may be expressed as

$$\frac{\text{PM}(\mathbf{r}, \mathbf{s}_0)}{\text{PM}(\mathbf{r}, \mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2 - r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \begin{matrix} s_0 \\ \geq \\ s_1 \end{matrix} \quad 1$$

or equivalently

$$r \begin{matrix} s_1 \\ \geq \\ s_0 \end{matrix} \frac{1}{2} A\sqrt{T}$$

The optimum threshold is $\frac{1}{2}A\sqrt{T}$.

b) The average probability of error is

$$\begin{aligned} P(e) &= \frac{1}{2}P(e|s_0) + \frac{1}{2}P(e|s_1) \\ &= \frac{1}{2} \int_{\frac{1}{2}A\sqrt{T}}^{\infty} f(r|s_0) dr + \frac{1}{2} \int_{-\infty}^{\frac{1}{2}A\sqrt{T}} f(r|s_1) dr \\ &= \frac{1}{2} \int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} dr + \frac{1}{2} \int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}} dr \\ &= \frac{1}{2} \int_{\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + \frac{1}{2} \int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= Q \left[\frac{1}{2} \sqrt{\frac{2}{N_0}} A\sqrt{T} \right] = Q \left[\sqrt{\text{SNR}} \right] \end{aligned}$$

where

$$\text{SNR} = \frac{\frac{1}{2}A^2T}{N_0}$$

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

Problem 7.19

Since the rate of transmission is $R = 10^5$ bits/sec, the bit interval T_b is 10^{-5} sec. The probability of error in a binary PAM system is

$$P(e) = Q \left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right]$$

where the bit energy is $\mathcal{E}_b = A^2T_b$. With $P(e) = P_2 = 10^{-6}$, we obtain

$$\sqrt{\frac{2\mathcal{E}_b}{N_0}} = 4.75 \implies \mathcal{E}_b = \frac{4.75^2 N_0}{2} = 0.112813$$

Thus

$$A^2T_b = 0.112813 \implies A = \sqrt{0.112813 \times 10^5} = 106.21$$

Problem 7.21

a) The optimum threshold is given by

$$\eta = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{1-p}{p} = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$$

b) The average probability of error is ($\eta = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$)

$$\begin{aligned} P(e) &= p(a_m = -1) \int_{\eta}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{\mathcal{E}_b})^2/N_0} dr \\ &\quad + p(a_m = 1) \int_{-\infty}^{\eta} \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{\mathcal{E}_b})^2/N_0} dr \\ &= \frac{2}{3} Q \left[\frac{\eta + \sqrt{\mathcal{E}_b}}{\sqrt{N_0/2}} \right] + \frac{1}{3} Q \left[\frac{\sqrt{\mathcal{E}_b} - \eta}{\sqrt{N_0/2}} \right] \\ &= \frac{2}{3} Q \left[\frac{\sqrt{2N_0/\mathcal{E}_b} \ln 2}{4} + \sqrt{\frac{2\mathcal{E}_b}{N_0}} \right] + \frac{1}{3} Q \left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \frac{\sqrt{2N_0/\mathcal{E}_b} \ln 2}{4} \right] \end{aligned}$$

Problem 7.32

a) The PDF of the noise n is

$$f(n) = \frac{\lambda}{2} e^{-\lambda|n|}$$

The optimal receiver uses the criterion

$$\frac{f(r|A)}{f(r|-A)} = e^{-\lambda[|r-A|-|r+A|]} \begin{matrix} A \\ \geq \\ -A \end{matrix} 1 \implies r \begin{matrix} A \\ \geq \\ -A \end{matrix} 0$$

The average probability of error is

$$\begin{aligned} P(e) &= \frac{1}{2} P(e|A) + \frac{1}{2} P(e|-A) \\ &= \frac{1}{2} \int_{-\infty}^0 f(r|A) dr + \frac{1}{2} \int_0^{\infty} f(r|-A) dr \\ &= \frac{1}{2} \int_{-\infty}^0 \lambda_2 e^{-\lambda|r-A|} dr + \frac{1}{2} \int_0^{\infty} \lambda_2 e^{-\lambda|r+A|} dr \\ &= \frac{\lambda}{4} \int_{-\infty}^{-A} e^{-\lambda|x|} dx + \frac{\lambda}{4} \int_A^{\infty} e^{-\lambda|x|} dx \\ &= \frac{\lambda}{4} \frac{1}{\lambda} e^{\lambda x} \Big|_{-\infty}^{-A} + \frac{\lambda}{4} \left(-\frac{1}{\lambda} \right) e^{-\lambda x} \Big|_A^{\infty} \\ &= \frac{1}{2} e^{-\lambda A} \end{aligned}$$

b) The variance of the noise is

$$\begin{aligned} \sigma_n^2 &= \frac{\lambda}{2} \int_{-\infty}^{\infty} e^{-\lambda|x|} x^2 dx \\ &= \lambda \int_0^{\infty} e^{-\lambda x} x^2 dx = \lambda \frac{2!}{\lambda^3} = \frac{2}{\lambda^2} \end{aligned}$$

Hence, the SNR is

$$\text{SNR} = \frac{A^2}{\frac{2}{\lambda^2}} = \frac{A^2 \lambda^2}{2}$$

and the probability of error is given by

$$P(e) = \frac{1}{2}e^{-\sqrt{\lambda^2 A^2}} = \frac{1}{2}e^{-\sqrt{2\text{SNR}}}$$

For $P(e) = 10^{-5}$ we obtain

$$\ln(2 \times 10^{-5}) = -\sqrt{2\text{SNR}} \implies \text{SNR} = 58.534 = 17.6741 \text{ dB}$$

If the noise was Gaussian, then

$$P(e) = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\text{SNR}}\right]$$

where SNR is the signal to noise ratio at the output of the matched filter. With $P(e) = 10^{-5}$ we find $\sqrt{\text{SNR}} = 4.26$ and therefore $\text{SNR} = 18.1476 = 12.594 \text{ dB}$. Thus the required signal to noise ratio is 5 dB less when the additive noise is Gaussian.
