ESE 532
HW#7 Solutions

Problem 4.35

1) The PDF of the random variable $Y = \alpha X$ is

$$f_Y(y) = \frac{1}{|\alpha|} f_X(\frac{y}{\alpha})$$

Hence,

$$\begin{aligned}
h(Y) &= -\int_{-\infty}^{\infty} f_Y(y)\log(f_Y(y))dy \\
&= -\int_{-\infty}^{\infty} \frac{1}{|\alpha|} f_X(\frac{y}{\alpha})\log\left(\frac{1}{|\alpha|} f_X(\frac{y}{\alpha})\right)dy \\
&= -\log\left(\frac{1}{|\alpha|}\right)\int_{-\infty}^{\infty} \frac{1}{|\alpha|} f_X(\frac{y}{\alpha})dy - \int_{-\infty}^{\infty} \frac{1}{|\alpha|} f_X(\frac{y}{\alpha})\log\left(f_X(\frac{y}{\alpha})\right)dy \\
&= -\log\left(\frac{1}{|\alpha|}\right) + h(X) = \log|\alpha| + h(X)
\end{aligned}$$

2) A similar relation does not hold if X is a discrete random variable. Suppose for example that X takes the values $\{x_1, x_2, \ldots, x_n\}$ with probabilities $\{p_1, p_2, \ldots, p_n\}$. Then, $Y = \alpha X$ takes the values $\{\alpha x_1, \alpha x_2, \ldots, \alpha x_n\}$ with probabilities $\{p_1, p_2, \ldots, p_n\}$, so that

$$H(Y) = -\sum_i p_i \log p_i = H(X)$$

Problem 4.40

1) The entropy of the source is

$$H(X) = -.25\log_2 .25 - .75\log_2 .75 = .8113 \text{ bits/symbol}$$

Thus, we can transmit the output of the source using $2000H(X) = 1623$ bits/sec with arbitrarily small probability of error.

2) Since $0 \le D \le \min\{p, 1-p\} = .25$ the rate distortion function for the binary memoryless source is

$$R(D) = H_b(p) - H_b(D) = H_b(.25) - H_b(.1) = .8113 - .4690 = .3423$$

Hence, the required number of bits per second is $2000R(D) = 685$.

3) For $D = .25$ the rate is $R(D) = 0$. We can reproduce the source at a distortion of $D = .25$ with no transmission at all by setting the reproduction vector to be the all zero vector.

Problem 4.41

1) For a zero-mean Gaussian source with variance σ^2 and with squared error distortion measure, the rate distortion function is given by

$$R(D) = \begin{cases} \frac{1}{2}\log\frac{\sigma^2}{D} & 0 \le D \le \sigma^2 \\ 0 & \text{otherwise} \end{cases}$$

With $R = 1$ and $\sigma^2 = 1$, we obtain

$$2 = \log\frac{1}{D} \Longrightarrow D = 2^{-2} = 0.25$$

2) If we set $D = 0.01$, then

$$R = \frac{1}{2}\log\frac{1}{0.01} = \frac{1}{2}\log 100 = 3.322 \text{ bits/sample}$$

Hence, the required transmission capacity is 3.322 bits per source symbol.

Problem 4.42

1) Since $R(D) = \log \frac{\lambda}{D}$ and $D = \frac{\lambda}{2}$, we obtain $R(D) = \log(\frac{\lambda}{\lambda/2}) = \log(2) = 1$ bit/sample.

2) The following figure depicts $R(D)$ for $\lambda = 0.1$, .2 and .3. As it is observed from the figure, an increase of the parameter λ increases the required rate for a given distortion.