with the value one. This will be referred to as the *consistency condition*. Obviously, the consistency condition is redundant for some closed paths. The minimum set of products needed to verify the consistency condition is investigated from a topological point of view in [18]. Note that the consistency condition for the queueing system of Fig. 5a is verified.

An important class of Markovian queueing networks is reversible [11]. For this class we can prove the following:

Corollary 1. *For reversible Markovian queueing networks the consistency condition is always verified.*

Proof: The detailed balance equations hold for reversible Markovian queueing networks. Hence, the consistency condition is equivalent with Kolmogorov's criterion for reversibility [11]. ▪

We have seen that if the consistency condition is verified the product of the path between any two nodes of the consistency graph is an invariant. For reversible queueing networks this reflects the property that the equilibrium probabilities of Markovian queueing networks are potentials [10]. This property is further explored in [19].

IV. Geometric Replication

The consistency condition developed in the previous section for a Markovian queueing network with M nodes and an arbitrary number of packets does not explicitly give a constructive method to find the set of partial balance equations. In this section *a method* for finding this set with applications to Markovian queueing networks arising in practice is given. This method generalizes the results derived in section II.

The generalized traffic flow equations for the Markovian queueing network defined above are given by:

$$\vartheta^{(j)}_{k_1 k_2 \cdots k_M} = \lambda^{(j)}_{k_1 k_2 \cdots k_M} + \sum_{i=1}^{M} r^{(ij)}_{k_1 k_2 \cdots k_i+1 \cdots k_M} \vartheta^{(i)}_{k_1 k_2 \cdots k_M} \quad . \tag{27}$$

for all $(k_1, k_2, \cdots, k_M) \varepsilon E_1 \times E_2 \times \cdots \times E_M$ and all j, $1 \leq j \leq M$, where $r = (r^{(ij)}_{k_1 k_2 \cdots k_M})$ is the set of state dependent routing probabilities for all nodes i, j, $1 \leq i, j \leq M$. As in the lower dimensional case, the generic M-dimensional cell is specified by the graph associated with the Markovian queueing network with only one packet. The corresponding graph translated to node $(k_1, k_2, \cdots, k_M)$ is shown in Fig. 6. The equilibrium probability that a packet is visiting queue l, $1 \leq l \leq M$, if only one packet is in the network amounts to:

$$p_{00 \cdots 1 \cdots 0} = \frac{\vartheta^{(j)}_{00 \cdots 0 \cdots 0}}{\mu^{(j)}_{00 \cdots 1 \cdots 0}} p_{00 \cdots 0 \cdots 0} \quad . \tag{28}$$

for all j, $1 \leq j \leq M$.

Recall that the state transition diagram of a single M/M/1 queue consists of 1-dimensional cells between each pair of adjacent nodes. These cells together with the two dimensional cells can also be used to construct a wide variety of state transition diagrams in two dimensions. Similarly, the one, two and three dimensional cells can be used to construct state transition diagrams in three dimensions. It is easy to see that for an M-dimensional state, space transition diagrams can be constructed based on 1, 2, ... , M dimensional cells. (For $M=2$ an example is given in Fig. 7). Alternatively, in an M-dimensional state transition diagram the l-cells, $1 \leq l \leq M$, can be interpreted as degenerate M-cells (M-cells with some of the transitions equal to zero).

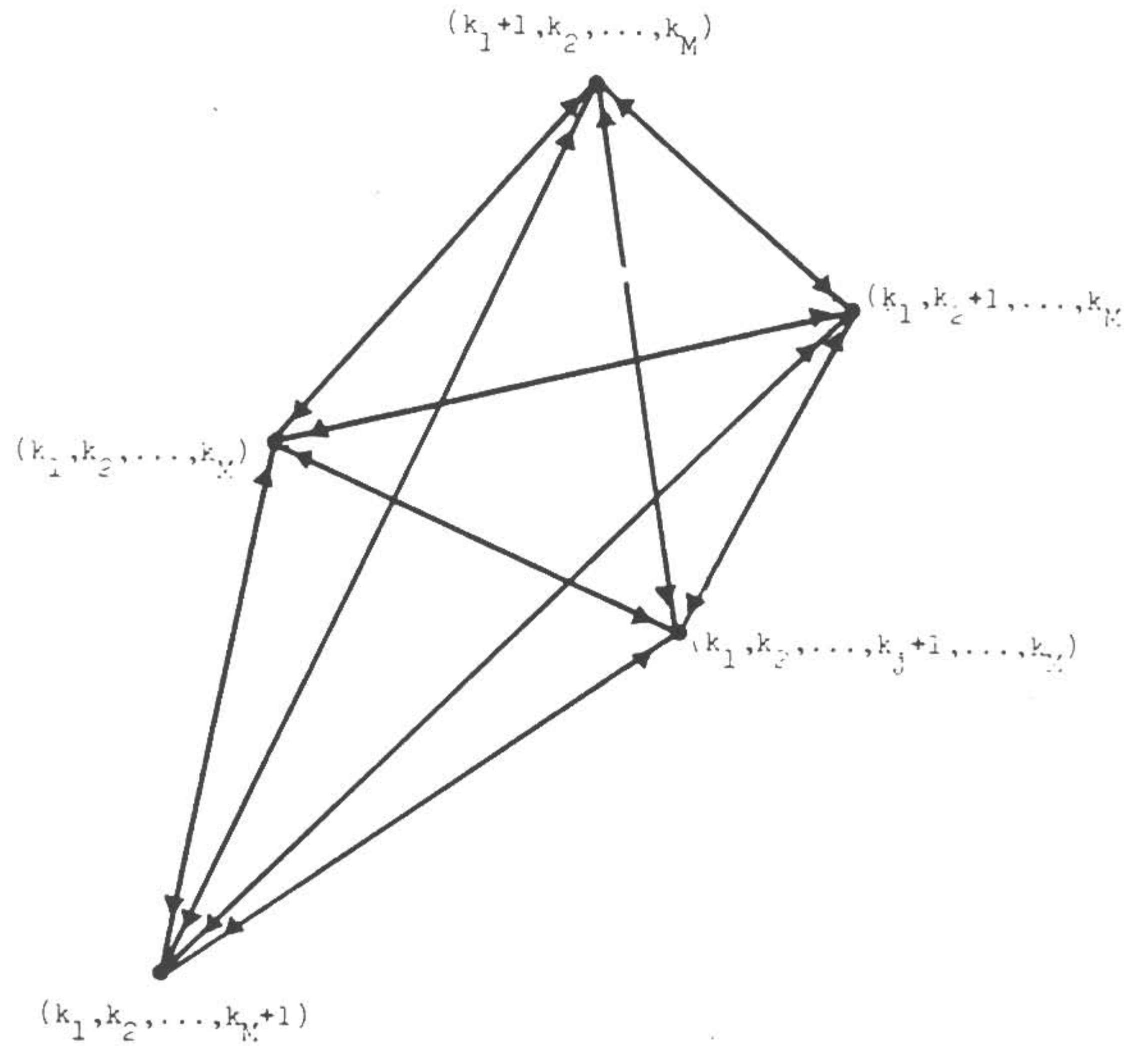

Fig. 6 M-dimensional cell.

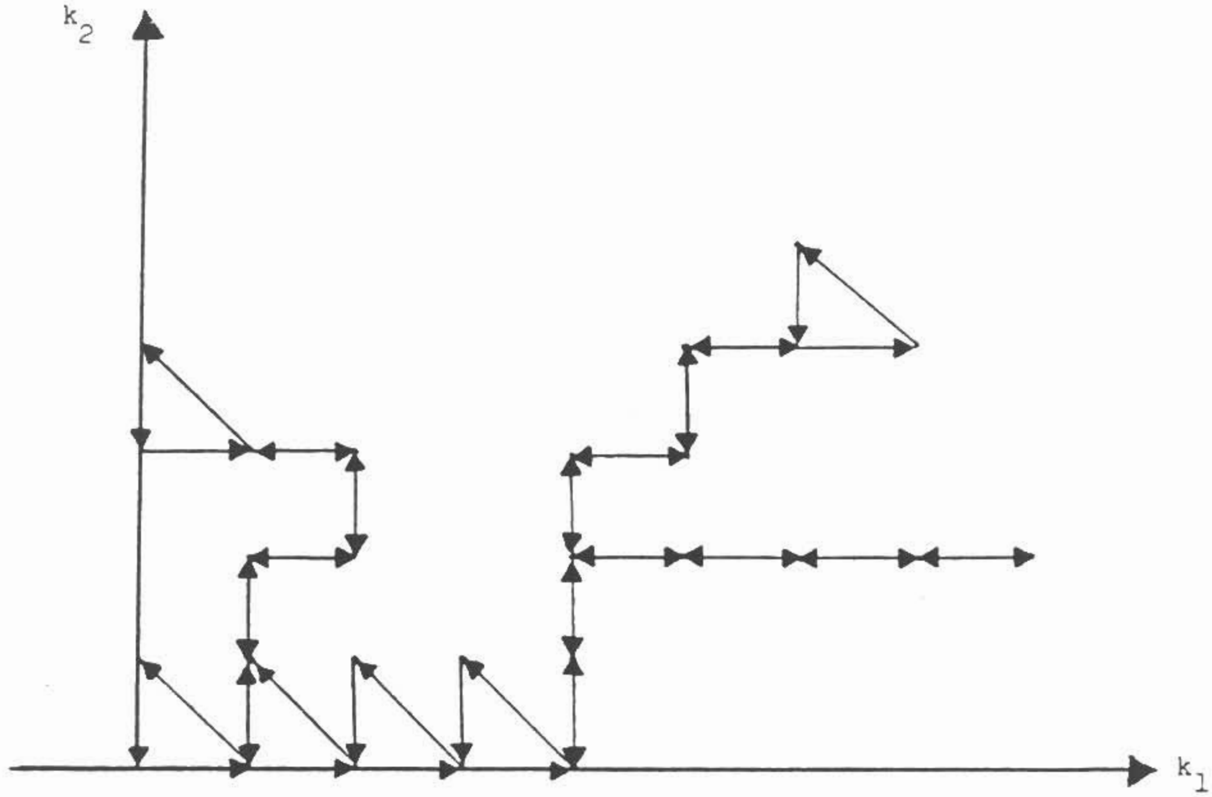

Fig. 7 State transition diagram consisting of one and two-dimensional cells.

Definition 2. The constructive process of aggregating M-dimensional cells into a complex is referred to as geometric replication.

This process is not necessarily the mere replication of M-dimensional cells having the same transitions. Rather, cells can be added to the state transition diagram that have different transition rates. The flows in these cells can be implemented using state dependent routing [35].

The result below proves that the process of geometric replication preserves the product form solution for networks with blocking and state dependent routing if an explicit consistency condition is satisfied. This condition requires that

(cc) the Markovian queueing network consisting of M parallel queues with arrival and departure rates $(\vartheta^{(j)}_{k_1k_2\cdots k_M}, \mu^{(j)}_{k_1k_2\cdots k_M})$, for all $(k_1,k_2,\cdots,k_M)\varepsilon E_1\times E_2\times\cdots\times E_M$ and all j, $1\le j\le M$, is reversible.

The inverse problem of determining a structure which provides desired probabilities is also of interest [1]. This problem can be cast into the framework of realization theory and is not dealt with in this paper.

Let us assume that the Markovian queueing network defined in the previous section has a state transition diagram constructed by geometric replication. Assume also that $\mu^{(j)}_{k_1k_2\cdots k_M} > 0$, if $k_j > 0$, for each j, $1\le j\le M$.

Theorem 2. *If the state transition diagram of a Markovian queueing network obtained by geometric replication satisfies the (cc) consistency condition, the equilibrium probabilities are given by:*

$$p_{k_1k_2\cdots k_M} = \prod_{j=1}^{M}\prod_{l_j=0}^{k_j} \frac{\vartheta^{(j)}_{l_1l_2\cdots l_j+1\cdots l_M}}{\mu^{(j)}_{l_1l_2\cdots l_j\cdots l_M}} p_{00\cdots 0} \tag{29}$$

for all $(k_1,k_2,\cdots,k_M)\ \varepsilon E_1\times E_2\times\cdots\times E_M$.

Proof: Using an inverse process to geometric replication, the state transition diagram can be decomposed into M-dimensional cells. The global balance equations of these cells considered in isolation are given by:

$$\lambda^{(j)}_{k_1k_2\cdots k_M} p_{k_1k_2\cdots k_M} + \sum_{\substack{i=1\\ i\ne j}}^{M} \eta^{(i)}_{k_1k_2\cdots k_i+1\cdots k_M}\, \mu^{(i)}_{k_1k_2\cdots k_i+1\cdots k_M}\, p_{k_1k_2\cdots k_i+1\cdots k_M} =$$

$$(1-\eta^{(j)}_{k_1k_2\cdots k_j+1\cdots k_M})\mu^{(j)}_{k_1k_2\cdots k_j+1\cdots k_M} p_{k_1k_2k_j+1\cdots k_M} \tag{30}$$

at node $(k_1,k_2,\cdots,k_j+1,\cdots,k_M)\varepsilon E_1\times E_2\times\cdots\times E_M$, for all j, $1\le j\le M$, and

$$\sum_{j=1}^{M}\lambda^{(j)}_{k_1k_2\cdots k_M}\, p_{k_1k_2\cdots k_M} =$$

$$\sum_{i=1}^{M}(1-\sum_{j=1}^{M}\eta^{(i)}_{k_1k_2\cdots k_j+1\cdots k_M})\ \mu^{(i)}_{k_1k_2\cdots k_i+1\cdots k_M}\, p_{k_1k_2\cdots k_i+1\cdots k_M} \tag{31}$$

at node $(k_1,k_2,\cdots,k_M)\varepsilon E_1\times E_2\times\cdots\times E_M$.

By summing the above global balance equations of the M-dimensional cells taken in *isolation* on obtains the global balance equations of the original state transition diagram. This can be shown by making the substitution $k_j\to k_j-1$ in (30), for all j, $1\le j\le M$, and summing the resulting equations together with equation (31). The result, as one can easily see, is the global balance equation specified by (24). Hence, the set of equations specified by (30) and (31) represents a set of *partial* balance equations for the original state transition diagram.

The partial balance equations at each node of the M-dimensional cells (see the set of equations (30)) have a simple product form solution based on an arbitrary reference probability that belongs to each cell. We have:

$$p_{k_1 k_2 \cdots k_j+1 \cdots k_M} = \frac{\vartheta^{(j)}_{k_1 k_2 \cdots k_j \cdots k_M}}{\mu^{(j)}_{k_1 k_2 \cdots k_j+1 \cdots k_M}} p_{k_1 k_2 \cdots k_j \cdots k_M} \quad , \tag{32}$$

for all $(k_1, k_2, \cdots, k_M) \varepsilon E_1 \times E_2 \times \cdots \times E_M$ and all j, $1 \leq j \leq M$. The algebraic values of the consistency graph associated with the partial balance equations (30) can be obtained from the set of equations (32) above. It can be easily seen that the set of equations (32) is consistent iff the (cc) condition is satisfied. In other words, the consistency condition of the partial balance equations is equivalent with Kolmogorov's reversibility criterion [11] for the Makovian network consisting of M parallel queues with arrival and departure rates $(\vartheta^{(j)}_{k_1 k_2 \cdots k_M}, \mu^{(j)}_{k_1 k_2 \cdots k_M})$, $1 \leq j \leq M$ [19]. Therefore, provided that the consistency condition is satisfied, the partial balance equations are equivalent to the global balance equations. Furthermore, the equilibrium probabilities can be recursively computed using the associated consistency graph or the set of equations (32). The reference probability $p_{00 \cdots 0}$ can be found using in addition to the set of equations (32) the law of conservation of probabilities. ▪

A wide variety of protocols with product form solution can be constructed by an incremental M-dimensional cell additions (geometric replications). The state transition diagram of reversible Markovian queueing networks can be obtained by geometrically replicating one-dimensional cells such that Kolmogorov's criterion remains valid. The result derived below shows that the networks studied by Jackson, Gordon and Newell have the same invariant probability measure [36]. While these results are known, the derivation below is new. The proof reflects the inherent geometric properties of the state transition diagrams. These can be obtained by geometric replication.

Corollary 2. *The equilibrium probability of a given state in a closed Jacksonian network with N packets is the same as the equilibrium probability of that state in the open network given that it contains exactly N packets. In addition both probabilities have a product form.*

Proof: The state transition diagram for both the open and closed Markovian queueing network can be obtained by geometric replication. In the general case, the state transition diagram of the open network is a complex composed of M-dimensional cells. The state transition diagram of the closed network consists of one-dimensional cells situated on a hyperplane. Consequently, the equilibrium probabilities have a product form and are equal. The above equality implies that the equilibrium probability of the closed network is the same as the trace probability induced by the conditional probability of the open network (see [25], section 1.4.4.). •

For large classes of state transition diagrams of practical interest the consistency condition is automatically satisfied. In particular, this is the case if the state transition diagram is composed of the same type of cells with the same transitions. The consistency conditions also holds for trees since such graphs do not have any closed paths. Transition diagrams consisting of identical replications of M-dimensional cells can be interconnected to form "supertrees". This construct guaranties that homogeneous superarcs consisting of an amalgamation of cells of the same type can only be uniquely reached from other nodes of the same graph.

Other possibilities are illustrated below. Figure 7 shows the realization of an arbitrary state transition diagram using only the one and two dimensional cells. A protocol with a minimum and maximum number of packets is given in Fig. 8. Fig. 9

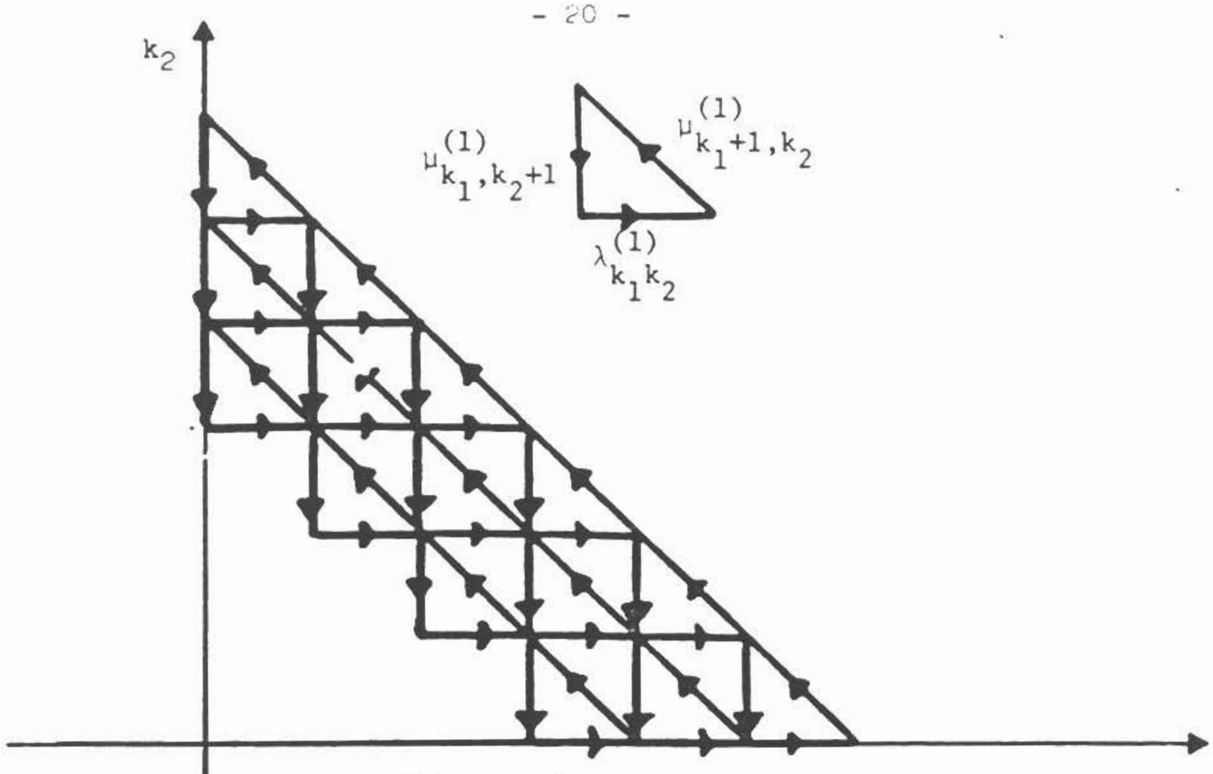

Figure 8 State transition diagram with a minimum and maximum number of packets.

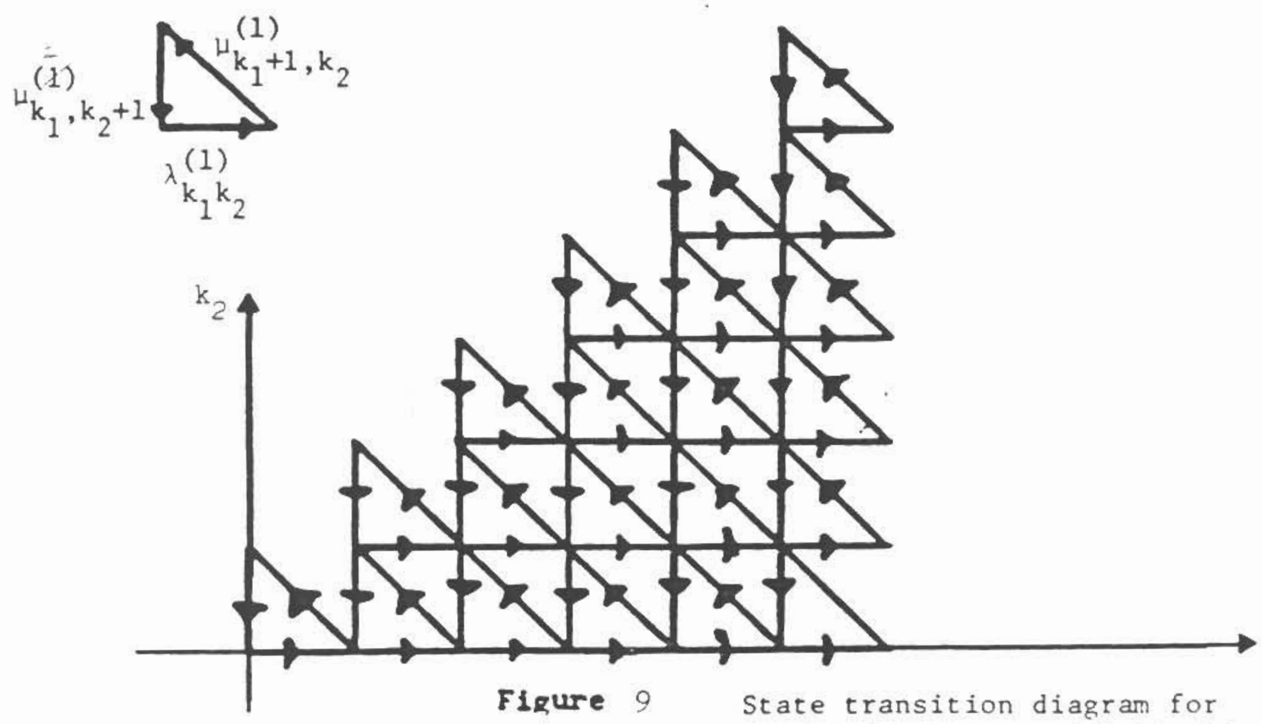

Figure 9 State transition diagram for two tandem queues with blocking and $k_2 \leq k_1+1$

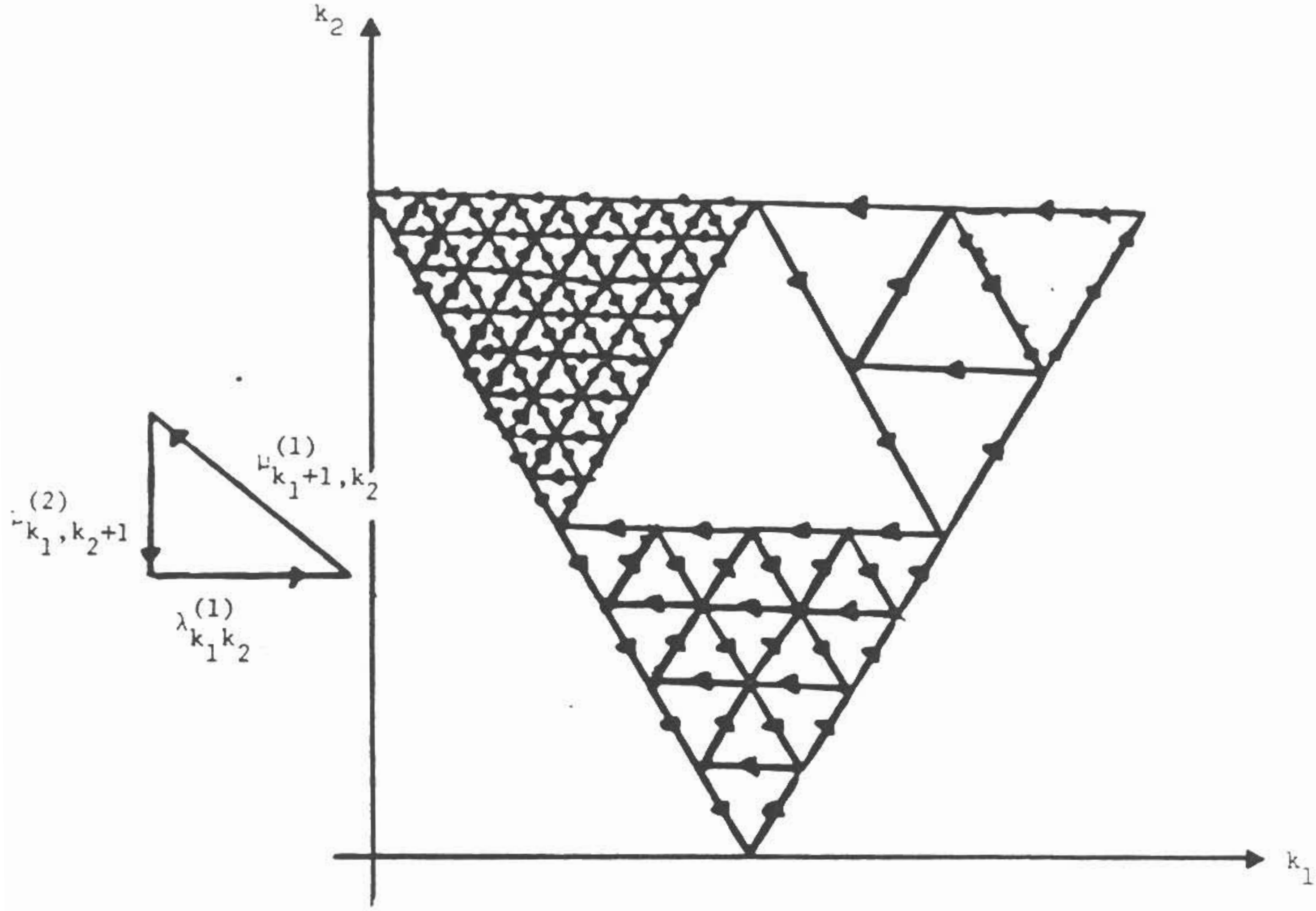

Fig. 10 An arbitrary state transition diagram.

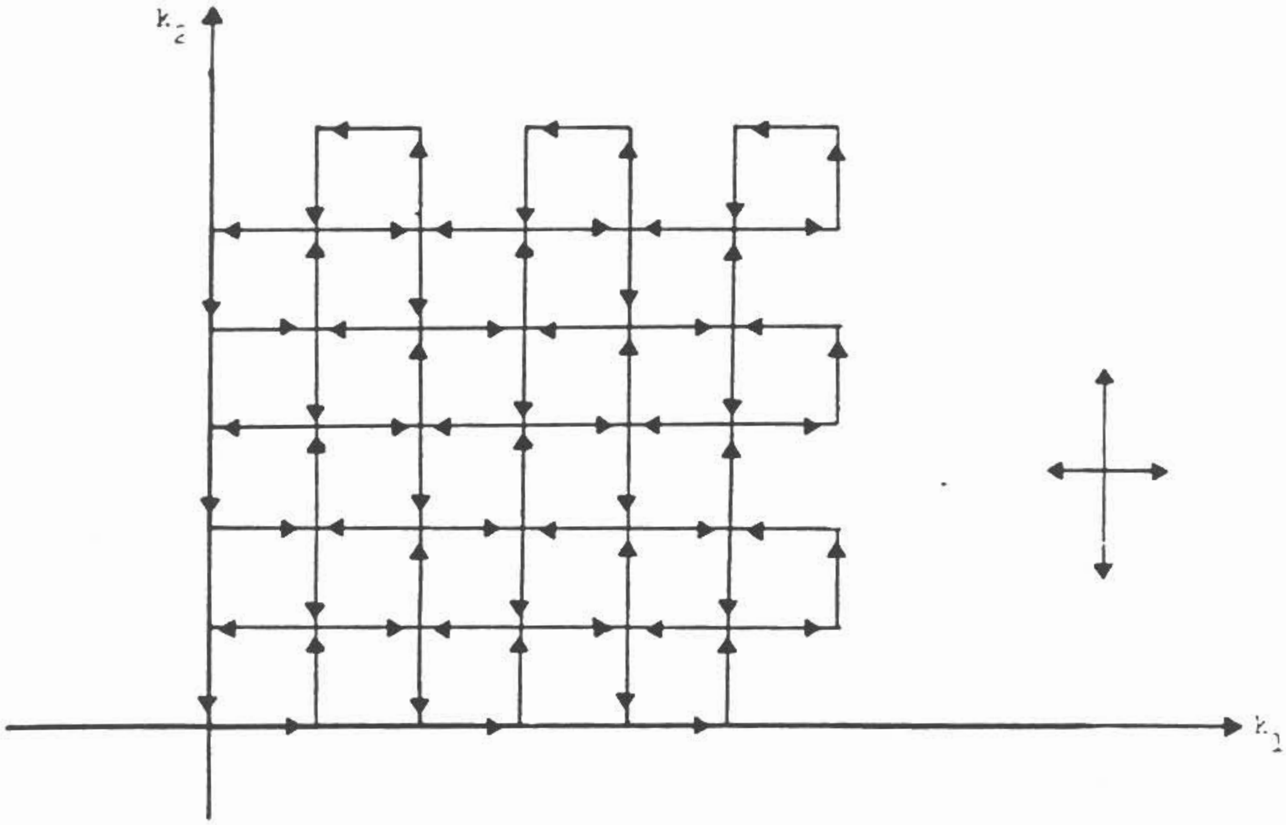

Fig. 11 Checkboard State Transition Diagram

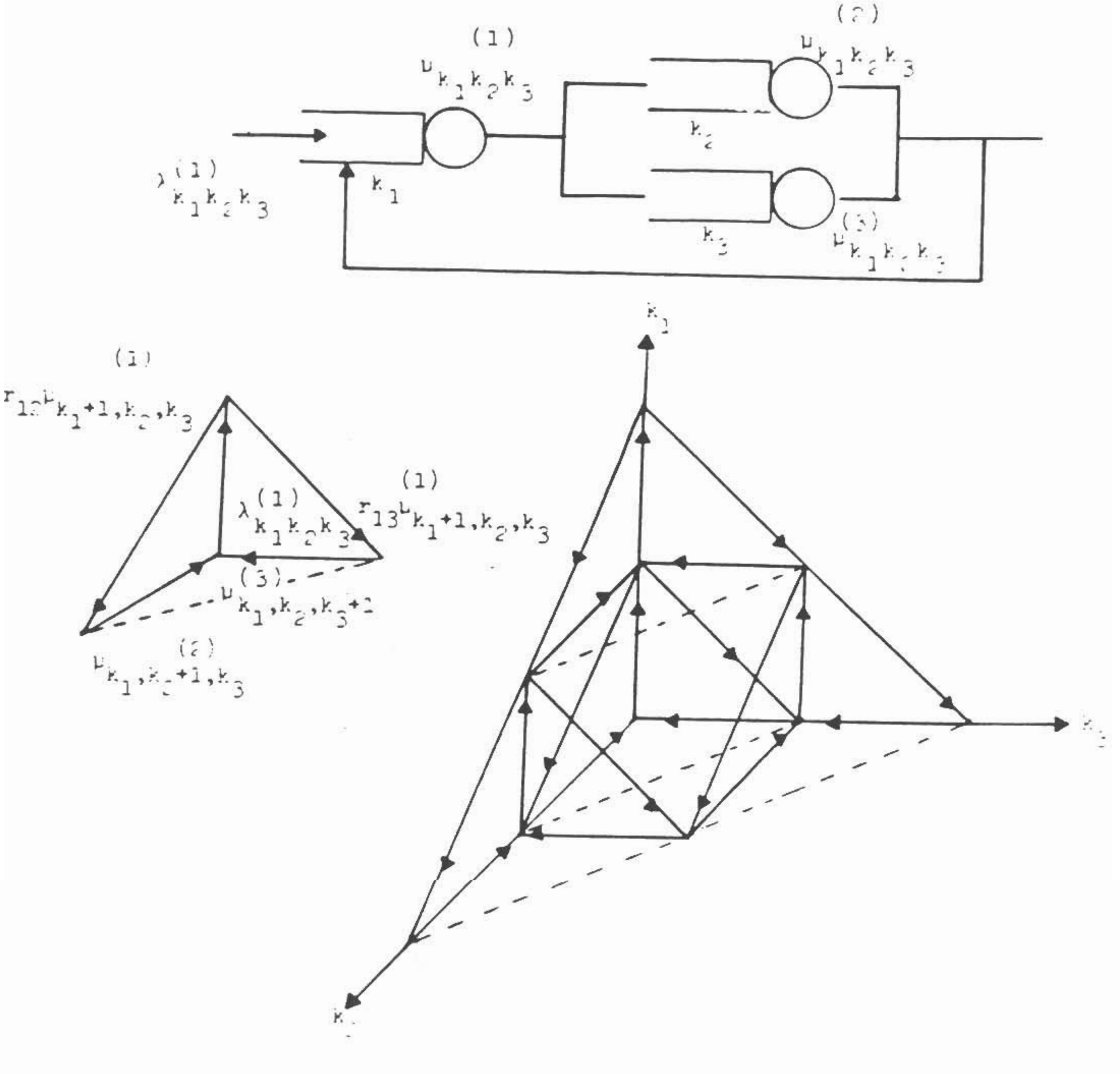

Fig. 12 State transitions for one queue in tandem with two parallel queues.

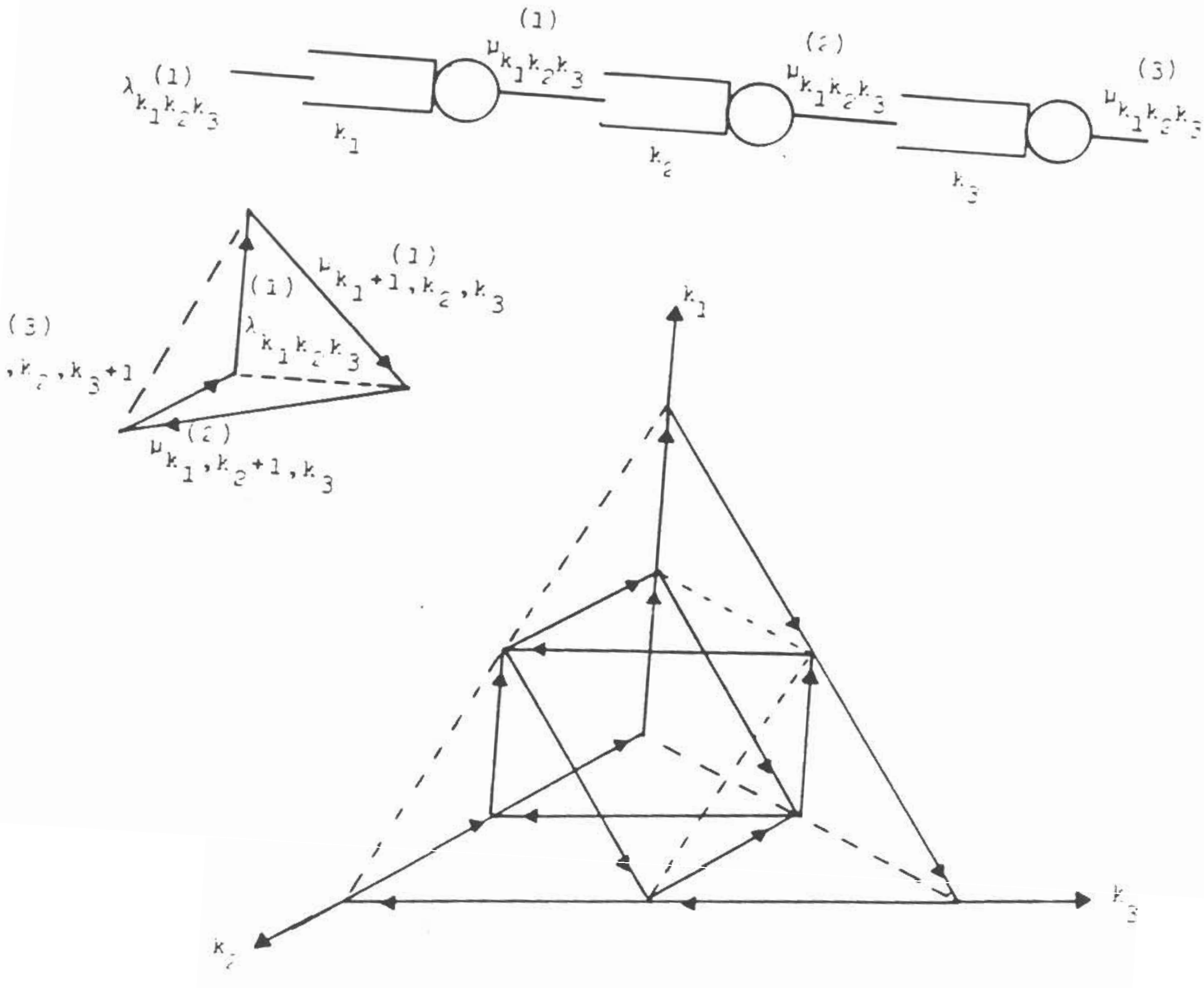

Figure 15. State transitions for three queues in tandem.

provides a protocol where the number of items in the second queue is less than the number of items in the first queue plus one. In Fig. 10 an arbitrary construction of two dimensional cells is presented. A grid of alternating rectangles is illustrated in Fig. 11. This example also suggests a different geometric replication using rectangles. Note the symmetry of the product form solution about the checkerboard diagonal. This grid pattern corresponds to a protocol where the two queues alternately are allowed to either receive arrivals or complete servicing. This can be seen by examining any interior node.

Let us now consider the three dimensional case. If a third queue is placed in parallel to the second queue in Fig. ?, then the tetrahedral state transition diagram of Fig. 12 follows. The addition of some transitions is necessary to produce a tetrahedral structure. A two token model [30] for a three tandem queueing network appears in Fig. 13. The transitions are along the edges of a tetrahedron (3-cell). Therefore, the tetrahedral constructions in a 3-space corresponds to the triangular grids of two dimensions. Here, the connections between elementary tetrahedrons are at the nodes of these tetrahedrons. The volume between the elements can be completely filled (packed) by inverted tetrahedral and octahedral solids.

V. Evaluation of the Equilibrium Probability Distribution

The consistency theorem (Theorem 1) has several immediate applications. Some of these will be presented in considerable detail in the sequel.

Consider the rectangular state transition grid that is associated with the case of two finite tandem queues (see Figure 14). As has been previously mentioned, the analysis of such blocking networks is difficult. However, if the horizontal transitions on the upper boundary and the vertical transitions on the side boundary are deleted, a triangular grid structure is discerned. The new structure can be constructed out of two-dimensional cells. Consequently, a product form solution results. Note that the nodes still form a rectangular geometry since only certain boundary transitions have been deleted. The existence of local conditions on boundaries that might lead to product form were alluded to by Newell ([27] page 391). See also Kaufman [9] and the references therein.

How does this new Markovian structure affect the physical protocol? It corresponds to a policy where, upon the blocking of a queue in the network, *only* that queue is allowed to continue service. This effectively precludes the interacting behavior of the usual blocking networks.

Consider a Markovian structure with a product form solution as obtained above. With an appropriate choice of transition rates, any two nodes may be connected by a pair of transitions (directionally opposite to each other) without changing the numerical values of the solution. The transition rates are simply set to create a new circulation around these two transitions which does not interact with other circulations. The magnitude of this new circulation can be arbitrarily set. Such additions (or removals) can also be made to structures without the type of product form solutions discussed in this paper. A non-interacting circulation involving any finite number of nodes can also be appended to (or removed from) a Markovian structure without changing the equilibrium values if the transition rates along the new circulation path are appropriately chosen.

This has several important consequences. Consider, for example, the Markovian structure for two tandem finite queues (say a $N \times N$ rectangular lattice). It was noted above that if the transitions along the two maximal boundaries were removed, a product form solution existed (see Fig. 14). Now instead, suppose that these transitions are left in place and a single new transition from $(N,0)$ to $(0,N)$ is added. This

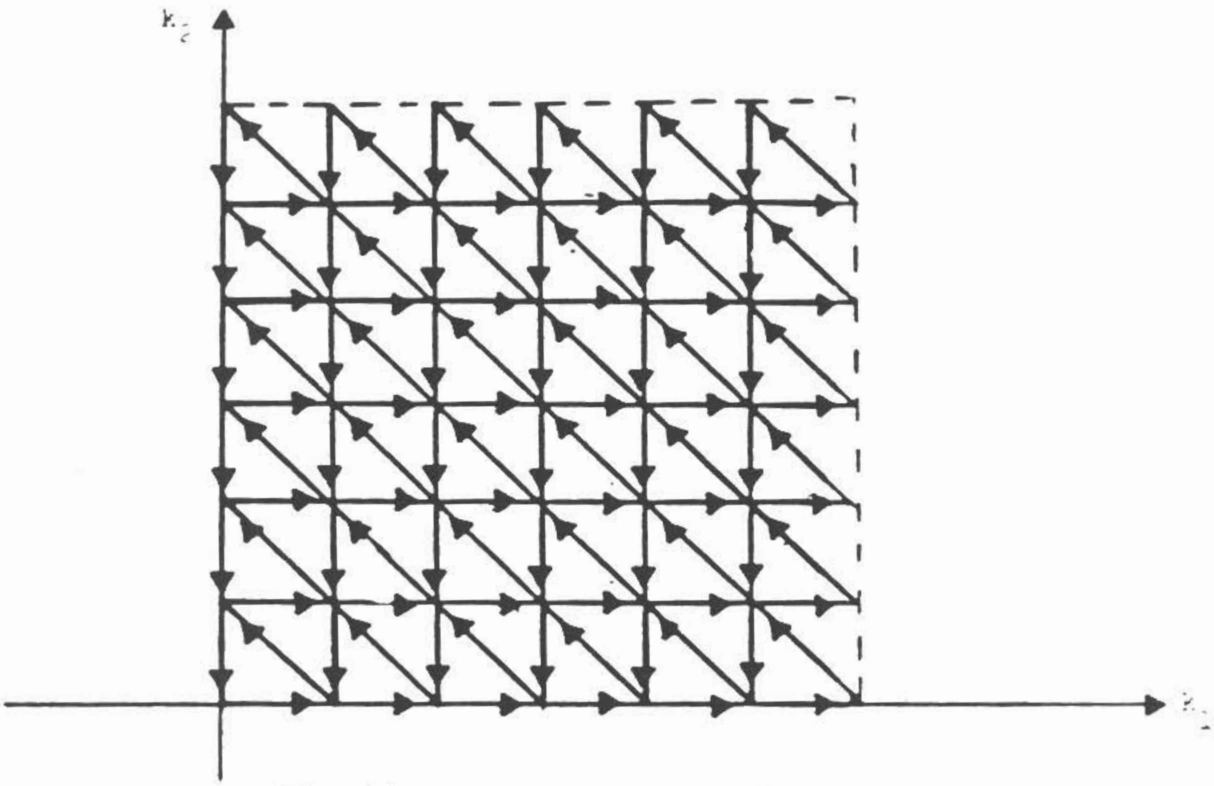

Fig. 14 State transistion diagram with product form solution.

completes a new circuit along the two maximal boundaries and back from $(N,0)$ to $(0,N)$. The transition rates along this new circulation path can now be set so that a new circular flow is established which does not involve adding or subtracting flow from the rest of the structure. The product form solution still holds around the two dimensional cells (triangles). Movement along the new circular path involves a ratio of adjacent transition rates. Physically, the new transition corresponds to a filled first queue sending all items to an empty second queue.

The concept of the addition or removal of non-interacting flows is intuitively simple. The equilibrium probabilities are akin to fixed potentials. If the transition rates are appropriately chosen, a new circulation is created which does not involve modifications of existing network flows. Because the potentials are assumed to be fixed, there is no limit to the flow magnitude. This can be set by simple scaling. Intuitive analogies in terms of other familiar networks are zero internal resistance or infinite capacity.

It should also be pointed out that even though the proposed structural modifications do not change the equilibrium probabilities, they do change the average time spent in a particular state. That is, there may be a faster movement between states, even though the proportional time spent in any state is the same.

The usual product form solutions involve a scaling of the probability of all network queues being empty. In the context of the previous model this is $p_{00...0}$. In considering the protocols described in section IV, however, it can be seen that some of them lack such a singular probability. Thus, product form solutions must be based on a reference probability.

To calculate an arbitrary probability in terms of a reference probability, the shortest lattice path between the two probabilities must first be determined. Then, either an algebraic or a numerical scaling can be recursively determined by moving along the path. In the case of a numerical calculation, the order and storage of intermediate products and divisions will affect the accuracy of the ultimate result. Alternatively, logarithms could be used. In order to calculate all the equilibrium probabilities, the shortest paths from each node to the reference node must be established. This can be accomplished by using a labeling shortest path algorithm on the consistency graph.

Deadlock [34] is one of the most serious problems that still defy mathematical analysis. One of the major difficulties is that queueing models for computer communication networks or computer systems with buffer size constraints cannot be used to model the phenomena of direct or indirect deadlock. In practice, if two buffers are filled and packets try to be exchanged between the two nodes direct deadlock occurs. Since in a Markovian queueing network simultaneous events occur with probability zero such an effect cannot be captured directly. One of the common methods to avoid deadlock is to prevent buffers from filling up at the same time. The basic protocol for two interconnected queues, using the geometric replication introduced in Section III, satisfies this relationship (see also Fig. 14: assume, however, that each arc has double arrows attached to it). Indeed, if one buffer fills up none of the others is permitted to serve unless the blocked queue reduces its size by one. In practice this requires the transmission of an interrupt signal to the other queue in order to freeze its server. A second disabling interrupt has to follow as soon as the sending buffer has cleared a packet. This protocol is easy to analyze since it has product form.

VI. Conclusion

The underlying geometry of the state transition diagram and a consistency condition have been identified as the conditions for recursive product form of the stationary probabilities for Markovian queueing networks. This geometry represents the common structure of the queueing networks previously studied by Jackson, Gordon and Newell and others. Certain classes of networks with blocking also have this property.

In a broader sense, this geometric formulation involves a graphic structure with positive values assigned to the arcs and a set of positive nodal values which sum to one. The rate of flow through an arc is the product of the arc value and the originating nodal value. Geometric replication has been used to obtain, although indirectly, a decomposition of the state transition diagram into its building blocks. Algebraically, this corresponds to a decomposition of the global balance equations into partial balance equations. The consistency of the latter equations can be verified using the consistency graph. The natural building block or cell has been shown to be topologically equivalent to the state transition diagram of the network accepting only one packet. This intuitively simple and pleasing construct is the basis for the product form of the equilibrium probabilities first established by Jackson [7].

Basic results obtained by other researchers in the past such as Muntz's $M => M$ property [24], the equilibrium distribution of the state seen by a packet when it jumps from one node to another [23], [33] and the distribution of sojourn times [22] can also be established within the framework of this paper. This requires, however, a somewhat more formal presentation and will be published elsewhere [20]. Extensions to Multiclass Markovian queueing networks can also be developed [17].

On the practical side this work complements existing work on queueing networks with blocking and state dependent routing. The geometrical method introduced is extremely powerful and simple to use. A simple check of the state transition diagram and of the consistency graph determines the existence, or lack thereof, of the product form solution. In addition, it also suggests *practical* ways for changing a protocol that did not originally have a product form solution. In this way, blocking models based on the usual queueing schematic representation may more closely model practical applications. It is hoped that a study of this class of Markovian structures may lead to insights concerning other Markovian structures not discussed here.

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