

Induced and Spontaneous Emission.

The transition probability per unit time between 2D subbands 2 and 3, induced by a monochromatic radiation $\mathbf{E} = \mathbf{e}_q \cos(\mathbf{q}\mathbf{r} - \Omega t)$ polarized in the direction \mathbf{e}_q at an angle α to the z axis (assume z perpendicular to the 2D plane), is given by

$$w_{\text{ind}} = \frac{2\pi}{\hbar^2} e^2 (E/2)^2 |z_{32}|^2 \cos^2 \alpha \delta(\Omega - \Omega_{32}), \quad (1)$$

where $z_{32} = \langle 3|z|2\rangle$ is the dipolar transition matrix element.[†] If the subband states have finite lifetimes (τ_3 and τ_2) due to collisions with impurities, phonons, etc., then we replace the δ function in Eq. (1) by a normalized lineshape function,

$$\gamma(\Omega) = \frac{\gamma_0/\pi}{(\Omega - \Omega_{32})^2 + \gamma_0^2}, \quad (2)$$

where $2\gamma_0 = \tau_3^{-1} + \tau_2^{-1}$ is the full width at half maximum.

Expressing the energy density of radiation ρ_q in the mode \mathbf{q} in terms of the number of quanta N_q in the volume V

$$\rho_q \equiv \frac{\varepsilon E^2}{2} = \frac{\hbar\Omega N_q}{V}, \quad (3)$$

we can re-write Eq. (1) in the form

$$w_{\text{ind}} = \frac{\pi e^2 \Omega |z_{32}|^2 \cos^2 \alpha}{\hbar \varepsilon V} N_q \delta(\Omega - \Omega_{32}). \quad (4)$$

Equation (4) describes the total induced transition rate per electron (either absorption or stimulated emission), due to the interaction with the single mode \mathbf{q} . The total spontaneous emission rate into the same mode is given by the Einstein relation, $w_{\text{sp}} = w_{\text{ind}}/N_q$,

$$w_{\text{sp}} = \frac{\pi e^2 \Omega |z_{32}|^2 \cos^2 \alpha}{\hbar \varepsilon V} \delta(\Omega - \Omega_{32}). \quad (5)$$

To calculate the total spontaneous emission rate, we must sum Eq. (5) over all modes of the resonator. For a slab waveguide only TM modes contribute, since for all TE waves $\alpha = 0$.

Let us *ignore* for simplicity the waveguide nature of the QCL structure, and take the dispersion relation in the simplest form $\Omega = \bar{c}q$, where $\bar{c} \equiv c/\bar{n}$ with \bar{n} being the refractive index of the medium. For any \mathbf{q} we can choose one mode in the direction $\mathbf{q} \times \hat{\mathbf{z}}$; this mode has $\alpha = 0$ and contributes nothing. For the other (orthogonal) mode we have $\alpha = 90^\circ - \theta$, where θ is the angle between \mathbf{q} and the z -axis. The summation gives:

$$\begin{aligned} w_{\text{sp}}^{\text{tot}} &= \frac{V}{(2\pi)^3} \int w_{\text{sp}}(\mathbf{q}) d\mathbf{q} = \frac{V}{(2\pi)^3} \int_0^\infty \frac{\Omega^2 d\Omega}{\bar{c}^3} \frac{\pi e^2 \Omega |z_{32}|^2}{\hbar \varepsilon V} \delta(\Omega - \Omega_{32}) \int_0^{2\pi} d\phi \int_0^\pi \sin^3 \theta d\theta \\ &= \frac{e^2 \Omega_{32}^3 |z_{32}|^2}{3\pi \hbar \varepsilon \bar{c}^3} \equiv \frac{e^2 R_0}{\hbar} \frac{\bar{n}}{3\pi} \frac{|z_{32}|^2 \Omega^3}{c^2} \equiv \frac{4\alpha \bar{n}}{3} \frac{|z_{32}|^2 \Omega^3}{c^2}, \end{aligned} \quad (6)$$

where $R_0 \equiv c\mu_0 = (c\varepsilon_0)^{-1} = 376.73 \Omega$ is the impedance of free space and $\alpha = e^2/\hbar c$ is the fine structure constant, $\alpha^{-1} \approx 137$. Equation (6) agrees with the standard expression for Einstein's A-coefficient for the case of dipolar electric transitions in a homogeneous medium, see, e.g., "Optical Electronics" by Yariv (4th Ed, p. 150).

[†] In this expression we have neglected the dependence of w on the electron wave vector \mathbf{k} , which arises from the conservation of energy and 2D quasimomentum during intersubband transitions. This dependence would be expressed by replacing $\delta(\Omega - \Omega_{32})$ by $\delta[\Omega - \Omega_{32} - \hbar(\mathbf{k}\cdot\mathbf{q})/m]$. Inclusion of this dependence for an electron gas having a distribution of \mathbf{k} (e.g., Fermi or Boltzmann) is entirely equivalent to the Doppler broadening of the atomic transitions in gases.

Gain.

Replacing the δ -function in Eq. (4) by the lineshape function (2) and expressing $|z_{32}|^2$ in terms of $w_{\text{sp}}^{\text{tot}} = \tau_{\text{sp}}^{-1}$ with the help of Eq. (6), we can put w_{ind} in the following form:

$$w_{\text{ind}} = \frac{3 \cos^2 \alpha \pi^2 \bar{c}^3}{\Omega^2 V \tau_{\text{sp}}} N_{\mathbf{q}} \gamma(\Omega) = \frac{\eta \bar{c}^2 P \gamma(\nu)}{8\pi h \nu^3 \tau_{\text{sp}}} = \frac{\eta \lambda^2 P \gamma(\nu)}{8\pi \bar{n}^2 h \nu \tau_{\text{sp}}}, \quad (7)$$

where $\lambda \equiv c/\nu$ is the wavelength in vacuum and P [W/cm^2] is the wave intensity. In the second equation in (7) we have used the relations

$$P \equiv \bar{c} \rho_{\mathbf{q}} = h\nu (N_{\mathbf{q}}/V) \bar{c} \quad \Omega \gamma(\Omega) = \nu \gamma(\nu) \quad \eta \equiv 3 \cos^2 \alpha \quad (8)$$

"Classical" derivation of the result (7) goes as follows:

Interpret (6) as the Einstein A coefficient. Then the corresponding B coefficient is of the form

$$B = A \times \frac{\bar{c}^3}{8\pi h \nu^3} = \frac{e^2 |z_{32}|^2}{6 \hbar^2 \epsilon} \equiv \frac{\bar{c}^3}{8\pi h \nu^3 \tau_{\text{sp}}}. \quad (9)$$

For a white spectrum with the energy density $\rho(\nu)$ per unit frequency, the rate of induced transitions per electron is given by $B\rho(\nu)$, as shown by Einstein. However, we are interested in the interaction with a single mode, where the correct result is undoubtedly (7). The only way to reconcile these two results is to say (waving hands like a windmill) that the appropriate density in this case is

$$\rho(\nu) = \rho_{\mathbf{q}} \eta \gamma(\nu) \quad \text{why?} \quad (10)$$

Using $w_{\text{ind}} = B\rho(\nu)$ with B given by (9) and $\rho(\nu)$ given by (10) produces (7) "as the doctor prescribed".

☛ It should be clearly understood that the use of the Einstein coefficients essentially **fixes the mode** for which the absorption is to be calculated. Indeed, the method deals with the probabilities, not amplitudes, and hence does not take into account the interference between different modes (e.g., in an isotropic medium, modes degenerate with respect to polarization). Inadequacy of the approach which uses Einstein's relation while interpreting A as the *total* spontaneous radiation rate into all modes, is carefully discussed by V. L. Ginzburg, *Theor. Physics and Astrophysics*, Chap. 10.

The total induced transition probability per unit volume is $w_{\text{ind}} \cdot (n_3 - n_2)/d$, where d is the superlattice period and n_3 and n_2 are the 2D densities in wells (levels) 3 and 2. Hence the power generated or absorbed per unit volume is given by

$$h\nu \cdot w_{\text{ind}} \cdot \frac{n_3 - n_2}{d} \equiv \frac{dP}{dx}, \quad (11)$$

where we have assumed that the power flows in the x direction (along with \mathbf{q}). From the defining relation, $dP/dx \equiv gP$, for the material gain g [cm^{-1}] and using Eqs. (7) and (11), we obtain a microscopic expression for g in the form:

$$g(\nu) = \frac{\eta \bar{c}^2}{8\pi \nu^2 \tau_{\text{sp}}} \frac{n_3 - n_2}{d} \gamma(\nu) = \frac{\pi}{3} \frac{e^2 R_0}{\hbar} \frac{\eta |z_{32}|^2}{\bar{n}} \frac{n_3 - n_2}{d} \nu \gamma(\nu). \quad (12)$$

Let us define the differential gain g'_n by $g = g'_n (n_3 - n_2)/d$. Taking $\eta \approx 3$ the differential gain at the peak frequency is given by

$$g'_n = \frac{e^2 R_0}{\hbar} \frac{|z_{32}|^2}{\bar{n}} \frac{\Omega}{\gamma_0} \equiv 4\pi \alpha \frac{|z_{32}|^2}{\bar{n}} \frac{\Omega}{\gamma_0}. \quad (13)$$

Taking $\bar{n} = 3.4$, $z_{32} = 15 \text{ \AA}$, $\hbar\Omega = 295 \text{ meV}$, and $\hbar\gamma_0 = 11.3 \text{ meV}$ (Faist's zero-temperature value), we find $g'_n(0) \approx 1.6 \cdot 10^{-14} \text{ cm}^2$. At finite temperatures, γ_0 increases and therefore g'_n decreases. By luminescence measurements in the range $0 \leq T \leq 300 \text{ K}$, Faist found the following behavior:

$$h\gamma_0 [\text{meV}]; \quad T [\text{K}]: \quad h\gamma_0 = 11.3 + 1.94 \cdot 10^{-2} T + 1.1 \cdot 10^{-4} T^2 \quad (14)$$

Rate equations.

Equations governing the carrier and photon dynamics in the QCL are of the form:

$$\text{electrons, well 3:} \quad \frac{dn_3}{dt} = J - \frac{n_3}{\tau_{32}} - \frac{n_3}{\tau_{\text{esc}}} - \bar{c} g S ; \quad (15a)$$

$$\text{electrons, well 2:} \quad \frac{dn_2}{dt} = \frac{n_3}{\tau_{32}} + \bar{c} g S - \frac{n_2 - n_g e^{-\Delta_{2g}/T_e}}{\tau_{2g}} ; \quad (15b)$$

$$\text{electrons, gr. alloy:} \quad \frac{dn_g}{dt} = \frac{n_3}{\tau_{\text{esc}}} + \frac{n_2 - n_g e^{-\Delta_{2g}/T_e}}{\tau_{2g}} - J ; \quad (15c)$$

$$\text{photons, [cm}^{-2}\text{]:} \quad \frac{dS}{dt} = \Gamma \bar{c} (g - \alpha_{\text{int}}) S - \frac{S}{\tau_{\text{ph}}} ; \quad (15d)$$

$$\text{neutrality:} \quad n_2 + n_3 + n_g = n_g^{(0)} \quad (16)$$

$$\text{energy balance:} \quad \frac{dT_e}{dt} = - \frac{n_g^{(0)} (T_e - T)}{\tau_e} + (\hbar\Omega + \Delta_{2g}) \cdot J - \hbar\Omega \cdot \bar{c} g S ; \quad (17)$$

where n_g denotes the free carrier concentration (per cm^2) in the graded alloy, and $n_g^{(0)}$ the total doping per period. Note, that these equations are not independent: if $n_g^{(0)}$ is constant, then Eq. (15c) follows from Eqs. (15a), (15b), and (16).

Consider the steady state situation. Note first that a necessary (though, of course, insufficient) condition for generation is $\tau_{32} > \tau_{2g}$. Indeed, at the threshold the ratio $\tau_{32}/\tau_{2g} = n_3/n_2 > 1$. Equation (15d) gives the amplitude condition for generation:

$$g_0 = (\bar{c} \Gamma \tau_{\text{ph}})^{-1} + \alpha_{\text{int}} . \quad (18)$$

Using expression (12) for the gain, we can write

$$n_3 - n_2 = \frac{g_0 d}{g'_n} = \frac{1}{4\pi\alpha} \frac{\bar{n} g_0 d}{|z_{32}|^2} \frac{\gamma_0}{\Omega} \equiv f(T_e) . \quad (19)$$

Note the effective temperature dependence in Eq. (19). In general, one can expect γ_0 to be dependent *both* on the state of the lattice (T) and on the electron kinetic energy (T_e). We shall assume that the empirical dependence Eq. (14) results from two contributions, both permitting us to consider γ_0 a function of T_e only. One contribution arises from the nonparabolicity of the conduction band, which implies that the emitted photon energy $\hbar\Omega$ depends on the kinetic energy of an electron making the transition. The other, more important, contribution arise from the interaction with optical phonons. We shall assume that the population of optical phonons is governed by an effective temperature *which is equal* to the T_e of the carriers. This situation is commonly referred to as the optical phonon bottleneck. Calculations show that the bottleneck assumption is a good approximation in InGaAs for mobile electron concentrations of order 10^{17} cm^{-3} and higher.

This may be surprising to some people, because dealing with a bipolar plasma, one normally does not expect a bottleneck at carrier concentrations below 10^{18} cm^{-3} . However, because of the lower electron mass, optical phonons that are "on speaking terms" with electrons occupy a much smaller volume in the Brillouin zone than it would be the case for holes of the same average energy. Consequently, the heat capacity of relevant optical phonons is lower and the bottleneck arises at substantially lower carrier concentrations.

Steady state characteristics.

Faist et al. have estimated the energy difference between the bottom level of the active layers and the graded-alloy level as $\Delta_{2g} = 79$ meV. Accordingly, for not too high temperatures we have $\exp(-\Delta_{2g}/T_e) \ll 1$ and the backflow can be neglected. The steady-state system of equations is then reduced to the following two equations:

$$\left[1 - \frac{\tau_{2g}}{\tau_3} \right] \cdot J - \bar{c} g_0 \cdot S = \frac{f(T_e)}{\tau_3} , \quad (20a)$$

$$\frac{n_g^{(0)} (T_e - T)}{\tau_\epsilon} = (\hbar\Omega + \Delta_{2g}) \cdot J - \hbar\Omega \cdot \bar{c} g_0 S , \quad (20b)$$

where we have denoted

$$\tau_3 \equiv \frac{\tau_{32} (\tau_{\text{esc}} + \tau_{2g})}{\tau_{\text{esc}} + \tau_{32}} , \quad (21a)$$

$$\begin{array}{ccc} \tau_3 \xrightarrow{\tau_{\text{esc}} \rightarrow 0} \tau_{2g} & \tau_{2g} < \tau_3 < \tau_{32} & \tau_3 \xrightarrow{\tau_{\text{esc}} \rightarrow \infty} \tau_{32} \end{array} \quad (21b)$$

For a given J [$\text{cm}^{-2} \cdot \text{sec}^{-1}$] Eqs. (20) yield both the temperature T_e and the photon density S . At the threshold $S = 0$ the equation for $T_e \equiv T_{\text{th}}$ is obtained by eliminating J from Eqs. (20):

$$\frac{\hbar\gamma_0 (T_{\text{th}})}{T_{\text{th}} - T} = \frac{4\pi\alpha}{\bar{n}} \frac{\tau_{\text{esc}} (\tau_{32} - \tau_{2g})}{\tau_\epsilon (\tau_{32} + \tau_{\text{esc}})} \frac{\hbar\Omega}{\hbar\Omega + \Delta_{2g}} \frac{n_g^{(0)} |Z_{32}|^2}{g_0 d} \equiv R . \quad (22)$$

Having determined T_{th} from Eq. (22),[†] we find J_{th} from

$$J_{\text{th}} = \frac{n_g^{(0)} (T_{\text{th}} - T)}{\tau_\epsilon (\hbar\Omega + \Delta_{2g})} . \quad (23)$$

Above the threshold, the equation determining $T_e (J)$ is of the form

$$[n_g^{(0)}/\tau_\epsilon] \delta T_e - (\hbar\Omega/\tau_3) \delta f = [\Delta_{2g} + (\tau_{2g}/\tau_3) \hbar\Omega] \delta J , \quad (24)$$

where we have denoted $\delta T_e \equiv T_e - T_{\text{th}}$, $\delta J \equiv J - J_{\text{th}}$, and $\delta f \equiv f(T_e) - f(T_{\text{th}})$. Having determined T_e from Eq. (24), the light-current relation is found from (20a).

An approximate close-form expression for the slope efficiency can be obtained by taking $\delta f \approx f'_T (T_{\text{th}}) \delta T_e$, where

$$f'_T = \frac{1}{4\pi\alpha} \frac{\bar{n} g_0 d}{|Z_{32}|^2} \frac{1}{\Omega} \frac{d\gamma_0}{dT_e} . \quad (25)$$

This gives

$$\bar{c} g_0 \frac{\partial S}{\partial J} = 1 - \frac{\tau_{2g}}{\tau_3} - \frac{f'_T [\Delta_{2g} + (\tau_{2g}/\tau_3) \hbar\Omega]}{n_g^{(0)} (\tau_3/\tau_\epsilon) - \hbar\Omega f'_T} . \quad (26)$$

We see that when τ_{esc} is sufficiently small, the efficiency may deviate from ideal even if $f'_T = 0$.

[†] Equation (22) makes sense for any form of the dependence $\gamma_0 (T_e)$. If $\gamma_0 (T_e) = \text{const}$, then there always is a solution. If $\gamma_0 (T_e)$ is a linear increasing function, then there may not be a solution when the value of the right-hand side of Eq. (22) is small. For a concave function, like in Eq. (14) there may be either no solutions at all or – if the value of the right-hand side R is sufficiently large – two solutions, $T_{\text{th}}^{(l)}$ and $T_{\text{th}}^{(u)}$, cf. Fig. 1. The corresponding currents $J_{\text{th}}^{(l)}$ and $J_{\text{th}}^{(u)}$ may be termed the lower and the upper thresholds. Lasing is only possible for $J_{\text{th}}^{(l)} < J < J_{\text{th}}^{(u)}$. For this effect to be real, however, the concavity of $\gamma_0 (T_e)$ should be physically motivated, which is hardly true in the case of Eq. (14).

Threshold.

Keeping in mind the footnote on p. 4, the threshold carrier temperature is determined by the first intersection of a line $(T_e - T) \cdot R$, where R is the right-hand side of Eq. (22), with the concave curve $\hbar\gamma_0(T_e)$. For a given R the highest lattice temperature $T = T_{\max}$ at which threshold is possible, is determined by the condition that the line is *tangent* to the curve.

$$\text{Hence} \quad \gamma_0'(T_{\text{th}}^{\text{cr}}) = R \quad \gamma_0(T_{\text{th}}^{\text{cr}}) = R \cdot (T_{\text{th}}^{\text{cr}} - T_{\max}) \quad (27)$$

Let us re-write the empirical relation (14) in a form where both γ_0 and T_e are measured in meV:

$$\gamma_0(T_e) = a T_e^2 + b T_e + c, \quad (28)$$

where $a = 1.5 \cdot 10^{-2} \text{ meV}^{-1}$; $b = 0.23$; $c = 11.3 \text{ meV}$.

From Eqs. (27) we then find $T_{\text{th}}^{\text{cr}} = (R - b)/2a$ and the maximum lattice temperature is:

$$T_{\max} = \frac{(R - b)^2}{4aR} - \frac{c}{R}. \quad (29)$$

From the inequality $T_{\max} > 0$ we find the following criterion

$$R > b + \sqrt{4ac} \approx 1.05 \quad (30)$$

for the threshold to be achievable *at any temperature*. The fact that the right-hand side of (30) is so close to unity is curious. We see that the necessary condition for lasing is that the dimensionless parameter R is greater than unity.

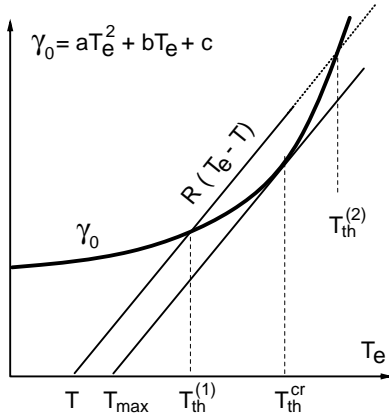


Fig. 1. For a concave function $\gamma_0(T_e)$, Eq. (22) may have either no solutions at all or – if the value of R is sufficiently large – two solutions. For a fixed value of R there is a maximum value T_{\max} of the lattice temperature T for which lasing is possible. Finally, there is a minimum value of R such that $T_{\max} > 0$. If R is less than this minimum value [which is about unity in the example of Eq. (28)] then there can be no lasing at any temperature.