HW # 2 Solution

Problem 2.4

For the first two questions we will need the integral $I = \int e^{ax}\cos^2 x dx$.

$$\begin{aligned}
I &= \frac{1}{a}\int \cos^2 x \, de^{ax} = \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a}\int e^{ax}\sin 2x \, dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}\int \sin 2x \, de^{ax} \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax}\cos 2x \, dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax}(2\cos^2 x - 1) \, dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax} \, dx - \frac{4}{a^2}I
\end{aligned}$$

Thus,

$$I = \frac{1}{4+a^2}\left[(a\cos^2 x + \sin 2x) + \frac{2}{a}\right]e^{ax}$$

1)

$$\begin{aligned}
E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_1^2(t)dx = \lim_{T\to\infty}\int_0^{\frac{T}{2}} e^{-2t}\cos^2 t dt \\
&= \lim_{T\to\infty}\frac{1}{8}\left[(-2\cos^2 t + \sin 2t) - 1\right]e^{-2t}\Big|_0^{\frac{T}{2}} \\
&= \lim_{T\to\infty}\frac{1}{8}\left[(-2\cos^2\frac{T}{2} + \sin T - 1)e^{-T} + 3\right] = \frac{3}{8}
\end{aligned}$$

Thus $x_1(t)$ is an energy-type signal and the energy content is 3/8

2)

$$\begin{aligned}
E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_2^2(t)dx = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-2t}\cos^2 t dt \\
&= \lim_{T\to\infty}\left[\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 t dt + \int_0^{\frac{T}{2}} e^{-2t}\cos^2 t dt\right]
\end{aligned}$$

But,

$$\begin{aligned}
\lim_{T\to\infty}\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 t dt &= \lim_{T\to\infty}\frac{1}{8}\left[(-2\cos^2 t + \sin 2t) - 1\right]e^{-2t}\Big|_{-\frac{T}{2}}^{0} \\
&= \lim_{T\to\infty}\frac{1}{8}\left[-3 + (2\cos^2\frac{T}{2} + 1 + \sin T)e^{T}\right] = \infty
\end{aligned}$$

since $2 + \cos\theta + \sin\theta > 0$. Thus, $E_x = \infty$ since as we have seen from the first question the second integral is bounded. Hence, the signal $x_2(t)$ is not an energy-type signal. To test if $x_2(t)$ is a power-type signal we find P_x.

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 dt + \lim_{T\to\infty}\frac{1}{T}\int_0^{\frac{T}{2}} e^{-2t}\cos^2 dt$$

But $\lim_{T\to\infty} \frac{1}{T}\int_0^{\frac{T}{2}} e^{-2t}\cos^2 dt$ is zero and

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 dt &= \lim_{T\to\infty} \frac{1}{8T}\left[2\cos^2\frac{T}{2} + 1 + \sin T\right] e^T \\
&> \lim_{T\to\infty} \frac{1}{T} e^T > \lim_{T\to\infty} \frac{1}{T}(1+T+T^2) > \lim_{T\to\infty} T = \infty
\end{aligned}$$

Thus the signal $x_2(t)$ is not a power-type signal.

3)

$$\begin{aligned}
E_x &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_3^2(t)dx = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \text{sgn}^2(t)dt = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \lim_{T\to\infty} T = \infty \\
P_x &= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \text{sgn}^2(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \lim_{T\to\infty} \frac{1}{T} T = 1
\end{aligned}$$

The signal $x_3(t)$ is of the power-type and the power content is 1.

4)
First note that

$$\lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} A\cos(2\pi f t)dt = \sum_{k=-\infty}^{\infty} A\int_{k-\frac{1}{2f}}^{k+\frac{1}{2f}} \cos(2\pi f t)dt = 0$$

so that

$$\begin{aligned}
\lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2\cos^2(2\pi f t)dt &= \lim_{T\to\infty} \frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} (A^2 + A^2\cos(2\pi 2 f t))dt \\
&= \lim_{T\to\infty} \frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 dt = \lim_{T\to\infty} \frac{1}{2} A^2 T = \infty
\end{aligned}$$

$$\begin{aligned}
E_x &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} (A^2\cos^2(2\pi f_1 t) + B^2\cos^2(2\pi f_2 t) + 2AB\cos(2\pi f_1 t)\cos(2\pi f_2 t))dt \\
&= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2\cos^2(2\pi f_1 t)dt + \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} B^2\cos^2(2\pi f_2 t)dt + \\
&\quad AB\lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} [\cos^2(2\pi(f_1+f_2) + \cos^2(2\pi(f_1-f_2)]dt \\
&= \infty + \infty + 0 = \infty
\end{aligned}$$

Thus the signal is not of the energy-type. To test if the signal is of the power-type we consider two cases $f_1 = f_2$ and $f_1 \neq f_2$. In the first case

$$\begin{aligned}
P_x &= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} (A+B)^2\cos^2(2\pi f_1)dt \\
&= \lim_{T\to\infty} \frac{1}{2T}(A+B)^2\int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \frac{1}{2}(A+B)^2
\end{aligned}$$

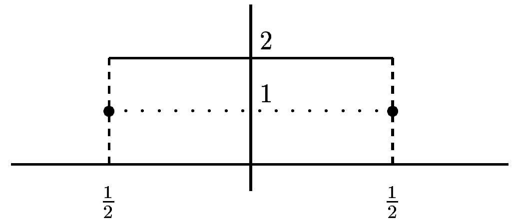
If $f_1 \neq f_2$ then

$$\begin{aligned}
P_x &= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} (A^2\cos^2(2\pi f_1 t) + B^2\cos^2(2\pi f_2 t) + 2AB\cos(2\pi f_1 t)\cos(2\pi f_2 t))dt \\
&= \lim_{T\to\infty} \frac{1}{T}\left[\frac{A^2 T}{2} + \frac{B^2 T}{2}\right] = \frac{A^2}{2} + \frac{B^2}{2}
\end{aligned}$$

Thus the signal is of the power-type and if $f_1 = f_2$ the power content is $(A+B)^2/2$ whereas if $f_1 \neq f_2$ the power content is $\frac{1}{2}(A^2 + B^2)$

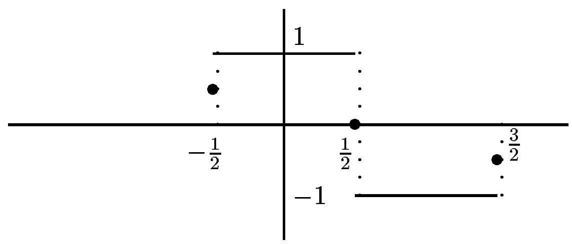
Problem 2.8

1) $x_1(t) = \Pi(t) + \Pi(-t)$. The signal $\Pi(t)$ is even so that $x_1(t) = 2\Pi(t)$



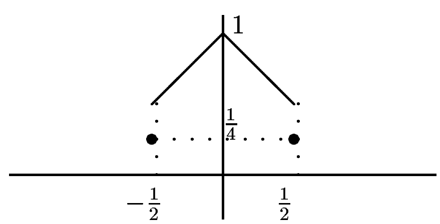
2)

$$x_2(t) = \Pi(t) - \Pi(t-1) = \begin{cases} 0, & t < -1/2 \\ 1/2, & t = 1/2 \\ 1, & -1/2 < t < 1/2 \\ 0, & t = 1/2 \\ -1, & 1/2 < t < 3/2 \\ -1/2, & t = 3/2 \\ 0, & 3/2 < t \end{cases}$$

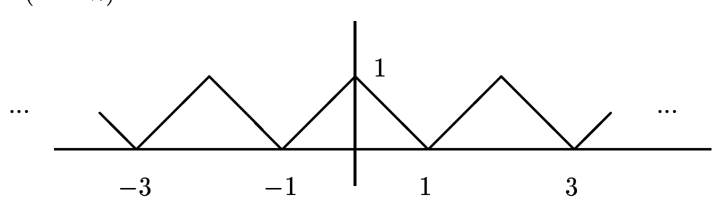


3)

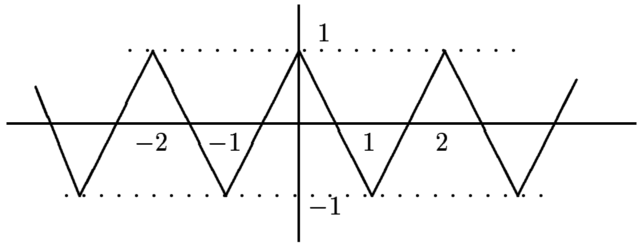
$$x_3(t) = \Lambda(t) \cdot \Pi(t) = \begin{cases} 0, & t < -1/2 \\ 1/4, & t = -1/2 \\ t+1, & -1/2 < t \leq 0 \\ -t+1, & 0 \leq t < 1/2 \\ 1/4, & t = 1/2 \\ 0, & 1/2 < t \end{cases}$$



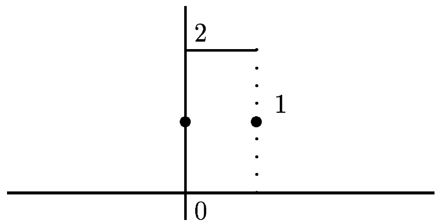
4) $x_4(t) = \sum_{n=-\infty}^{\infty} \Lambda(t - 2n)$



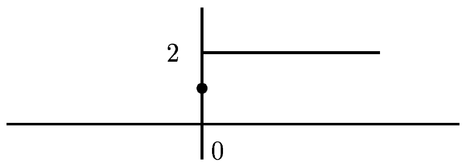
5) $x_5(t) = \sum_{n=-\infty}^{\infty} (-1)^n \Lambda(t - n)$



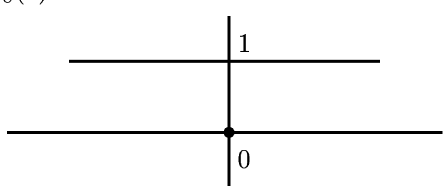
6) $x_6(t) = \text{sgn}(t) + \text{sgn}(1-t)$. Note that $x_6(0) = 1$, $x_6(1) = 1$



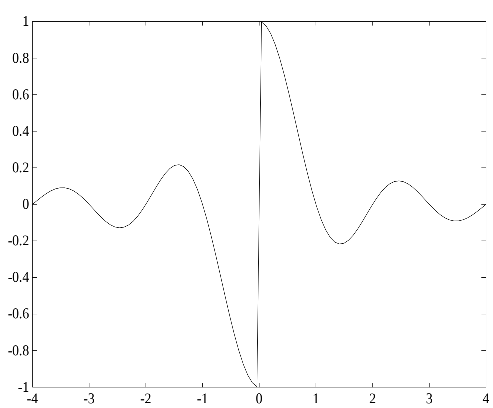
7) $x_7(t) = 1 + \text{sgn}(t)$. Note that $x_7(0) = 1$.



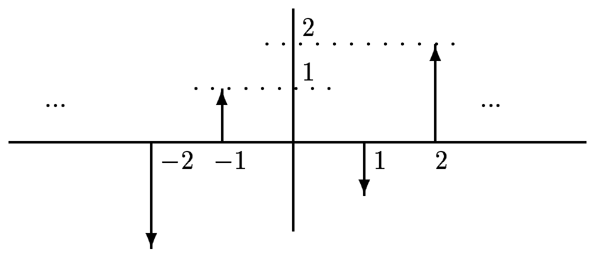
8) $x_8(t) = \text{sgn}^2(t)$. Note that $x_8(0) = 0$



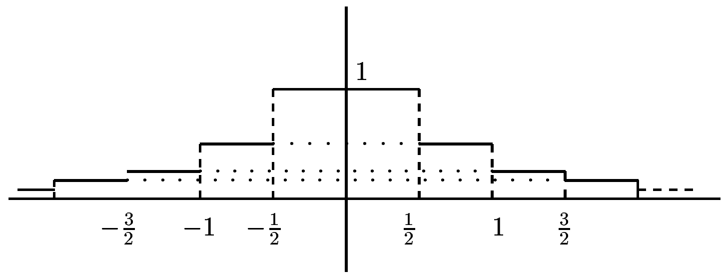
9) $x_9(t) = \text{sinc}(t)\text{sgn}(t)$. Note that $x_9(0) = 0$.



10) $x_{10}(t) = \sum_{n=-\infty}^{\infty} (-1)^n n \delta(t-n)$



11) $x_{11}(t) = \sum_{n=1}^{\infty} \frac{1}{2^n}\Pi(\frac{t}{n})$ Note that for $|t| < 1/2$, $x_{11}(t) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$



Problem 2.30

In order for the signals $\psi_n(t)$ to constitute an orthonormal set of signals in $[\alpha, \alpha + T_0]$ the following condition should be satisfied

$$\langle \psi_n(t), \psi_m(t)\rangle = \int_{\alpha}^{\alpha+T_0} \psi_n(t)\psi_m^*(t)dt = \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

But

$$\begin{aligned} \langle \psi_n(t), \psi_m(t)\rangle &= \int_{\alpha}^{\alpha+T_0} \frac{1}{\sqrt{T_0}} e^{j2\pi \frac{n}{T_0}t} \frac{1}{\sqrt{T_0}} e^{-j2\pi \frac{m}{T_0}t} dt \\ &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} e^{j2\pi \frac{(n-m)}{T_0}t} dt \end{aligned}$$

If $n = m$ then $e^{j2\pi \frac{(n-m)}{T_0}t} = 1$ so that

$$\langle \psi_n(t), \psi_n(t)\rangle = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} dt = \frac{1}{T_0} t\Big|_{\alpha}^{\alpha+T_0} = 1$$

When $n \neq m$ then,

$$\langle \psi_n(t), \psi_m(t)\rangle = \frac{1}{j2\pi(n-m)} e^x \Big|_{j2\pi(n-m)\alpha/T_0}^{j2\pi(n-m)(\alpha+T_0)/T_0} = 0$$

Thus, $\langle \psi_n(t), \psi_n(t)\rangle = \delta_{mn}$ which proves that $\psi_n(t)$ constitute an orthonormal set of signals.

Problem 2.31

$$x_1(t) = 1 \Rightarrow \|x_1(t)\| = \left(\int_0^1 1dt\right)^{\frac{1}{2}} = 1 \Rightarrow i_1(t) = \frac{x_1(t)}{\|x_1(t)\|} = 1$$

$$\begin{aligned} x_2(t) \cdot i_1(t) &= \int_0^1 t dt = \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2} \\ v_2(t) &= x_2(t) - \langle x_2(t), i_1(t)\rangle i_1(t) = t - \frac{1}{2} \\ \|v_2(t)\| &= \left(\int_0^1 (t-\frac{1}{2})^2 dt\right)^{\frac{1}{2}} = \left(\frac{1}{3}t^3\Big|_0^1 + \frac{1}{4}t\Big|_0^1 - \frac{1}{2}t^2\Big|_0^1\right)^{\frac{1}{2}} = (\frac{1}{12})^{\frac{1}{2}} \\ i_2(t) &= \frac{v_2(t)}{\|v_2(t)\|} = \sqrt{12}(t - \frac{1}{2}) \end{aligned}$$

$$
\begin{aligned}
x_3(t)\cdot i_1(t) &= \int_0^1 t^2 dt = \frac{1}{3}t^3\Big|_0^1 = \frac{1}{3} \\
x_3(t)\cdot i_2(t) &= \int_0^1 t^2\sqrt{12}(t-\frac{1}{2})dt = \sqrt{12}\left(\frac{1}{4}t^4 - \frac{1}{6}t^3\right)\Big|_0^1 = \frac{1}{\sqrt{12}} \\
v_3(t) &= x_3(t) - \langle x_3(t), i_1(t)\rangle i_1(t) - \langle x_3(t), i_1(t)\rangle i_1(t) \\
&= t^2 - \frac{1}{3} - \frac{1}{\sqrt{12}}\sqrt{12}(t-\frac{1}{2}) = t^2 - t + \frac{1}{6} \\
\|v_3(t)\| &= \left(\int_0^1 (t^2 - t + 1/6)^2 dt\right)^{\frac{1}{2}} \\
&= \left(\int_0^1 t^4 + t^2 + \frac{1}{36} - 2t^3 + \frac{1}{3}t^2 - \frac{1}{3}t\right)^{\frac{1}{2}} \\
&= \frac{1}{5}t^5\Big|_0^1 + \frac{4}{9}t^3\Big|_0^1 + \frac{1}{36}t\Big|_0^1 - \frac{1}{2}t^4\Big|_0^1 - \frac{1}{6}t^2\Big|_0^1 = \frac{1}{180} \\
i_3(t) &= \frac{v_3(t)}{\|v_3(t)\|} = 180t^2 - 180^t + 30
\end{aligned}
$$

The higher order polynomial functions are obtained in the same way.

Problem 2.33

1)

$$
\begin{aligned}
\epsilon^2 &= \int_{-\infty}^{\infty}\left|x(t) - \sum_{i=1}^N \alpha_i\phi_i(t)\right|^2 dt \\
&= \int_{-\infty}^{\infty}\left(x(t) - \sum_{i=1}^N \alpha_i\phi_i(t)\right)\left(x^*(t) - \sum_{j=1}^N \alpha_j^*\phi_j^*(t)\right) dt \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^N \alpha_i \int_{-\infty}^{\infty}\phi_i(t)x^*(t)dt - \sum_{j=1}^N \alpha_j^* \int_{-\infty}^{\infty}\phi_j^*(t)x(t)dt \\
&\quad + \sum_{i=1}^N\sum_{j=1}^N \alpha_i\alpha_j^* \int_{-\infty}^{\infty}\phi_i(t)\phi_j^* dt \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt + \sum_{i=1}^N |\alpha_i|^2 - \sum_{i=1}^N \alpha_i \int_{-\infty}^{\infty}\phi_i(t)x^*(t)dt - \sum_{j=1}^N \alpha_j^* \int_{-\infty}^{\infty}\phi_j^*(t)x(t)dt
\end{aligned}
$$

Completing the square in terms of α_i we obtain

$$
\epsilon^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^N \left|\int_{-\infty}^{\infty}\phi_i^*(t)x(t)dt\right|^2 + \sum_{i=1}^N \left|\alpha_i - \int_{-\infty}^{\infty}\phi_i^*(t)x(t)dt\right|^2
$$

The first two terms are independent of α's and the last term is always positive. Therefore the minimum is achieved for

$$
\alpha_i = \int_{-\infty}^{\infty}\phi_i^*(t)x(t)dt
$$

which causes the last term to vanish.

2) With this choice of α_i's

$$
\begin{aligned}
\epsilon^2 &= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^N \left|\int_{-\infty}^{\infty}\phi_i^*(t)x(t)dt\right|^2 \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^N |\alpha_i|^2
\end{aligned}
$$

Problem 2.34

1) The signal $x_1(t)$ is periodic with period $T_0 = 2$. Thus

$$
\begin{aligned}
x_{1,n} &= \frac{1}{2}\int_{-1}^{1}\Lambda(t)e^{-j2\pi\frac{n}{2}t}dt = \frac{1}{2}\int_{-1}^{1}\Lambda(t)e^{-j\pi nt}dt \\
&= \frac{1}{2}\int_{-1}^{0}(t+1)e^{-j\pi nt}dt + \frac{1}{2}\int_{0}^{1}(-t+1)e^{-j\pi nt}dt \\
&= \frac{1}{2}\left(\frac{j}{\pi n}te^{-j\pi nt} + \frac{1}{\pi^2 n^2}e^{-j\pi nt}\right)\Bigg|_{-1}^{0} + \frac{j}{2\pi n}e^{-j\pi nt}\Bigg|_{-1}^{0} \\
&\quad -\frac{1}{2}\left(\frac{j}{\pi n}te^{-j\pi nt} + \frac{1}{\pi^2 n^2}e^{-j\pi nt}\right)\Bigg|_{0}^{1} + \frac{j}{2\pi n}e^{-j\pi nt}\Bigg|_{0}^{1} \\
&\quad \frac{1}{\pi^2 n^2} - \frac{1}{2\pi^2 n^2}(e^{j\pi n} + e^{-j\pi n}) = \frac{1}{\pi^2 n^2}(1-\cos(\pi n))
\end{aligned}
$$

When $n = 0$ then

$$x_{1,0} = \frac{1}{2}\int_{-1}^{1}\Lambda(t)dt = \frac{1}{2}$$

Thus

$$x_1(t) = \frac{1}{2} + 2\sum_{n=1}^{\infty}\frac{1}{\pi^2 n^2}(1-\cos(\pi n))\cos(\pi nt)$$

2) $x_2(t) = 1$. It follows then that $x_{2,0} = 1$ and $x_{2,n} = 0$, $\forall n \neq 0$.

3) The signal is periodic with period $T_0 = 1$. Thus

$$
\begin{aligned}
x_{3,n} &= \frac{1}{T_0}\int_0^{T_0} e^t e^{-j2\pi nt}dt = \int_0^1 e^{(-j2\pi n+1)t}dt \\
&= \frac{1}{-j2\pi n+1}e^{(-j2\pi n+1)t}\Bigg|_0^1 = \frac{e^{(-j2\pi n+1)}-1}{-j2\pi n+1} \\
&= \frac{e-1}{1-j2\pi n} = \frac{e-1}{\sqrt{1+4\pi^2 n^2}}(1+j2\pi n)
\end{aligned}
$$

4) The signal $\cos(t)$ is periodic with period $T_1 = 2\pi$ whereas $\cos(2.5t)$ is periodic with period $T_2 = 0.8\pi$. It follows then that $\cos(t)+\cos(2.5t)$ is periodic with period $T = 4\pi$. The trigonometric Fourier series of the even signal $\cos(t)+\cos(2.5t)$ is

$$
\begin{aligned}
\cos(t)+\cos(2.5t) &= \sum_{n=1}^{\infty}\alpha_n\cos(2\pi\frac{n}{T_0}t) \\
&= \sum_{n=1}^{\infty}\alpha_n\cos(\frac{n}{2}t)
\end{aligned}
$$

By equating the coefficients of $\cos(\frac{n}{2}t)$ of both sides we observe that $a_n = 0$ for all n unless $n = 2, 5$ in which case $a_2 = a_5 = 1$. Hence $x_{4,2} = x_{4,5} = \frac{1}{2}$ and $x_{4,n} = 0$ for all other values of n.

5) The signal $x_5(t)$ is periodic with period $T_0 = 1$. For $n = 0$

$$x_{5,0} = \int_0^1(-t+1)dt = (-\frac{1}{2}t^2+t)\Bigg|_0^1 = \frac{1}{2}$$

For $n \neq 0$

$$x_{5,n} = \int_0^1(-t+1)e^{-j2\pi nt}dt$$

$$
\begin{aligned}
&= -\left(\frac{j}{2\pi n}te^{-j2\pi nt} + \frac{1}{4\pi^2 n^2}e^{-j2\pi nt}\right)\Big|_0^1 + \frac{j}{2\pi n}e^{-j2\pi nt}\Big|_0^1 \\
&= -\frac{j}{2\pi n}
\end{aligned}
$$

Thus,

$$
x_5(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\sin 2\pi nt
$$

6) The signal $x_6(t)$ is periodic with period $T_0 = 2T$. We can write $x_6(t)$ as

$$
\begin{aligned}
x_6(t) &= \sum_{n=-\infty}^{\infty} \delta(t - n2T) - \sum_{n=-\infty}^{\infty} \delta(t - T - n2T) \\
&= \frac{1}{2T}\sum_{n=-\infty}^{\infty} e^{j\pi\frac{n}{T}t} - \frac{1}{2T}\sum_{n=-\infty}^{\infty} e^{j\pi\frac{n}{T}(t-T)} \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{2T}(1 - e^{-j\pi n})e^{j2\pi\frac{n}{2T}t}
\end{aligned}
$$

However, this is the Fourier series expansion of $x_6(t)$ and we identify $x_{6,n}$ as

$$
x_{6,n} = \frac{1}{2T}(1 - e^{-j\pi n}) = \frac{1}{2T}(1 - (-1)^n) = \begin{cases} 0 & n \text{ even} \\ \frac{1}{T} & n \text{ odd} \end{cases}
$$

7) The signal is periodic with period T. Thus,

$$
\begin{aligned}
x_{7,n} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta'(t)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{1}{T}(-1)\frac{d}{dt}e^{-j2\pi\frac{n}{T}t}\Big|_{t=0} = \frac{j2\pi n}{T^2}
\end{aligned}
$$

8) The signal $x_8(t)$ is real even and periodic with period $T_0 = \frac{1}{2f_0}$. Hence, $x_{8,n} = a_{8,n}/2$ or

$$
\begin{aligned}
x_{8,n} &= 2f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0 t)\cos(2\pi n 2f_0 t)dt \\
&= f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1+2n)t)dt + f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1-2n)t)dt \\
&= \frac{1}{2\pi(1+2n)}\sin(2\pi f_0(1+2n)t)\Big|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} + \frac{1}{2\pi(1-2n)}\sin(2\pi f_0(1-2n)t)\Big|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} \\
&= \frac{(-1)^n}{\pi}\left[\frac{1}{(1+2n)} + \frac{1}{(1-2n)}\right]
\end{aligned}
$$

9) The signal $x_9(t) = \cos(2\pi f_0 t) + |\cos(2\pi f_0 t)|$ is even and periodic with period $T_0 = 1/f_0$. It is equal to $2\cos(2\pi f_0 t)$ in the interval $[-\frac{1}{4f_0}, \frac{1}{4f_0}]$ and zero in the interval $[\frac{1}{4f_0}, \frac{3}{4f_0}]$. Thus

$$
\begin{aligned}
x_{9,n} &= 2f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0 t)\cos(2\pi n f_0 t)dt \\
&= f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1+n)t)dt + f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1-n)t)dt
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{2\pi(1+n)}\sin(2\pi f_0(1+n)t)\Big|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} + \frac{1}{2\pi(1-n)}\sin(2\pi f_0(1-n)t)\Big|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} \\
&= \frac{1}{\pi(1+n)}\sin(\frac{\pi}{2}(1+n)) + \frac{1}{\pi(1-n)}\sin(\frac{\pi}{2}(1-n))
\end{aligned}$$

Thus $x_{9,n}$ is zero for odd values of n unless $n = \pm 1$ in which case $x_{9,\pm 1} = \frac{1}{2}$. When n is even ($n = 2l$) then

$$x_{9,2l} = \frac{(-1)^l}{\pi}\left[\frac{1}{1+2l} + \frac{1}{1-2l}\right]$$

Problem 2.42

1) Using the Fourier transform pair

$$e^{-\alpha|t|} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2 + (2\pi f)^2} = \frac{2\alpha}{4\pi^2}\frac{1}{\frac{\alpha^2}{4\pi^2} + f^2}$$

and the duality property of the Fourier transform: $X(f) = \mathcal{F}[x(t)] \Rightarrow x(-f) = \mathcal{F}[X(t)]$ we obtain

$$\left(\frac{2\alpha}{4\pi^2}\right)\mathcal{F}\left[\frac{1}{\frac{\alpha^2}{4\pi^2} + t^2}\right] = e^{-\alpha|f|}$$

With $\alpha = 2\pi$ we get the desired result

$$\mathcal{F}\left[\frac{1}{1+t^2}\right] = \pi e^{-2\pi|f|}$$

2)

$$\begin{aligned}
\mathcal{F}[x(t)] &= \mathcal{F}[\Pi(t-3) + \Pi(t+3)] \\
&= \text{sinc}(f)e^{-j2\pi f3} + \text{sinc}(f)e^{j2\pi f3} \\
&= 2\text{sinc}(f)\cos(2\pi 3f)
\end{aligned}$$

3)

$$\begin{aligned}
\mathcal{F}[x(t)] &= \mathcal{F}[\Lambda(2t+3) + \Lambda(3t-2)] \\
&= \mathcal{F}[\Lambda(2(t+\frac{3}{2})) + \Lambda(3(t-\frac{2}{3})] \\
&= \frac{1}{2}\text{sinc}^2(\frac{f}{2})e^{j\pi f3} + \frac{1}{3}\text{sinc}^2(\frac{f}{3})e^{-j2\pi f\frac{2}{3}}
\end{aligned}$$

4) $T(f) = \mathcal{F}[\text{sinc}^3(t)] = \mathcal{F}[\text{sinc}^2(t)\text{sinc}(t)] = \Lambda(f) \star \Pi(f)$. But

$$\Pi(f) \star \Lambda(f) = \int_{-\infty}^{\infty}\Pi(\theta)\Lambda(f-\theta)d\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}}\Lambda(f-\theta)d\theta = \int_{f-\frac{1}{2}}^{f+\frac{1}{2}}\Lambda(v)dv$$

For $\quad f \leq -\frac{3}{2} \Longrightarrow T(f) = 0$

For $\quad -\frac{3}{2} < f \leq -\frac{1}{2} \Longrightarrow T(f) = \int_{-1}^{f+\frac{1}{2}}(v+1)dv = (\frac{1}{2}v^2 + v)\Big|_{-1}^{f+\frac{1}{2}} = \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8}$

For $\quad -\frac{1}{2} < f \leq \frac{1}{2} \Longrightarrow T(f) = \int_{f-\frac{1}{2}}^{0}(v+1)dv + \int_{0}^{f+\frac{1}{2}}(-v+1)dv$

$$= (\frac{1}{2}v^2 + v)\Big|_{f-\frac{1}{2}}^{0} + (-\frac{1}{2}v^2 + v)\Big|_{0}^{f+\frac{1}{2}} = -f^2 + \frac{3}{4}$$

For $\frac{1}{2} < f \leq \frac{3}{2} \Longrightarrow T(f) = \int_{f-\frac{1}{2}}^{1} (-v+1)dv = (-\frac{1}{2}v^2 + v)\Big|_{f-\frac{1}{2}}^{1} = \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8}$

For $\frac{3}{2} < f \Longrightarrow T(f) = 0$

Thus,

$$T(f) = \begin{cases} 0 & f \leq -\frac{3}{2} \\ \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8} & -\frac{3}{2} < f \leq -\frac{1}{2} \\ -f^2 + \frac{3}{4} & -\frac{1}{2} < f \leq \frac{1}{2} \\ \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8} & \frac{1}{2} < f \leq \frac{3}{2} \\ 0 & \frac{3}{2} < f \end{cases}$$

5)

$$\mathcal{F}[t\text{sinc}(t)] = \frac{1}{\pi}\mathcal{F}[\sin(\pi t)] = \frac{j}{2\pi}\left[\delta(f+\frac{1}{2}) - \delta(f-\frac{1}{2})\right]$$

The same result is obtain if we recognize that multiplication by t results in differentiation in the frequency domain. Thus

$$\mathcal{F}[t\text{sinc}] = \frac{j}{2\pi}\frac{d}{df}\Pi(f) = \frac{j}{2\pi}\left[\delta(f+\frac{1}{2}) - \delta(f-\frac{1}{2})\right]$$

6)

$$\begin{aligned} \mathcal{F}[t\cos(2\pi f_0 t)] &= \frac{j}{2\pi}\frac{d}{df}\left(\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right) \\ &= \frac{j}{4\pi}\left(\delta'(f-f_0) + \delta'(f+f_0)\right) \end{aligned}$$

7)

$$\mathcal{F}[e^{-\alpha|t|}\cos(\beta t)] = \frac{1}{2}\left[\frac{2\alpha}{\alpha^2 + (2\pi(f-\frac{\beta}{2\pi}))^2} + \frac{2\alpha}{\alpha^2 + (2\pi(f+\frac{\beta}{2\pi}))^2}\right]$$

8)

$$\begin{aligned} \mathcal{F}[te^{-\alpha|t|}\cos(\beta t)] &= \frac{j}{2\pi}\frac{d}{df}\left(\frac{\alpha}{\alpha^2 + (2\pi(f-\frac{\beta}{2\pi}))^2} + \frac{\alpha}{\alpha^2 + (2\pi(f+\frac{\beta}{2\pi}))^2}\right) \\ &= -j\left[\frac{2\alpha\pi(f-\frac{\beta}{2\pi})}{\left(\alpha^2 + (2\pi(f-\frac{\beta}{2\pi}))^2\right)^2} + \frac{2\alpha\pi(f+\frac{\beta}{2\pi})}{\left(\alpha^2 + (2\pi(f+\frac{\beta}{2\pi}))^2\right)^2}\right] \end{aligned}$$

Problem 2.47

1) Clearly

$$\begin{aligned} x_1(t+kT_0) &= \sum_{n=-\infty}^{\infty} x(t+kT_0-nT_0) = \sum_{n=-\infty}^{\infty} x(t-(n-k)T_0) \\ &= \sum_{m=-\infty}^{\infty} x(t-mT_0) = x_1(t) \end{aligned}$$

where we used the change of variable $m = n - k$.

2)

$$x_1(t) = x(t) \star \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

This is because

$$\int_{-\infty}^{\infty} x(\tau) \sum_{n=-\infty}^{\infty} \delta(t-\tau-nT_0)d\tau = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-nT_0)d\tau = \sum_{n=-\infty}^{\infty} x(t-nT_0)$$

3)

$$\begin{aligned}
\mathcal{F}[x_1(t)] &= \mathcal{F}[x(t) \star \sum_{n=-\infty}^{\infty} \delta(t-nT_0)] = \mathcal{F}[x(t)]\mathcal{F}[\sum_{n=-\infty}^{\infty} \delta(t-nT_0)] \\
&= X(f)\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f-\frac{n}{T_0}) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_0})\delta(f-\frac{n}{T_0})
\end{aligned}$$

Problem 2.48
1) By Parseval's theorem

$$\int_{-\infty}^{\infty} \text{sinc}^5(t)dt = \int_{-\infty}^{\infty} \text{sinc}^3(t)\text{sinc}^2(t)dt = \int_{-\infty}^{\infty} \Lambda(f)T(f)df$$

where

$$T(f) = \mathcal{F}[\text{sinc}^3(t)] = \mathcal{F}[\text{sinc}^2(t)\text{sinc}(t)] = \Pi(f) \star \Lambda(f)$$

But

$$\Pi(f) \star \Lambda(f) = \int_{-\infty}^{\infty} \Pi(\theta)\Lambda(f-\theta)d\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Lambda(f-\theta)d\theta = \int_{f-\frac{1}{2}}^{f+\frac{1}{2}} \Lambda(v)dv$$

For $f \leq -\frac{3}{2} \Longrightarrow T(f) = 0$

For $-\frac{3}{2} < f \leq -\frac{1}{2} \Longrightarrow T(f) = \int_{-1}^{f+\frac{1}{2}} (v+1)dv = (\frac{1}{2}v^2+v)\Big|_{-1}^{f+\frac{1}{2}} = \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8}$

For $-\frac{1}{2} < f \leq \frac{1}{2} \Longrightarrow T(f) = \int_{f-\frac{1}{2}}^{0} (v+1)dv + \int_{0}^{f+\frac{1}{2}} (-v+1)dv$

$$= (\frac{1}{2}v^2+v)\Big|_{f-\frac{1}{2}}^{0} + (-\frac{1}{2}v^2+v)\Big|_{0}^{f+\frac{1}{2}} = -f^2 + \frac{3}{4}$$

For $\frac{1}{2} < f \leq \frac{3}{2} \Longrightarrow T(f) = \int_{f-\frac{1}{2}}^{1} (-v+1)dv = (-\frac{1}{2}v^2+v)\Big|_{f-\frac{1}{2}}^{1} = \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8}$

For $\frac{3}{2} < f \Longrightarrow T(f) = 0$

Thus,

$$T(f) = \begin{cases} 0 & f \leq -\frac{3}{2} \\ \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8} & -\frac{3}{2} < f \leq -\frac{1}{2} \\ -f^2 + \frac{3}{4} & -\frac{1}{2} < f \leq \frac{1}{2} \\ \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8} & \frac{1}{2} < f \leq \frac{3}{2} \\ 0 & \frac{3}{2} < f \end{cases}$$

Hence,

$$\int_{-\infty}^{\infty} \Lambda(f)T(f)df = \int_{-1}^{-\frac{1}{2}} (\frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8})(f+1)df + \int_{-\frac{1}{2}}^{0} (-f^2 + \frac{3}{4})(f+1)df$$

$$
\begin{aligned}
& + \int_0^{\frac{1}{2}} (-f^2 + \frac{3}{4})(-f+1)df + \int_{\frac{1}{2}}^1 (\frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8})(-f+1)df \\
= & \ \frac{41}{64}
\end{aligned}
$$

2)

$$
\begin{aligned}
\int_0^\infty e^{-\alpha t}\text{sinc}(t)dt &= \int_{-\infty}^\infty e^{-\alpha t}u_{-1}(t)\text{sinc}(t)dt \\
&= \int_{-\infty}^\infty \frac{1}{\alpha + j2\pi f}\Pi(f)df = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\alpha + j2\pi f}df \\
&= \frac{1}{j2\pi}\ln(\alpha + j2\pi f)\Big|_{-1/2}^{1/2} = \frac{1}{j2\pi}\ln(\frac{\alpha + j\pi}{\alpha - j\pi}) = \frac{1}{\pi}\tan^{-1}\frac{\pi}{\alpha}
\end{aligned}
$$

3)

$$
\begin{aligned}
\int_0^\infty e^{-\alpha t}\text{sinc}^2(t)dt &= \int_{-\infty}^\infty e^{-\alpha t}u_{-1}(t)\text{sinc}^2(t)dt \\
&= \int_{-\infty}^\infty \frac{1}{\alpha + j2\pi f}\Lambda(f)df df \\
&= \int_{-1}^0 \frac{f+1}{\alpha + j\pi f}df + \int_0^1 \frac{-f+1}{\alpha + j\pi f}df
\end{aligned}
$$

But $\int \frac{x}{a+bx}dx = \frac{x}{b} - \frac{a}{b^2}\ln(a+bx)$ so that

$$
\begin{aligned}
\int_0^\infty e^{-\alpha t}\text{sinc}^2(t)dt &= (\frac{f}{j2\pi} + \frac{\alpha}{4\pi^2}\ln(\alpha + j2\pi f))\Big|_{-1}^0 \\
& \quad -(\frac{f}{j2\pi} + \frac{\alpha}{4\pi^2}\ln(\alpha + j2\pi f))\Big|_0^1 + \frac{1}{j2\pi}\ln(\alpha + j2\pi f)\Big|_{-1}^1 \\
&= \frac{1}{\pi}\tan^{-1}(\frac{2\pi}{\alpha}) + \frac{\alpha}{2\pi^2}\ln(\frac{\alpha}{\sqrt{\alpha^2 + 4\pi^2}})
\end{aligned}
$$

4)

$$
\begin{aligned}
\int_0^\infty e^{-\alpha t}\cos(\beta t)dt &= \int_{-\infty}^\infty e^{-\alpha t}u_{-1}(t)\cos(\beta t)dt \\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{1}{\alpha + j2\pi f}(\delta(f - \frac{\beta}{2\pi}) + \delta(f + \frac{\beta}{2\pi}))dt \\
&= \frac{1}{2}[\frac{1}{\alpha + j\beta} + \frac{1}{\alpha - j\beta}] = \frac{\alpha}{\alpha^2 + \beta^2}
\end{aligned}
$$