ESE 532
HW#8 Solutions

Problem 10.20
The codewords of the linear code of Example 10.5.1 are

$$\begin{aligned}
\mathbf{c}_1 &= [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ] \\
\mathbf{c}_2 &= [\ 1\ \ 0\ \ 1\ \ 0\ \ 0\ ] \\
\mathbf{c}_3 &= [\ 0\ \ 1\ \ 1\ \ 1\ \ 1\ ] \\
\mathbf{c}_4 &= [\ 1\ \ 1\ \ 0\ \ 1\ \ 1\ ]
\end{aligned}$$

Since the code is linear the minimum distance of the code is equal to the minimum weight of the codewords. Thus,

$$d_{\min} = w_{\min} = 2$$

There is only one codeword with weight equal to 2 and this is $\mathbf{c}_2$.

Problem 10.21
The parity check matrix of the code in Example 10.5.3 is

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The codewords of the code are

$$\begin{aligned}
\mathbf{c}_1 &= [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ] \\
\mathbf{c}_2 &= [\ 1\ \ 0\ \ 1\ \ 0\ \ 0\ ] \\
\mathbf{c}_3 &= [\ 0\ \ 1\ \ 1\ \ 1\ \ 1\ ] \\
\mathbf{c}_4 &= [\ 1\ \ 1\ \ 0\ \ 1\ \ 1\ ]
\end{aligned}$$

Any of the previous codewords when postmultiplied by $\mathbf{H}^t$ produces an all-zero vector of length 3. For example

$$\begin{aligned}
\mathbf{c}_2\mathbf{H}^t &= [\ 1 \oplus 1\ \ \ 0\ \ \ 0\ ] = [\ 0\ \ \ 0\ \ \ 0\ ] \\
\mathbf{c}_4\mathbf{H}^t &= [\ 1 \oplus 1\ \ \ 1 \oplus 1\ \ \ 1 \oplus 1\ ] = [\ 0\ \ \ 0\ \ \ 0\ ]
\end{aligned}$$

Problem 10.22
The following table lists all the codewords of the (7,4) Hamming code along with their weight. Since the Hamming codes are linear $d_{\min} = w_{\min}$. As it is observed from the table the minimum weight is 3 and therefore $d_{\min} = 3$.

No.	Codewords	Weight
1	0000000	0
2	1000110	3
3	0100011	3
4	0010101	3
5	0001111	4
6	1100101	4
7	1010011	4
8	1001001	3
9	0110110	4
10	0101100	3
11	0011010	3
12	1110000	3
13	1101010	4
14	1011100	4
15	0111001	4
16	1111111	7

Problem 10.23

The parity check matrix $\mathbf{H}$ of the (15,11) Hamming code consists of all binary sequences of length 4, except the all zero sequence. The systematic form of the matrix $\mathbf{H}$ is

$$\mathbf{H} = [\ \mathbf{P}^t \quad | \quad \mathbf{I}_4\ ] = \left(\begin{array}{ccccccccccc|cccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

The corresponding generator matrix is

$$\mathbf{G} = [\ \mathbf{I}_{11} \quad | \quad \mathbf{P}\ ] = \left(\begin{array}{ccccccccccc|cccc} 1 & & & & & & & & & & & 1 & 1 & 0 & 0 \\ & 1 & & & & & & & & & & 1 & 0 & 1 & 0 \\ & & 1 & & & & & \mathbf{0} & & & & 1 & 0 & 0 & 1 \\ & & & 1 & & & & & & & & 0 & 1 & 1 & 0 \\ & & & & 1 & & & & & & & 0 & 1 & 0 & 1 \\ & & & & & 1 & & & & & & 0 & 0 & 1 & 1 \\ & & & & & & 1 & & & & & 1 & 1 & 1 & 0 \\ & & & & & & & 1 & & & & 1 & 1 & 0 & 1 \\ & & & \mathbf{0} & & & & & 1 & & & 1 & 0 & 1 & 1 \\ & & & & & & & & & 1 & & 0 & 1 & 1 & 1 \\ & & & & & & & & & & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

Problem 10.24

Let C be an (n, k) linear block code with parity check matrix $\mathbf{H}$. We can express the parity check matrix in the form

$$\mathbf{H} = [\ \mathbf{h}_1 \quad \mathbf{h}_2 \quad \cdots \quad \mathbf{h}_n\ ]$$

where $\mathbf{h}_i$ is an $n-k$ dimensional column vector. Let $\mathbf{c} = [c_1 \cdots c_n]$ be a codeword of the code C with l nonzero elements which we denote as $c_{i_1}, c_{i_2}, \ldots, c_{i_l}$. Clearly $c_{i_1} = c_{i_2} = \ldots = c_{i_l} = 1$ and since $\mathbf{c}$ is a codeword

$$\begin{aligned} \mathbf{c}\mathbf{H}^t = 0 &= c_1\mathbf{h}_1 + c_2\mathbf{h}_2 + \cdots + c_n\mathbf{h}_n \\ &= c_{i_1}\mathbf{h}_{i_1} + c_{i_2}\mathbf{h}_{i_2} + \cdots + c_{i_l}\mathbf{h}_{i_l} \\ &= \mathbf{h}_{i_1} + \mathbf{h}_{i_2} + \cdots + \mathbf{h}_{i_l} = 0 \end{aligned}$$

This proves that l column vectors of the matrix $\mathbf{H}$ are linear dependent. Since for a linear code the minimum value of l is $w_{\min}$ and $w_{\min} = d_{\min}$, we conclude that there exist $d_{\min}$ linear dependent column vectors of the matrix $\mathbf{H}$.
Now we assume that the minimum number of column vectors of the matrix $\mathbf{H}$ that are linear dependent is $d_{\min}$ and we will prove that the minimum weight of the code is $d_{\min}$. Let $\mathbf{h}_{i_1}, \mathbf{h}_{i_2}, \ldots, \mathbf{h}_{d_{\min}}$ be a set of linear dependent column vectors. If we form a vector $\mathbf{c}$ with non-zero components at positions $i_1, i_2, \ldots, i_{d_{\min}}$, then

$$\mathbf{c}\mathbf{H}^t = c_{i_1}\mathbf{h}_{i_1} + \cdots + c_{i_{d_{\min}}} = 0$$

which implies that $\mathbf{c}$ is a codeword with weight $d_{\min}$. Therefore, the minimum distance of a code is equal to the minimum number of columns of its parity check matrix that are linear dependent.
For a Hamming code the columns of the matrix $\mathbf{H}$ are non-zero and distinct. Thus, no two columns $\mathbf{h}_i$, $\mathbf{h}_j$ add to zero and since $\mathbf{H}$ consists of all the $n-k$ tuples as its columns, the sum $\mathbf{h}_i + \mathbf{h}_j = \mathbf{h}_m$ should also be a column of $\mathbf{H}$. Then,

$$\mathbf{h}_i + \mathbf{h}_j + \mathbf{h}_m = 0$$

and therefore the minimum distance of the Hamming code is 3.

Problem 10.25

The generator matrix of the $(n, 1)$ repetition code is a $1 \times n$ matrix, consisted of the non-zero codeword. Thus,

$$\mathbf{G} = \left[\ 1 \quad | \quad 1 \quad \cdots \quad 1\ \right]$$

This generator matrix is already in systematic form, so that the parity check matrix is given by

$$\mathbf{H} = \left(\begin{array}{c|cccc} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{array}\right)$$