# ESE 532
# HW#3 Solutions

**Problem 3.35**

$$m_X(t) = E[A + Bt] = E[A] + E[B]t = 0$$

where the last equality follows from the fact that $A$, $B$ are uniformly distributed over $[-1 \;\; 1]$ so that $E[A] = E[B] = 0$.

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1)X(t_2)] = E[(A + Bt_1)(A + Bt_2)] \\ &= E[A^2] + E[AB]t_2 + E[BA]t_1 + E[B^2]t_1 t_2 \end{aligned}$$

The random variables $A$, $B$ are independent so that $E[AB] = E[A]E[B] = 0$. Furthermore

$$E[A^2] = E[B^2] = \int_{-1}^{1} x^2 \frac{1}{2} dx = \frac{1}{6} x^3 \Big|_{-1}^{1} = \frac{1}{3}$$

Thus

$$R_X(t_1, t_2) = \frac{1}{3} + \frac{1}{3} t_1 t_2$$

**Problem 3.38**

The process is not wide sense stationary for the autocorrelation function depends on the values of $t_1$, $t_2$ and not on their difference. To see this suppose that $t_1 = t_2 = t$. If the process was wide sense stationary, then $R_X(t, t) = R_X(0)$. However, $R_X(t, t) = \sigma^2 t$ and it depends on $t$ as it is opposed to $R_X(0)$ which is independent of $t$.

**Problem 3.40**

1) $f(\tau)$ cannot be the autocorrelation function of a random process for $f(0) = 0 < f(1/4f_0) = 1$. Thus the maximum absolute value of $f(\tau)$ is not achieved at the origin $\tau = 0$.

2) $f(\tau)$ cannot be the autocorrelation function of a random process for $f(0) = 0$ whereas $f(\tau) \neq 0$ for $\tau \neq 0$. The maximum absolute value of $f(\tau)$ is not achieved at the origin.

3) $f(0) = 1$ whereas $f(\tau) > f(0)$ for $|\tau| > 1$. Thus $f(\tau)$ cannot be the autocorrelation function of a random process.

4) $f(\tau)$ is even and the maximum is achieved at the origin $(\tau = 0)$. We can write $f(\tau)$ as

$$f(\tau) = 1.2\Lambda(\tau) - \Lambda(\tau - 1) - \Lambda(\tau + 1)$$

Taking the Fourier transform of both sides we obtain

$$\mathcal{S}(f) = 1.2\text{sinc}^2(f) - \text{sinc}^2(f)\left(e^{-j2\pi f} + e^{j2\pi f}\right) = \text{sinc}^2(f)(1.2 - 2\cos(2\pi f))$$

As we observe the power spectrum $\mathcal{S}(f)$ can take negative values, i.e. for $f = 0$. Thus $f(\tau)$ can not be the autocorrelation function of a random process.

**Problem 3.51**

**1)** $\mathcal{S}_X(f) = \frac{N_0}{2}$, $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$. The autocorrelation function and the power spectral density of the output are given by

$$R_Y(t) = R_X(\tau) \star h(\tau) \star h(-\tau), \qquad \mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2$$

With $H(f) = \Pi(\frac{f}{2B})$ we have $|H(f)|^2 = \Pi^2(\frac{f}{2B}) = \Pi(\frac{f}{2B})$ so that

$$\mathcal{S}_Y(f) = \frac{N_0}{2}\Pi(\frac{f}{2B})$$

Taking the inverse Fourier transform of the previous we obtain the autocorrelation function of the output

$$R_Y(\tau) = 2B\frac{N_0}{2}\text{sinc}(2B\tau) = BN_0\text{sinc}(2B\tau)$$

**2)** The output random process $Y(t)$ is a zero mean Gaussian process with variance

$$\sigma^2_{Y(t)} = E[Y^2(t)] = E[Y^2(t+\tau)] = R_Y(0) = BN_0$$

The correlation coefficient of the jointly Gaussian processes $Y(t+\tau)$, $Y(t)$ is

$$\rho_{Y(t+\tau)Y(t)} = \frac{COV(Y(t+\tau)Y(t))}{\sigma_{Y(t+\tau)}\sigma_{Y(t)}} = \frac{E[Y(t+\tau)Y(t)]}{BN_0} = \frac{R_Y(\tau)}{BN_0}$$

With $\tau = \frac{1}{2B}$, we have $R_Y(\frac{1}{2B}) = \text{sinc}(1) = 0$ so that $\rho_{Y(t+\tau)Y(t)} = 0$. Hence the joint probability density function of $Y(t)$ and $Y(t+\tau)$ is

$$f_{Y(t+\tau)Y(t)} = \frac{1}{2\pi BN_0} e^{-\frac{Y^2(t+\tau)+Y^2(t)}{2BN_0}}$$

Since the processes are Gaussian and uncorrelated they are also independent.

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**Problem 3.62**

$h(t) = e^{-\beta t}u_{-1}(t) \Rightarrow H(f) = \frac{1}{\beta+j2\pi f}$. The power spectral density of the input process is $\mathcal{S}_X(f) = \mathcal{F}[e^{-\alpha|\tau|}] = \frac{2\alpha}{\alpha^2+4\pi^2f^2}$. If $\alpha = \beta$, then

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \frac{2\alpha}{(\alpha^2+4\pi^2f^2)^2}$$

If $\alpha \neq \beta$, then

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \frac{2\alpha}{(\alpha^2+4\pi^2f^2)(\beta^2+4\pi^2f^2)}$$

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Please let me know if you find any mistake in this solution

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