ESE 535 —*Information Theory and Reliable Communications*

Electrical Engineering Department
SUNY at Stony Brook

Final Exam — N. Phamdo
Closed Books, Closed Notes, 3 Hours — May 12, 1999

---

1. **(30 Pts Total)** Decide whether each of the following statements is true or false. Prove the validity of those that are true and give counterexamples or arguments based on known facts to disprove those that are false.

   (a) **(6 pts)** Let X and Y denote two discrete random variables, and let $g(\cdot)$ be a function of Y. Then $H(X|g(Y)) \geq H(X|Y)$.

   (b) **(6 pts)** If $0 < p < q < 1$, then the capacity (in bits/channel use) of a binary symmetric channel with crossover probability p is strictly greater than the capacity (in bits/channel use) of a binary symmetric channel with crossover probability q.

   (c) **(6 pts)** Let $\mathcal{X}$ and $\mathcal{Y}$ be two discrete memoryless sources with the same entropy rate, $H(\mathcal{X}) = H(\mathcal{Y})$. Let $R_{\mathcal{X},n}$ and $R_{\mathcal{Y},n}$ be the rates (in bits/sample) of the n-order Huffman codes for $\mathcal{X}$ and $\mathcal{Y}$, respectively. Then

   $$\lim_{n\to\infty} R_{\mathcal{X},n} = \lim_{n\to\infty} R_{\mathcal{Y},n}$$

   (d) **(6 pts)** There exists a binary symmetric channel for which the capacity $C = \pi/3$ bits/channel use.

   (e) **(6 pts)** Consider an i.i.d. Gaussian source (mean μ, variance σ^2) with the squared-error distortion measure and distortion-rate function, $D(R)$. There exists a one-dimensional ($n = 1$) rate-distortion code with rate $R = 0$ and average squared-error distortion $D(0)$. [Hint: The distortion-rate function is the inverse of the rate-distortion function.]

2. **(30 Pts Total)** Consider a stationary and ergodic source $\{X_n\}_{n=1}^{\infty}$ on the binary alphabet $\mathscr{X} = \{-1, +1\}$. Assume a reproduction alphabet $\hat{\mathscr{X}} = \mathbb{R}$ and a squared-error distortion measure. From experimental measurements, we have determined that $\Pr\{X_n = -1\} = \Pr\{X_n = +1\} = 1/2$, i.e., the source is uniformly distributed. However, we do not know whether the source has memory or not. From years of research and development, we have been able to construct a 40-dimensional quantizer, $Q(\cdot)$, of rate $R = 1/4$ and average squared-error distortion of $D = 1/4$.

   (a) **(10 pts)** Given $Q(\cdot)$, find an 80-dimensional quantizer of rate $R = 5/8$ and distortion $D = 1/8$.

   (b) **(10 pts)** Given $Q(\cdot)$, find an 80-dimensional quantizer of rate $R = 1/8$ and distortion $D = 3/8$.

(c) **(10 pts)** Based on the information provided above, find the tightest upper-bound on the rate-distortion function of the source for the squared-error distortion measure.

3. **(30 Pts Total)** Consider a discrete-time, finite-alphabet and *stationary* process $\{U_n\}_{n=0}^{\infty}$. Define

$$H_{L|L}(U) \triangleq \frac{1}{L} H(U_{2L}, \cdots, U_{L+1} | U_L, \cdots, U_1).$$

(a) **(6 pts)** Show that $H_{L|L}(U)$ is nonincreasing with L.

(b) **(6 pts)** Prove that

$$\lim_{L\to\infty} H_{L|L}(U) = H(\mathcal{U}),$$

where $H(\mathcal{U})$ is the source entropy rate:

$$H(\mathcal{U}) \triangleq \lim_{L\to\infty} \frac{1}{L} H(U_L, \cdots, U_1) = \lim_{L\to\infty} H(U_{2L}|U_{2L-1}, \cdots, U_1).$$

(c) **(6 pts)** For the binary stationary Markov source with transition probability

$$Pr\{U_{n+1} = j | U_n = i\} = \begin{cases} 0.05 & \text{if } j = 1 \text{ and } i = 0, \\ 0.50 & \text{if } j = 0 \text{ and } i = 1, \end{cases}$$

compute its entropy rate $H(\mathcal{U})$.

(d) **(6 pts)** For the Markov source in (c), design first-, and second-order binary Huffman codes. Determine in each case the average code rate. If you use an n-th order Huffman code, what will be the value of the average code rate as $n \to \infty$?

(e) **(6 pts)** Is it possible to *reliably* transmit this Markov source (of part (c)) across a discrete memoryless channel with capacity $C = 0.5$ bits/channel use (assuming one source sample per channel use)?

4. **(30 Pts Total)** Consider an additive colored Gaussian noise channel:

$$Y_i = X_i + Z_i, \qquad i = 1, 2, \ldots,$$

where the input has power constraint $E[X_i^2] \le S$ and Z_i is stationary with zero mean and power spectral density

$$S_Z(f) = \begin{cases} 1/4 & \text{if } |f| \le 1/4, \\ (1 - 2|f|)/2 & \text{if } 1/4 < |f| \le 1/2. \end{cases}$$

Determine the capacity of the channel as a function of S.

Have an enjoyable summer !!!
