ESE 532
HW#6 Solutions

Problem 4.56
Suppose that the transmitted sequence is $\mathbf{x}$. If an error occurs at the i^{th} bit of the sequence, then the received sequence $\mathbf{x}'$ is

$$\mathbf{x}' = \mathbf{x} + [0 \ldots 010 \ldots 0]$$

where addition is modulo 2. Thus the error sequence is $e_i = [0 \ldots 010 \ldots 0]$, which in natural binary coding has the value 2^{i-1}. If the spacing between levels is Δ, then the error introduced by the channel is $2^{i-1}\Delta$.

2)

$$\begin{aligned} D_{\text{channel}} &= \sum_{i=1}^{\nu} p(\text{error in } i \text{ bit}) \cdot (2^{i-1}\Delta)^2 \\ &= \sum_{i=1}^{\nu} p_b \Delta^2 4^{i-1} = p_b \Delta^2 \frac{1-4^\nu}{1-4} \\ &= p_b \Delta^2 \frac{4^\nu - 1}{3} \end{aligned}$$

3) The total distortion is

$$\begin{aligned} D_{\text{total}} &= D_{\text{channel}} + D_{\text{quantiz.}} = p_b \Delta^2 \frac{4^\nu - 1}{3} + \frac{x_{\max}^2}{3 \cdot N^2} \\ &= p_b \frac{4 \cdot x_{\max}^2}{N^2} \frac{4^\nu - 1}{3} + \frac{x_{\max}^2}{3 \cdot N^2} \end{aligned}$$

or since $N = 2^\nu$

$$D_{\text{total}} = \frac{x_{\max}^2}{3 \cdot 4^\nu}(1 + 4p_b(4^\nu - 1)) = \frac{x_{\max}^2}{3N^2}(1 + 4p_b(N^2 - 1))$$

4)

$$\text{SNR} = \frac{E[X^2]}{D_{\text{total}}} = \frac{E[X^2] 3N^2}{x_{\max}^2 (1 + 4p_b(N^2 - 1))}$$

If we let $\breve{X} = \frac{X}{x_{\max}}$, then $\frac{E[X^2]}{x_{\max}^2} = E[\breve{X}^2] = \overline{\breve{X}^2}$. Hence,

$$\text{SNR} = \frac{3N^2 \overline{\breve{X}^2}}{1 + 4p_b(N^2 - 1)} = \frac{3 \cdot 4^\nu \overline{\breve{X}^2}}{1 + 4p_b(4^\nu - 1)}$$

Problem 4.57
1)

$$g(x) = \frac{\log(1 + \mu \frac{|x|}{x_{\max}})}{\log(1 + \mu)} \text{sgn}(x)$$

Differentiating the previous using natural logarithms, we obtain

$$g'(x) = \frac{1}{\ln(1+\mu)} \frac{\mu / x_{\max}}{(1 + \mu \frac{|x|}{x_{\max}})} \text{sgn}^2(x)$$

Since, for the μ-law compander $y_{\max} = g(x_{\max}) = 1$, we obtain

$$D \approx \frac{y_{\max}^2}{3 \times 4^\nu} \int_{-\infty}^{\infty} \frac{f_X(x)}{[g'(x)]^2} dx$$

$$
\begin{aligned}
&= \frac{x_{\max}^2[\ln(1+\mu)]^2}{3\times 4^\nu \mu^2}\int_{-\infty}^{\infty}\left(1+\mu^2\frac{|x|^2}{x_{\max}^2}+2\mu\frac{|x|}{x_{\max}}\right)f_X(x)dx \\
&= \frac{x_{\max}^2[\ln(1+\mu)]^2}{3\times 4^\nu \mu^2}\left[1+\mu^2 E[\breve{X}^2]+2\mu E[|\breve{X}|]\right] \\
&= \frac{x_{\max}^2[\ln(1+\mu)]^2}{3\times N^2 \mu^2}\left[1+\mu^2 E[\breve{X}^2]+2\mu E[|\breve{X}|]\right]
\end{aligned}
$$

where $N^2 = 4^\nu$ and $\breve{X} = X/x_{\max}$.

2)

$$
\begin{aligned}
\text{SQNR} &= \frac{E[X^2]}{D} \\
&= \frac{E[X^2]}{x_{\max}^2}\frac{\mu^2 3\cdot N^2}{[\ln(1+\mu)]^2(\mu^2 E[\breve{X}^2]+2\mu E[|\breve{X}|]+1)} \\
&= \frac{3\mu^2 N^2 E[\breve{X}^2]}{[\ln(1+\mu)]^2(\mu^2 E[\breve{X}^2]+2\mu E[|\breve{X}|]+1)}
\end{aligned}
$$

3) Since $\text{SQNR}_{\text{unif}} = 3\cdot N^2 E[\breve{X}^2]$, we have

$$
\begin{aligned}
\text{SQNR}_{\mu_{\text{law}}} &= \text{SQNR}_{\text{unif}}\frac{\mu^2}{[\ln(1+\mu)]^2(\mu^2 E[\breve{X}^2]+2\mu E[|\breve{X}|]+1)} \\
&= \text{SQNR}_{\text{unif}} G(\mu,\breve{X})
\end{aligned}
$$

where we identify

$$
G(\mu,\breve{X}) = \frac{\mu^2}{[\ln(1+\mu)]^2(\mu^2 E[\breve{X}^2]+2\mu E[|\breve{X}|]+1)}
$$

3) The truncated Gaussian distribution has a PDF given by

$$
f_Y(y) = \frac{K}{\sqrt{2\pi}\sigma_x}e^{-\frac{x^2}{2\sigma_x^2}}
$$

where the constant K is such that

$$
K\int_{-4\sigma_x}^{4\sigma_x}\frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{x^2}{2\sigma_x^2}}dx = 1 \Longrightarrow K = \frac{1}{1-2Q(4)} = 1.0001
$$

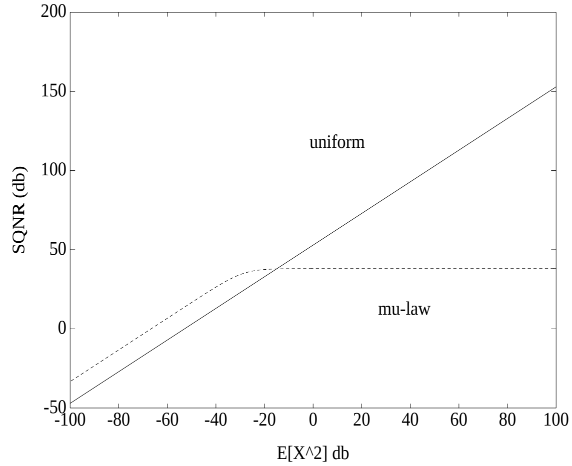
Hence,

$$
\begin{aligned}
E[|\breve{X}|] &= \frac{K}{\sqrt{2\pi}\sigma_x}\int_{-4\sigma_x}^{4\sigma_x}\frac{|x|}{4\sigma_x}e^{-\frac{x^2}{2\sigma_x^2}}dx \\
&= \frac{2K}{4\sqrt{2\pi}\sigma_x^2}\int_0^{4\sigma_x} xe^{-\frac{x^2}{2\sigma_x^2}}dx \\
&= \frac{K}{2\sqrt{2\pi}\sigma_x^2}\left[-\sigma_x^2 e^{-\frac{x^2}{2\sigma_x^2}}\right]_0^{4\sigma_x} \\
&= \frac{K}{2\sqrt{2\pi}}(1-e^{-2}) = 0.1725
\end{aligned}
$$

In the next figure we plot $10\log_{10}\text{SQNR}_{\text{unif}}$ and $10\log_{10}\text{SQNR}_{mu-\text{law}}$ vs. $10\log_{10}E[\breve{X}^2]$ when the latter varies from -100 to 100 db. As it is observed the μ-law compressor is insensitive to the dynamic range of the input signal for $E[\breve{X}^2] > 1$.

Problem 4.58
The optimal compressor has the form

$$
g(x) = y_{\max}\left[\frac{2\int_{-\infty}^{x}[f_X(v)]^{\frac{1}{3}}dv}{\int_{-\infty}^{\infty}[f_X(v)]^{\frac{1}{3}}dv} - \right]
$$



where $y_{\max} = g(x_{\max}) = g(1)$.

$$\begin{aligned}\int_{-\infty}^{\infty} [f_X(v)]^{\frac{1}{3}} dv &= \int_{-1}^{1} [f_X(v)]^{\frac{1}{3}} dv = \int_{-1}^{0} (v+1)^{\frac{1}{3}} dv + \int_{0}^{1} (-v+1)^{\frac{1}{3}} dv \\ &= 2\int_0^1 x^{\frac{1}{3}} dx = \frac{3}{2}\end{aligned}$$

If $x \leq 0$, then

$$\begin{aligned}\int_{-\infty}^{x} [f_X(v)]^{\frac{1}{3}} dv &= \int_{-1}^{x} (v+1)^{\frac{1}{3}} dv = \int_{0}^{x+1} z^{\frac{1}{3}} dz = \frac{3}{4} z^{\frac{4}{3}} \Big|_0^{x+1} \\ &= \frac{3}{4}(x+1)^{\frac{4}{3}}\end{aligned}$$

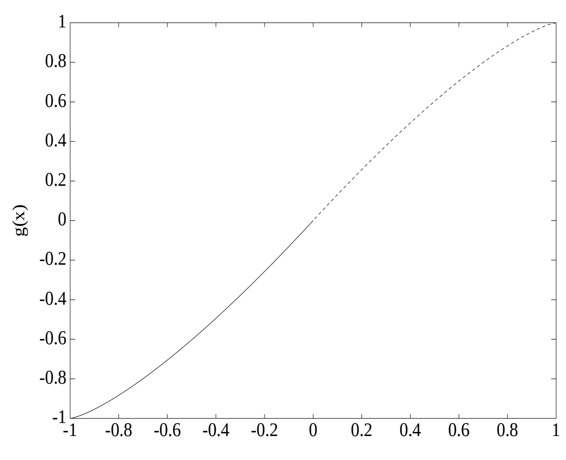
If $x > 0$, then

$$\begin{aligned}\int_{-\infty}^{x} [f_X(v)]^{\frac{1}{3}} dv &= \int_{-1}^{0} (v+1)^{\frac{1}{3}} dv + \int_{0}^{x} (-v+1)^{\frac{1}{3}} dv = \frac{3}{4} + \int_{1-x}^{1} z^{\frac{1}{3}} dz \\ &= \frac{3}{4} + \frac{3}{4}\left(1 - (1-x)^{\frac{4}{3}}\right)\end{aligned}$$

Hence,

$$g(x) = \begin{cases} g(1)\left[(x+1)^{\frac{4}{3}} - 1\right] & -1 \leq x < 0 \\ g(1)\left[1 - (1-x)^{\frac{4}{3}}\right] & 0 \leq x \leq 1 \end{cases}$$

The next figure depicts $g(x)$ for $g(1) = 1$. Since the resulting distortion is (see Equation 4.6.17)



$$D = \frac{1}{12 \times 4^{\nu}} \left[\int_{-infty}^{\infty} [f_X(x)]^{\frac{1}{3}} dx \right]^3 = \frac{1}{12 \times 4^{\nu}} \left(\frac{3}{2} \right)^3$$

we have

$$\text{SQNR} = \frac{E[X^2]}{D} = \frac{32}{9} \times 4^{\nu} E[X^2] = \frac{32}{9} \times 4^{\nu} \cdot \frac{1}{6} = \frac{16}{27} 4^{\nu}$$

Problem 4.59

The sampling rate is $f_s = 44100$ meaning that we take 44100 samples per second. Each sample is quantized using 16 bits so the total number of bits per second is 44100×16. For a music piece of duration 50 min $= 3000$ sec the resulting number of bits is

$$44100 \times 16 \times 3000 = 2.1168 \times 10^9$$
