ESE 535 —*Information Theory and Reliable Communications*

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SUNY at Stony Brook

Final Exam — N. Phamdo

Closed Books, Closed Notes, 3 Hours — May 11, 1998

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1. **(25 Pts Total)** Decide whether each of the following statements is true or false. Prove the validity of those that are true and give counterexamples or arguments based on known facts to disprove those that are false.

   (a) **(5 pts)** There exists a binary random variable, X, with entropy $H(X) = \sqrt{2}$.

   (b) **(5 pts)** The Kraft inequality can sometimes be used to show that a code is not uniquely decodable, but it can never be used to show that a code *is* uniquely decodable.

   (c) **(5 pts)** If X and Y are real-valued independent random variables, then $h(X + Y) \geq h(X)$.

   (d) **(5 pts)** Given an i.i.d. Gaussian source which produces 5 samples/second and a binary symmetric channel (BSC) which is used 30 times/second with crossover probability 0.11, there exists a coding scheme with expected squared-error distortion (in SDR) of 16 dB or better. [Hint: $h_b(0.11) = 0.5$.]

   (e) **(5 pts)** Let $\mathcal{X} = \{X_n\}_{n=1}^{\infty}$ be a discrete i.i.d. source with alphabet $\mathscr{X}$. If $\mathcal{Y} = \{Y_n\}_{n=1}^{\infty}$ is a discrete (strictly) stationary source on the same alphabet, $\mathscr{X}$, and is such that for all letters $a \in \mathscr{X}$,

   $$\Pr\{X_1 = a\} = \Pr\{Y_1 = a\},$$

   then $H(\mathcal{Y}) \leq H(\mathcal{X})$.

2. **(20 Pts Total)** Consider a stationary and ergodic source $\{X_n\}_{n=1}^{\infty}$ on the binary alphabet $\mathscr{X} = \{-1, +1\}$. Assume a reproduction alphabet $\widehat{\mathscr{X}} = \mathbb{R}$ and a squared-error distortion measure. From experimental measurements, we have determined that $\Pr\{X_n = -1\} = \Pr\{X_n = +1\} = 1/2$, i.e., the source is uniformly distributed. However, we do not know whether the source has memory or not. From years of research and development, we have been able to construct a 40-dimensional quantizer, $Q(\cdot)$, of rate $R = 1/4$ and average squared-error distortion of $D = 1/4$.

   (a) **(10 pts)** Given $Q(\cdot)$, find an 80-dimensional quantizer of rate $R = 5/8$ and distortion $D = 1/8$.

   (b) **(10 pts)** Based on the information provided above, find the tightest upper-bound on the rate-distortion function of the source for the squared-error distortion measure.

3. **(25 Pts Total)** A discrete memoryless channel (DMC) is given by

$$Y = X + Z \pmod 4,$$

where X is the channel input, Y is the channel output and Z is the noise. The noise distribution is given by

$$\Pr\{Z = 0\} = \Pr\{Z = 1\} = 1/2.$$

The channel has identical input and output alphabets, $\mathcal{X} = \{0, 1, 2, 3\}$.

(a) **(5 pts)** Determine the channel probability transition matrix.

(b) **(10 pts)** Determine the capacity, C, of this channel.

(c) **(10 pts)** Find a sequence of $(2^{nC}, n)$ codes for which the maximal probability of error $\lambda_{max} = 0$ for all n. [Give both encoder and decoder.]

4. **(30 Pts Total)** Consider a parallel Gaussian noise channel:

$$Y_i = X_i + Z_i, \qquad i = 1, 2, \ldots, n,$$

where $Z_i \sim \mathcal{N}(0, i^2)$. Furthermore, Z_i and Z_j are independent for $i \neq j$. The input power is constrained by

$$\sum_{i=1}^{n} \frac{E[X_i^2]}{i} \leq 7.$$

Find the capacity of the channel and the values of $E[X_i^2]$ that achieve capacity for the cases of $n = 2$, $n = 4$ and $n = 354$.

[Hint: First scale the input and noise on each channel (by $\sqrt{i}$) to reduce to the previously solved problem.]

Have an enjoyable summer !!!
