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ESE 532  
HW#9 Solutions

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**Problem 10.26**

1) The parity check matrix  $\mathbf{H}_e$  of the extended code is an  $(n+1-k) \times (n+1)$  matrix. The codewords of the extended code have the form

$$\mathbf{c}_{e,i} = [ \mathbf{c}_i \mid x ]$$

where  $x$  is 0 if the weight of  $\mathbf{c}_i$  is even and 1 if the weight of  $\mathbf{c}_i$  is odd. Since  $\mathbf{c}_{e,i}\mathbf{H}_e^t = [\mathbf{c}_i|x]\mathbf{H}_e^t = 0$  and  $\mathbf{c}_i\mathbf{H}^t = 0$ , the first  $n-k$  columns of  $\mathbf{H}_e^t$  can be selected as the columns of  $\mathbf{H}^t$  with a zero added in the last row. In this way the choice of  $x$  is immaterial. The last column of  $\mathbf{H}_e^t$  is selected in such a way that the even-parity condition is satisfied for every codeword  $\mathbf{c}_{e,i}$ . Note that if  $\mathbf{c}_{e,i}$  has even weight, then

$$c_{e,i_1} + c_{e,i_2} + \dots + c_{e,i_{n+1}} = 0 \implies \mathbf{c}_{e,i} [ 1 \ 1 \ \dots \ 1 ]^t = 0$$

for every  $i$ . Therefore the last column of  $\mathbf{H}_e^t$  is the all-one vector and the parity check matrix of the extended code has the form

$$\mathbf{H}_e = (\mathbf{H}_e^t)^t = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

2) The original code has minimum distance equal to 3. But for those codewords with weight equal to the minimum distance, a 1 is appended at the end of the codewords to produce even parity. Thus, the minimum weight of the extended code is 4 and since the extended code is linear, the minimum distance is  $d_{e,\min} = w_{e,\min} = 4$ .

3) The coding gain of the extended code is

$$G_{\text{coding}} = d_{e,\min} R_c = 4 \times \frac{3}{7} = 1.7143$$

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**Problem 10.27**

If no coding is employed, we have

$$p_b = Q \left[ \sqrt{\frac{2\mathcal{E}_b}{N_0}} \right] = Q \left[ \sqrt{\frac{P}{RN_0}} \right]$$

where

$$\frac{P}{RN_0} = \frac{10^{-6}}{10^4 \times 2 \times 10^{-11}} = 5$$

Thus,

$$p_b = Q[\sqrt{5}] = 1.2682 \times 10^{-2}$$

and therefore, the error probability for 11 bits is

$$P_{\text{error in 11 bits}} = 1 - (1 - p_b)^{11} \approx 0.1310$$

If coding is employed, then since the minimum distance of the (15, 11) Hamming code is 3,

$$p_e \leq (M-1)Q \left[ \sqrt{\frac{d_{\min}\mathcal{E}_s}{N_0}} \right] = 10Q \left[ \sqrt{\frac{3\mathcal{E}_s}{N_0}} \right]$$

where

$$\frac{\mathcal{E}_s}{N_0} = R_c \frac{\mathcal{E}_b}{N_0} = R_c \frac{P}{RN_0} = \frac{11}{15} \times 5 = 3.6667$$

Thus

$$p_e \leq 10Q[\sqrt{3 \times 3.6667}] \approx 4.560 \times 10^{-3}$$

As it is observed the probability of error decreases by a factor of 28. If hard decision is employed, then

$$p_e \leq (M - 1) \sum_{i=\frac{d_{\min}+1}{2}}^{d_{\min}} \binom{d_{\min}}{i} p_b^i (1 - p_b)^{d_{\min}-i}$$

where  $M = 10$ ,  $d_{\min} = 3$  and  $p_b = Q\left[\sqrt{R_c \frac{P}{RN_0}}\right] = 2.777 \times 10^{-2}$ . Hence,

$$p_e = 10 \times (3 \times p_b^2(1 - p_b) + p_b^3) = 0.0227$$

In this case coding has decreased the error probability by a factor of 6.

### Problem 10.28

The following table shows the standard array for the (7,4) Hamming code.

		$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
		1000000	0100000	0010000	0001000	0000100	0000010	0000001
$\mathbf{c}_1$	0000000	1000000	0100000	0010000	0001000	0000100	0000010	0000001
$\mathbf{c}_2$	1000110	0000110	1100110	1010110	1001110	1000010	1000100	1000111
$\mathbf{c}_3$	0100011	1100011	0000011	0110011	0101011	0100111	0100001	0100010
$\mathbf{c}_4$	0010101	1010101	0110101	0000101	0011101	0010001	0010111	0010100
$\mathbf{c}_5$	0001111	1001111	0101111	0011111	0000111	0001011	0001101	0001110
$\mathbf{c}_6$	1100101	0100101	1000101	1110101	1101101	1100001	1100111	1100100
$\mathbf{c}_7$	1010011	0010011	1110011	1000011	1011011	1010111	1010001	1010010
$\mathbf{c}_8$	1001001	0001001	1101001	1011001	1000001	1001101	1001011	1001000
$\mathbf{c}_9$	0110110	1110110	0010110	0100110	0111110	0110010	0110100	0110111
$\mathbf{c}_{10}$	0101100	1101100	0001100	0111100	0100100	0101000	0101110	0101101
$\mathbf{c}_{11}$	0011010	1011010	0111010	0001010	0010010	0011110	0011000	0011011
$\mathbf{c}_{12}$	1110000	0110000	1010000	1100000	1111000	1110100	1110010	1110001
$\mathbf{c}_{13}$	1101010	0101010	1001010	1111010	1100010	1101110	1101000	1101011
$\mathbf{c}_{14}$	1011100	0011100	1111100	1001100	1010100	1011000	1011110	1011101
$\mathbf{c}_{15}$	0111001	1111001	0011001	0101001	0110001	0111101	0111011	0111000
$\mathbf{c}_{16}$	1111111	0111111	1011111	1101111	1110111	1111011	1111101	1111110

As it is observed the received vector  $\mathbf{y} = [1110100]$  is in the 7<sup>th</sup> column of the table under the error vector  $\mathbf{e}_5$ . Thus, the received vector will be decoded as

$$\mathbf{c} = \mathbf{y} + \mathbf{e}_5 = [1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0] = \mathbf{c}_{12}$$