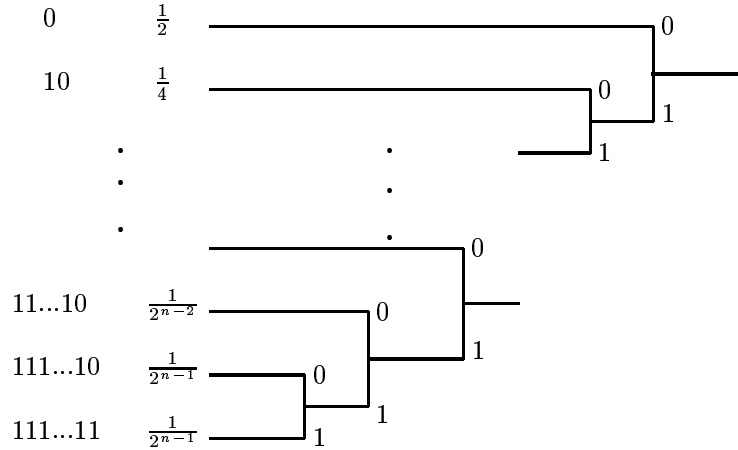

ESE 532
HW#5 Solutions

Problem 4.23

The following figure shows the design of the Huffman code. Note that at each step of the algorithm the branches with the lowest probabilities (that merge together) are those at the bottom of the tree.



The entropy of the source is

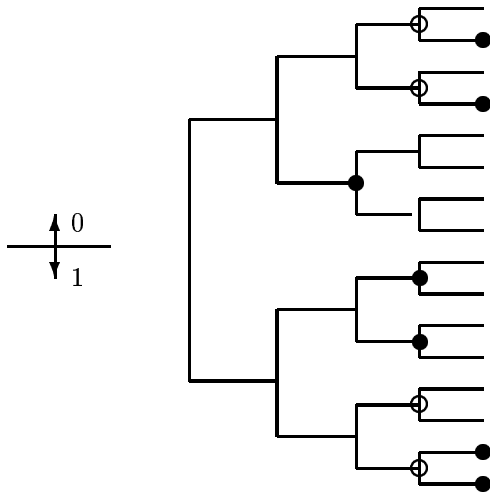
$$\begin{aligned}
 H(X) &= \sum_{i=1}^{n-1} \frac{1}{2^i} \log_2 2^i + \frac{1}{2^{n-1}} \log_2 2^{n-1} \\
 &= \sum_{i=1}^{n-1} \frac{1}{2^i} i \log_2 2 + \frac{1}{2^{n-1}} (n-1) \log_2 2 \\
 &= \sum_{i=1}^{n-1} \frac{i}{2^i} + \frac{n-1}{2^{n-1}}
 \end{aligned}$$

In the way that the code is constructed, the first codeword (0) has length one, the second codeword (10) has length two and so on until the last two codewords (111...10, 111...11) which have length $n-1$. Thus, the average codeword length is

$$\begin{aligned}
 \bar{R} &= \sum_{x \in \mathcal{X}} p(x) l(x) = \sum_{i=1}^{n-1} \frac{i}{2^i} + \frac{n-1}{2^{n-1}} \\
 &= 2 \left(1 - (1/2)^{n-1} \right) = H(X)
 \end{aligned}$$

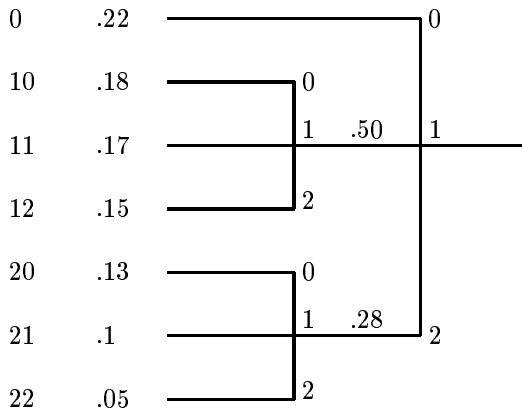
Problem 4.24

The following figure shows the position of the codewords (black filled circles) in a binary tree. Although the prefix condition is not violated the code is not optimum in the sense that it uses more bits than is necessary. For example the upper two codewords in the tree (0001, 0011) can be substituted by the codewords (000, 001) (unfilled circles) reducing in this way the average codeword length. Similarly codewords 1111 and 1110 can be substituted by codewords 111 and 110.



Problem 4.25

The following figure depicts the design of a ternary Huffman code.



The average codeword length is

$$\begin{aligned} \bar{R}(X) &= \sum_x p(x)l(x) = .22 + 2(.18 + .17 + .15 + .13 + .10 + .05) \\ &= 1.78 \quad (\text{ternary symbols/output}) \end{aligned}$$

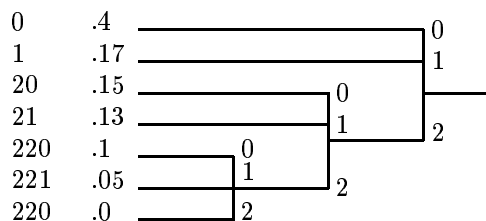
For a fair comparison of the average codeword length with the entropy of the source, we compute the latter with logarithms in base 3. Hence,

$$H(X) = -\sum_x p(x) \log_3 p(x) = 1.7047$$

As it is expected $H(X) \leq \bar{R}(X)$.

Problem 4.26

If D is the size of the code alphabet, then the Huffman coding scheme takes D source outputs and it merges them to 1 symbol. Hence, we have a decrease of output symbols by $D - 1$. In K steps of the algorithm the decrease of the source outputs is $K(D - 1)$. If the number of the source outputs is $K(D - 1) + D$, for some K , then we are in a good position since we will be left with D symbols for which we assign the symbols $0, 1, \dots, D - 1$. To meet the above condition with a ternary code the number of the source outputs should be $2K + 3$. In our case that the number of source outputs is six we can add a dummy symbol with zero probability so that $7 = 2 \cdot 2 + 3$. The following figure shows the design of the ternary Huffman code.



Problem 4.27

Parsing the sequence by the rules of the Lempel-Ziv coding scheme we obtain the phrases

0, 00, 1, 001, 000, 0001, 10, 00010, 0000, 0010, 00000, 101, 00001,
000000, 11, 01, 0000000, 110, ...

The number of the phrases is 18. For each phrase we need 5 bits plus an extra bit to represent the new source output.

Dictionary Location	Dictionary Contents	Codeword
1 00001	0	00000 0
2 00010	00	00001 0
3 00011	1	00000 1
4 00100	001	00010 1
5 00101	000	00010 0
6 00110	0001	00101 1
7 00111	10	00011 0
8 01000	00010	00110 0
9 01001	0000	00101 0
10 01010	0010	00100 0
11 01011	00000	01001 0
12 01100	101	00111 1
13 01101	00001	01001 1
14 01110	000000	01011 0
15 01111	11	00011 1
16 10000	01	00001 1
17 10001	0000000	01110 0
18 10010	110	01111 0

Problem 4.46

1) From Table 4.2 we find that for a unit variance Gaussian process, the optimal level spacing for a 16-level uniform quantizer is .3352. This number has to be multiplied by σ to provide the optimal level spacing when the variance of the process is σ^2 . In our case $\sigma^2 = 10$ and $\Delta = \sqrt{10} \cdot 0.3352 = 1.060$. The quantization levels are

$$\begin{aligned}\hat{x}_1 = -\hat{x}_{16} &= -7 \cdot 1.060 - \frac{1}{2} \cdot 1.060 = -7.950 \\ \hat{x}_2 = -\hat{x}_{15} &= -6 \cdot 1.060 - \frac{1}{2} \cdot 1.060 = -6.890 \\ \hat{x}_3 = -\hat{x}_{14} &= -5 \cdot 1.060 - \frac{1}{2} \cdot 1.060 = -5.830 \\ \hat{x}_4 = -\hat{x}_{13} &= -4 \cdot 1.060 - \frac{1}{2} \cdot 1.060 = -4.770 \\ \hat{x}_5 = -\hat{x}_{12} &= -3 \cdot 1.060 - \frac{1}{2} \cdot 1.060 = -3.710 \\ \hat{x}_6 = -\hat{x}_{11} &= -2 \cdot 1.060 - \frac{1}{2} \cdot 1.060 = -2.650 \\ \hat{x}_7 = -\hat{x}_{10} &= -1 \cdot 1.060 - \frac{1}{2} \cdot 1.060 = -1.590 \\ \hat{x}_8 = -\hat{x}_9 &= -\frac{1}{2} \cdot 1.060 = -0.530\end{aligned}$$

The boundaries of the quantization regions are given by

$$\begin{aligned}a_1 = a_{15} &= -7 \cdot 1.060 = -7.420 \\ a_2 = a_{14} &= -6 \cdot 1.060 = -6.360 \\ a_3 = a_{13} &= -5 \cdot 1.060 = -5.300 \\ a_4 = a_{12} &= -4 \cdot 1.060 = -4.240 \\ a_5 = a_{11} &= -3 \cdot 1.060 = -3.180 \\ a_6 = a_{10} &= -2 \cdot 1.060 = -2.120 \\ a_7 = a_9 &= -1 \cdot 1.060 = -1.060 \\ a_8 &= 0\end{aligned}$$

2) The resulting distortion is $D = \sigma^2 \cdot 0.01154 = 0.1154$.

3) The entropy is available from Table 4.2. Nevertheless we will rederive the result here. The probabilities of the 16 outputs are

$$\begin{aligned}
 p(\hat{x}_1) = p(\hat{x}_{16}) &= Q\left(\frac{a_{15}}{\sqrt{10}}\right) = 0.0094 \\
 p(\hat{x}_2) = p(\hat{x}_{15}) &= Q\left(\frac{a_{14}}{\sqrt{10}}\right) - Q\left(\frac{a_{15}}{\sqrt{10}}\right) = 0.0127 \\
 p(\hat{x}_3) = p(\hat{x}_{14}) &= Q\left(\frac{a_{13}}{\sqrt{10}}\right) - Q\left(\frac{a_{14}}{\sqrt{10}}\right) = 0.0248 \\
 p(\hat{x}_4) = p(\hat{x}_{13}) &= Q\left(\frac{a_{12}}{\sqrt{10}}\right) - Q\left(\frac{a_{13}}{\sqrt{10}}\right) = 0.0431 \\
 p(\hat{x}_5) = p(\hat{x}_{12}) &= Q\left(\frac{a_{11}}{\sqrt{10}}\right) - Q\left(\frac{a_{12}}{\sqrt{10}}\right) = 0.0674 \\
 p(\hat{x}_6) = p(\hat{x}_{11}) &= Q\left(\frac{a_{10}}{\sqrt{10}}\right) - Q\left(\frac{a_{11}}{\sqrt{10}}\right) = 0.0940 \\
 p(\hat{x}_7) = p(\hat{x}_{10}) &= Q\left(\frac{a_9}{\sqrt{10}}\right) - Q\left(\frac{a_{10}}{\sqrt{10}}\right) = 0.1175 \\
 p(\hat{x}_8) = p(\hat{x}_9) &= Q\left(\frac{a_8}{\sqrt{10}}\right) - Q\left(\frac{a_9}{\sqrt{10}}\right) = 0.1311
 \end{aligned}$$

Hence, the entropy of the quantized source is

$$H(\hat{X}) = -\sum_{i=1}^{16} 6p(\hat{x}_i) \log_2 p(\hat{x}_i) = 3.6025$$

This is the minimum number of bits per source symbol required to represent the quantized source.

4) Substituting $\sigma^2 = 10$ and $D = 0.1154$ in the rate-distortion bound, we obtain

$$R = \frac{1}{2} \log_2 \frac{\sigma^2}{D} = 3.2186$$

5) The distortion of the 16-level optimal quantizer is $D_{16} = \sigma^2 \cdot 0.01154$ whereas that of the 8-level optimal quantizer is $D_8 = \sigma^2 \cdot 0.03744$. Hence, the amount of increase in SQNR (db) is

$$10 \log_{10} \frac{\text{SQNR}_{16}}{\text{SQNR}_8} = 10 \cdot \log_{10} \frac{0.03744}{0.01154} = 5.111 \text{ db}$$

Problem 4.48

Using Table 4.3 we find the quantization regions and the quantized values for $N = 16$. These values should be multiplied by $\sigma = P_X^{1/2} = \sqrt{10}$, since Table 4.3 provides the optimum values for a unit variance Gaussian source.

$$\begin{aligned}
 a_1 = -a_{15} &= -\sqrt{10} \cdot 2.401 = -7.5926 \\
 a_2 = -a_{14} &= -\sqrt{10} \cdot 1.844 = -5.8312 \\
 a_3 = -a_{13} &= -\sqrt{10} \cdot 1.437 = -4.5442 \\
 a_4 = -a_{12} &= -\sqrt{10} \cdot 1.099 = -3.4753 \\
 a_5 = -a_{11} &= -\sqrt{10} \cdot 0.7996 = -2.5286 \\
 a_6 = -a_{10} &= -\sqrt{10} \cdot 0.5224 = -1.6520 \\
 a_7 = -a_9 &= -\sqrt{10} \cdot 0.2582 = -0.8165 \\
 a_8 &= 0
 \end{aligned}$$

The quantized values are

$$\begin{aligned}
 \hat{x}_1 = -\hat{x}_{16} &= -\sqrt{10} \cdot 2.733 = -8.6425 \\
 \hat{x}_2 = -\hat{x}_{15} &= -\sqrt{10} \cdot 2.069 = -6.5428 \\
 \hat{x}_3 = -\hat{x}_{14} &= -\sqrt{10} \cdot 1.618 = -5.1166
 \end{aligned}$$

$$\begin{aligned}
\hat{x}_4 = -\hat{x}_{13} &= -\sqrt{10} \cdot 1.256 = -3.9718 \\
\hat{x}_5 = -\hat{x}_{12} &= -\sqrt{10} \cdot 0.9424 = -2.9801 \\
\hat{x}_6 = -\hat{x}_{11} &= -\sqrt{10} \cdot 0.6568 = -2.0770 \\
\hat{x}_7 = -\hat{x}_{10} &= -\sqrt{10} \cdot 0.3881 = -1.2273 \\
\hat{x}_8 = -\hat{x}_9 &= -\sqrt{10} \cdot 0.1284 = -0.4060
\end{aligned}$$

The resulting distortion is $D = 10 \cdot 0.009494 = 0.09494$. From Table 4.3 we find that the minimum number of bits per source symbol is $H(\hat{X}) = 3.765$. Setting $D = 0.09494$, $\sigma^2 = 10$ in $R = \frac{1}{2} \log_2 \frac{\sigma^2}{D}$ we obtain $R = 3.3594$. Thus, the minimum number of bits per source symbol is slightly larger than the predicted one from the rate-distortion bound.

Please let me know if you find any mistake in this solution
