Mathematical Deficiencies of Numerically Simplified Dynamic Robot Models

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Abstract—**The on-line evaluation of the complete dynamic model in robot control is time-consuming and often dictates the use of simplified robot models in the controller. The impact of the simplifications on the controller performance cannot be easily quantified and experiments often contradict simulation results. The objective of this note is to demonstrate that numerically simplified models can violate fundamental physical principles and destroy the structure of the controller.**

I. INTRODUCTION

The complexity of the highly coupled and nonlinear robot dynamics has led to the development of model-based robot control algorithms which are surveyed in [1]. Model-based robot controller design is founded upon classical control engineering concepts and requires the exact knowledge of robot dynamics. In practice, the on-line evaluation of the complete dynamic model is time consuming and often dictates the use of simplified robot models in the controller [1]–[3]. These simplified models fail to characterize the complex robot dynamics and result in oscillations or overshoot of the end-effector. The impact of the simplifications on the controller performance cannot be easily quantified and experiments often contradict simulation results [4].

The objective of this note is to demonstrate that the underlying deficiency of numerically simplified models is that they ignore classical mechanics and lead to models that do not characterize any *realizable* manipulator. We show that numerically simplified models can violate fundamental physical principles and destroy the structure of robot dynamics. The controller is then required to drive a nonsense mechanism through large corrective torques that may saturate the actuators.

The note is organized as follows. In Section II, we review robot dynamics and the associated nomenclature. In Section III, we address the need for simplified dynamic robot models and highlight a systematic simplification procedure. Then, in Section IV, we demonstrate the thesis of the note through a practical example. Finally, in Section V, we summarize our results and identify several research issues.

II. ROBOT DYNAMICS

Robot dynamics are concerned with the mathematical formulation of the equations of robot motion. In the robotics literature, the closed-form dynamic model of an open-chain robot with N rigid links is

$$D(q)\ddot{q}+h(q,\dot{q})=F(t) \tag{1}$$

where $D(q)=[d_{ij}]\in R^{N\times N}$ is the *positive-definite* inertial matrix; $h(q,\dot{q})\in R^N$ is the coupling vector that incorporates centrifugal and Coriolis, gravitational and frictional effects; and $F(t)\in R^N$ is the vector of actuating (motor) joint forces/torques.

The inertial matrix in (1) exhibits noteworthy physical and mathematical properties which have abundant implications for control system analysis and design [5]. The most important one, from the control engineering point of view, is the positive-definiteness of the quadratic form

$$K=\frac{1}{2}\dot{q}^TD(q)q \tag{2}$$

which signifies the kinetic energy of the manipulator. Specifically, it has been shown in [6] that the aforementioned property can guarantee the stability (under fairly general conditions) of point-to-point controllers for the system in (1).

The importance of dynamic modeling in robot control stems from its *inverse dynamics* application, i.e., the computation of the joint forces/torques that are required to drive a manipulator along a desired trajectory. In *model-based* robot control, the actuating joint force/torque signal is computed as

$$F(t)=\hat{D}(q)u(t)+\hat{h}(q,\dot{q}) \tag{3}$$

where the caret "ˆ" signifies the estimated inertial matrix and coupling vector and $u(t)$ is the commanded acceleration. Stability is guaranteed as long as $\hat{D}(q)$ remains positive-definite [6, p. 137]. It is important to stress that the invertibility of $\hat{D}(q)$ is a critical assumption in the development of other robot control algorithms as well, such as computed-torque [4], sliding-mode [6, p. 160], robust control [6, p. 182], and adaptive control [7, p. 259] schemes. The thesis of this note, therefore, is valid for all robot control algorithms that require an invertible inertial matrix $\hat{D}(q)$ in the implementation.

In a preliminary report [8], we alerted the community to the fact that arbitrarily simplified models may result in a singular $\hat{D}(q)$. That report was based on a simple, planar manipulator paradigm with contrived numerical data to support our point. In this note, we expand upon the same theme and demonstrate how a systematic simplification procedure can also destroy the positive-definiteness of $\hat{D}(q)$. We illustrate our argument for the JPL/Stanford manipulator with numerical data obtained from the robotics literature.

III. MODEL SIMPLIFICATION

A salient issue of inverse dynamics applications is computational efficiency. Computational efficiency is inversely proportional to the amount of time needed to generate the driving forces/torques from the inverse dynamics model. Most systematic attempts to simplify the robot model have been based upon (numerical) significance analysis of the dynamic coefficients in (1). In the sequence, we codify a numerical simplification procedure that has been advocated in [9]. Although we restrict our attention to the inertial coefficients, the procedure applies equally to all the dynamic coefficients in (1).

Let $d(q_1, q_2, \cdots, q_N)$ denote a generic inertial coefficient. In general, its functional dependency on the joint coordinates manifests itself as a *finite* summation of terms

$$d(q_1,q_2,\cdots,q_N)=\sum_{i=1}^{M}k_i\alpha_i(q_1,q_2,\cdots,q_N) \tag{4}$$

where each α_i is written as a product of: i) some sines and cosines (and their powers) of the *rotational* joint coordinates; and of ii) some (normalized) *translational* joint coordinates. Accordingly, the *constants* k_i depend on the geometrical and mass properties of the links as well as the range of the translational joint coordinates. Without going into the details of the representation, it suffices to note that

$$|\alpha_i(q_1,q_2,\cdots,q_N)|\le 1,\quad \forall i. \tag{5}$$

For expository convenience, and without loss of generality, we assume that the terms in (4) have been ordered so that

$$|k_1|\le|k_2|\le\cdots\le|k_M|=k_{\max}. \tag{6}$$

The simplification then proceeds through the following steps.

Step 1: Write the inertial coefficient under consideration as

$$d(q_1,q_2,\cdots,q_N)=k_{\max}\sum_{i=1}^{M}p_i\alpha_i(q_1,q_2,\cdots,q_N)$$

where $p_i\equiv k_i/k_{\max}$ so $|p_i|\le 1$, $\forall i$.

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Step 2: Define the modeling tolerance ϵ, where $0 < \epsilon < 1$.

Step 3: Find the greatest integer $L \leq M$ for which

$$\sum_{i=1}^{L} |p_i| \leq \epsilon.$$

Step 4: Define $\hat{p}_i$ as

$$\hat{p}_i = \begin{cases} p_i & \text{if } i > L \\ 0 & \text{if } i \leq L. \end{cases}$$

Step 5: Evaluate the simplified coefficient as

$$\hat{d}(q_1, q_2, \cdots, q_N) = k_{\max} \sum_{i=1}^{M} \hat{p}_i \alpha_i(q_1, q_2, \cdots, q_N).$$

This simplification procedure retains only the numerically significant terms to reduce the computational complexity and enable real-time implementation. In [9] it is noted that the error between the actual and the simplified coefficient is bounded by the product $k_{\max}\epsilon$

$$\begin{aligned} &|d(q_1, q_2, \cdots, q_N) - \hat{d}(q_1, q_2, \cdots, q_N)| \\ &= k_{\max} \left| \sum_{i=1}^{M} (p_i - \hat{p}_i)\alpha_i(q_1, q_2, \cdots, q_N) \right| \\ &\leq k_{\max} \sum_{i=1}^{M} |p_i - \hat{p}_i| \\ &= k_{\max} \sum_{i=1}^{L} |p_i| \\ &\leq k_{\max}\epsilon. \end{aligned}$$

The significance of this upper bound for the simplification error is often overestimated and may be, at times, misleading. The underlying deficiency of numerical simplification is that it fails to account for the configuration dependence (or the inter-relationships) of the inertial coefficients. In the following section, we demonstrate the loss of the positive-definiteness of the quadratic form in (2) via numerical simplification.

IV. An Example

In this section we demonstrate the thesis of our note through a common three degree-of-freedom robot, namely the positioning system of the JPL/Stanford arm [3]. The joint variables of this arm imitate the spherical coordinates, thus resulting in a spherical workspace.

The positioning system of the JPL/Stanford manipulator has the following inertial matrix:

$$\boldsymbol{D}(q_2, q_3) = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & 0 \\ d_{13} & 0 & d_{33} \end{pmatrix} \tag{7}$$

where

$$\begin{aligned} d_{11} &= 196 + 356 \sin^2 q_2 + 755.04 \left(\frac{q_3}{44}\right)^2 \\ &\quad \cdot \sin^2 q_2 - 875.16 \left(\frac{q_3}{44}\right) \sin^2 q_2 \qquad (\text{oz} \cdot \text{in}^2) \\ d_{12} &= 64 \cos q_2 - 108.68 \left(\frac{q_3}{44}\right) \cos q_2 \qquad (\text{oz} \cdot \text{in}^2) \\ d_{22} &= 668 + 755.04 \left(\frac{q_3}{44}\right)^2 - 875.16 \left(\frac{q_3}{44}\right) \qquad (\text{oz} \cdot \text{in}^2) \\ d_{13} &= -2.47 \sin q_2 \qquad (\text{oz} \cdot \text{in}) \\ d_{33} &= 0.45 \qquad (\text{oz}) \end{aligned}$$

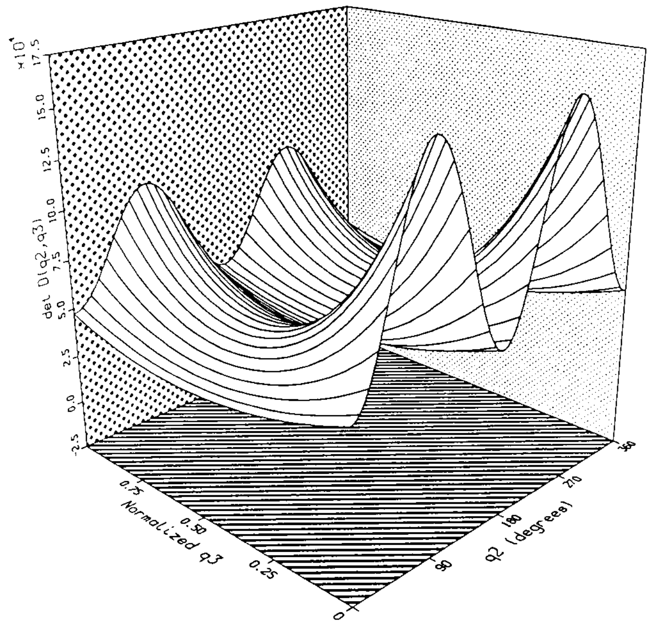

Fig. 1. Actual $|\boldsymbol{D}(q_2, q_3)|$ plot.

and $0 \leq q_2 \leq 360°$, $0 \leq q_3 \leq 44$ in [3]. We seek to obtain the simplified inertial matrix with a tolerance of 25 percent (i.e., $\epsilon = 0.25$). At this tolerance, only d_{11} is affected by the simplification procedure. Indeed

$$d_{11} = 875.16 \left[0.224 + 0.407 \sin^2 q_2 + 0.863 \left(\frac{q_3}{44}\right)^2 \sin^2 q_2 - \left(\frac{q_3}{44}\right) \sin^2 q_2 \right],$$

hence

$$\hat{d}_{11} = 875.16 \left[0.407 \sin^2 q_2 + 0.863 \left(\frac{q_3}{44}\right)^2 \sin^2 q_2 - \left(\frac{q_3}{44}\right) \sin^2 q_2 \right]$$

and $|d_{11} - \hat{d}_{11}| \leq 218.79 \text{ oz}\cdot\text{in}^2$. An analytical evaluation of the modeling error, however, shows that it is not small; the average error of $\hat{d}_{11}$ is 196 $\text{oz}\cdot\text{in}^2$ (55 percent w.r.t. the range of d_{11}, which is 356 $\text{oz}\cdot\text{in}^2$).

The simplified inertial matrix is written then as

$$\hat{\boldsymbol{D}}(q_2, q_3) = \begin{pmatrix} \hat{d}_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & 0 \\ d_{13} & 0 & d_{33} \end{pmatrix}. \tag{8}$$

We now compute the determinant of the actual inertial matrix in (7) as

$$|\boldsymbol{D}(q_2, q_3)| = d_{11}d_{22}d_{33} - d_{12}^2 d_{33} - d_{13}^2 d_{22} \qquad (\text{oz}^3 \cdot \text{in}^4)$$

and plot it in Fig. 1. Similarly, we compute the determinant of the simplified inertial matrix in (8) as

$$|\hat{\boldsymbol{D}}(q_2, q_3)| = \hat{d}_{11}d_{22}d_{33} - d_{12}^2 d_{33} - d_{13}^2 d_{22} \qquad (\text{oz}^3 \cdot \text{in}^4)$$

and plot it in Fig. 2. We observe that the simplification error for $\hat{d}_{11}$ is compounded, leading to significant errors for $|\hat{\boldsymbol{D}}(q_2, q_3)|$. The associated maximum and average errors are 48 323 $\text{oz}^3\cdot\text{in}^4$ and 42 521 $\text{oz}^3\cdot\text{in}^4$ (32 and 28 percent w.r.t. the range of $|\boldsymbol{D}(q_2, q_3)|$, respectively, which is 151 856 $\text{oz}^3\cdot\text{in}^4$). We also believe that the simplified determinant $|\hat{\boldsymbol{D}}(q_2, q_3)|$ is shifted downwards compared to the actual determinant $|\boldsymbol{D}(q_2, q_3)|$. For certain joint variables q_2 and q_3, the simplified model leads to negative values of $|\hat{\boldsymbol{D}}(q_2, q_3)|$, thus destroying the positive-definiteness of the quadratic form in (2). More importantly, zero values of $|\hat{\boldsymbol{D}}(q_2, q_3)|$ will increase the magnitude of the driving vector in (5) to infinity and cause the controller to collapse.

In practice, the locus of points in the workspace for which $|\hat{\boldsymbol{D}}(q_2, q_3)| = 0$ must be excluded in the trajectory planning stage. Furthermore, to

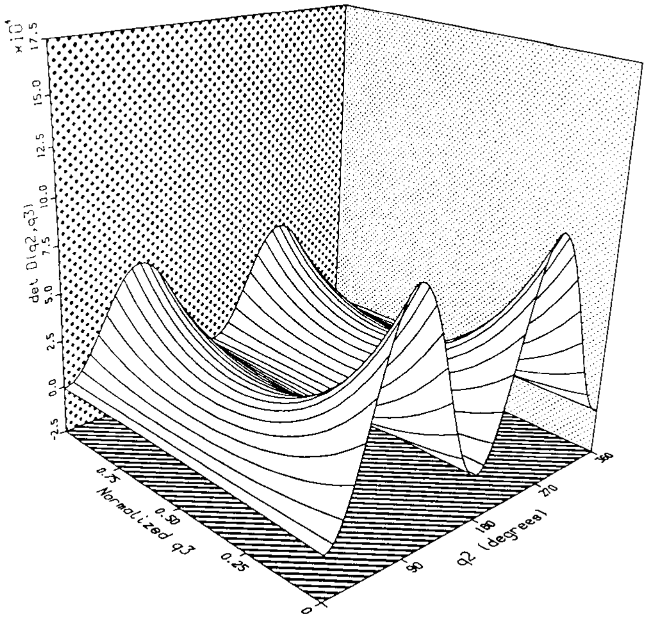

Fig. 2. Simplified $|\hat{D}(q_2, q_3)|$ plot.

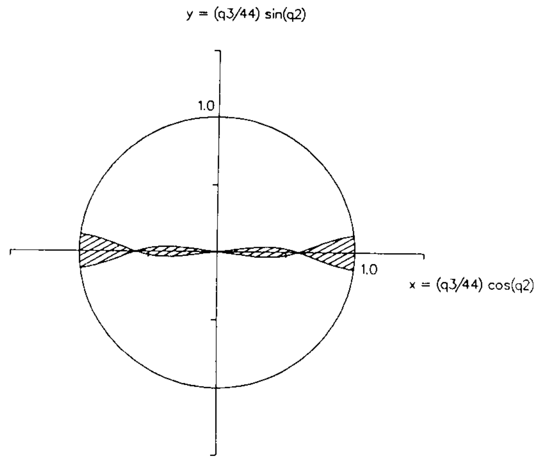

Fig. 3. Actual versus usable workspace.

avoid zero-crossings, the entire volume swept by the shaded area in Fig. 3 must be defined as off-bounds for the arm. This reduces the usable workspace by approximately 1 percent and effectively separates it into two disjoint parts. Since near-zero values of $|\hat{D}(q_2, q_3)|$ will saturate the actuators and corrupt the performance of the controller, a buffer zone must be instituted to guard against such values [10]. Such a buffer zone will reduce the usable workspace further.

Our example thus demonstrates that numerical simplification of the model can lead to significant difficulties for the controller *and* the trajectory planner. We note that, in general, remedial measures to alleviate the mathematical deficiencies of the simplified model cannot be easily instituted. It should be appreciated that the mathematical evaluation of the positive-definiteness of a six degree-of-freedom inertial matrix (whose elements are functions of up to five joint variables [1]) is an extremely complex, if not intractable, task. Furthermore, the analytical evaluation of the usable workspace is a computationally intensive procedure, placing an additional burden on the real-time trajectory planner.

V. Conclusions

In this note we have highlighted the issues leading to model simplifications for robot control and summarized a common numerical simplification procedure. By applying this systematic procedure to a realistic example, we have demonstrated the mathematical deficiencies of the simplification process and their detrimental effect on the controller. Care, therefore, should be exercised in using simplified models in the design of robot controllers. In addition, future research efforts should focus upon the development of systematic simplification procedures that preserve the inherent properties of the dynamic robot model.

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A Class of Stability Regions for Which a Kharitonov-Like Theorem Holds

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***Abstract*—This paper deals with families of complex polynomials whose coefficients lie within given intervals. In particular, the paper is concerned with the problem of determining if all polynomials in a family have the property that all of their roots lie within a given region. Towards this end, the paper defines a notion of a "Kharitonov region." Roughly speaking, a Kharitonov region is a region in the complex plane with the following property: given any suitable family of polynomials, in order to determine if all polynomials in the family have all of their roots in the region, it suffices to check only the vertex polynomials of the family. The main result of this paper is a sufficient condition for a given region to be a Kharitonov region.**

I. Introduction

This paper is concerned with the stability properties of a family of polynomials in which all of the coefficients lie within intervals. Research in this area was initiated by Kharitonov who showed that every polynomial in a given family will be Hurwitz if and only if each vertex polynomial of the family is Hurwitz. Kharitonov's theorem was originally proved for the case of polynomials with real coefficients in [1] and subsequently generalized to the complex case in [2]; see also [3] and [12]. In this paper, we consider a generalization of Kharitonov's theorem to

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