Multicast Connection Capacity of WDM Switching Networks With Limited Wavelength Conversion

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***Abstract*—Currently, many bandwidth-intensive applications require multicast services for efficiency purposes. In particular, as wavelength division multiplexing (WDM) technique emerges as a promising solution to meet the rapidly growing demands on bandwidth in present communication networks, supporting multicast at the WDM layer becomes an important yet challenging issue. In this paper, we introduce a systematic approach to analyzing the multicast connection capacity of WDM switching networks with limited wavelength conversion. We focus on the practical all-optical limited wavelength conversion with a small conversion degree d (e.g., $d = 2$ or 3), where an incoming wavelength can be switched to one of the d outgoing wavelengths. We then compare the multicast performance of the network with limited wavelength conversion to that of no wavelength conversion and full wavelength conversion. Our results demonstrate that limited wavelength conversion with small conversion degrees provides a considerable fraction of the performance improvement obtained by full wavelength conversion over no wavelength conversion. We also present an economical multistage switching architecture for limited wavelength conversion. Our results indicate that the multistage switching architecture along with limited wavelength conversion of small degrees is a cost-effective design for WDM multicast switching networks.**

***Index Terms*—Limited wavelength conversion, multicast, multicast connection capacity, multistage networks, optical networks, switching networks, wavelength conversion, wavelength division multiplexing (WDM).**

I. Introduction

CURRENT trends in communications indicate that *multicast*, the capability of efficiently sending a stream of information from a single source node to multiple destination nodes, is becoming increasingly popular in networking applications. Typical multicast applications include video lectures, multi-person conferences, e-commerce, and multipoint-LAN interconnections which allow large organizations to treat their many geographically-distributed LANs as a single network.

As wavelength division multiplexing (WDM) technique emerges as a promising solution to meet the rapidly growing demands on bandwidth, and multicast can be supported more efficiently in optical domain by utilizing the inherent light splitting capability of optical switches than copying data in electronic domain, it is important to have an in-depth study on the behaviors of WDM networks under multicast traffic. This topic has recently started to receive much attention in the optical networking community [1]–[7].

In all-optical WDM networks, the data remain in the optical domain throughout their paths except at the end nodes. Such paths are termed *lightpaths*. In the simplest WDM networks, a connection between two nodes must use the same wavelength within the lightpath. This requirement is referred to as the *wavelength continuity constraint*. A critical measure for improving network performance of WDM networks is the *wavelength conversion*.

A wavelength converter is a device which converts an incoming wavelength to a different outgoing wavelength without loss of any information modulated on the incoming wavelength. A *full wavelength converter* is able to convert any input wavelength to any output wavelength, so that a connection can be established between two nodes if there exists at least one wavelength at each node. The advantages of full wavelength conversion have been extensively investigated in the literature [4], [8]–[13]. It has been shown that full wavelength conversion may improve the network performance by resolving the wavelength conflicts along lightpaths.

However, in reality wavelength converters are often limited in the conversions that can be performed. A realistic all-optical wavelength converter may only be able to convert to a limited number of output wavelengths for any given input wavelength [14]–[17]. Such wavelength converters are referred to as *limited wavelength converters*. For example, a fast tunable all-optical wavelength converter capable of converting a wavelength toward two different wavelengths for a total conversion interval of 20 nm is reported in [18]. It is possible to realize full wavelength conversion optoelectronically. However, introducing these OEO conversion-based wavelength converters into networks loses the advantages of all-optical transmission and switching.

Realizing this limitation, researchers have begun studying limited wavelength conversion in a systematic way to quantify its advantage versus no wavelength conversion and full wavelength conversion. Yates *et al.* [14] presented a simple, approximate probabilistic analysis for two-hop paths in a WDM network. Tripathi and Sivarajan [15] provided an approximate analytical model to calculate the blocking probabilities for limited wavelength conversion under an arbitrary network topology. Recently, Ramaswami and Sasaki [19] provided a nonprobabilistic analysis of the problem for ring networks and, under certain restrictions, for tree networks and networks of arbitrary topology. Sharma and Varvarigos [16] analyzed limited wavelength conversion in regular all-optical WDM torus

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and hypercube networks. Note that these models and analyses mainly focus on the unicast traffic pattern. More recently, Qin and Yang [17] studied the permutation performance of WDM switching networks with limited wavelength conversion and revealed that by introducing a small degree of limited wavelength conversion, the performance of a WDM switching network can be significantly improved. In another separate work, Yang *et al.* [4] also investigated WDM switching networks with no and full wavelength conversion for multicast communication. In this paper, we will extend the ideas in [4] and [17] to the analysis of multicast performance of WDM switching networks with realistic limited wavelength conversion of small degrees. We will investigate quantitatively what advantages limited wavelength conversion can offer to the multicast performance of the network.

The rest of the paper is organized as follows. In Section II, we provide some necessary definitions and notations for multicast WDM switching networks. In Sections III and IV, we present our analytical method to calculate the multicast connection capacity under limited wavelength conversion. Based on these analyses, we obtain in Section V some numerical results and provide comparisons and discussions. In Section VI, we present an economical multistage switching architecture for limited wavelength conversion. Our conclusions appear in Section VII.

II. MULTICAST WDM SWITCHING NETWORKS

The WDM switching network considered in this paper consists of N input ports and N output ports as shown in Fig. 1. Each port connects to the switching network via a fiber link carrying k wavelengths. We denote the k wavelengths as $\{\lambda_0, \lambda_1, \ldots, \lambda_{k-1}\}$. Also, each input port is equipped with k fixed-tuned optical transmitters, and each output port is equipped with k fixed-tuned optical receivers. The switching unit is assumed to be a crossbar-like switching fabric and is nonblocking from a space-switching point of view. We also assume the switching network is multicast-capable and may be implemented using *light splitters, light combiners*, and *optical crossconnect elements* such as semiconductor optical amplifier (SOA) gates or micro-electro-mechanical systems (MEMS). In the presence of wavelength converters, we assume these wavelength converters, either with full or limited wavelength conversion capability, are placed at the output side in a dedicated way (i.e., each output port has k wavelength converters), so that a wavelength may be converted to another wavelength right before it enters the multiplexer (MUX) stage before the output fiber.

The WDM switching network (or simply a WDM network or a WDM switch) considered in this paper can be used as a photonic centralized switch or a switching (routing) node inside a mesh-type wavelength-routed network.

A. Multicast Connections

Multicast connections need to be established in an all-optical network when the information from a single source is to be distributed to a number of destinations. Generally speaking, a multicast connection in a WDM network uses a wavelength at an input port and one or more wavelengths at a set of output ports.

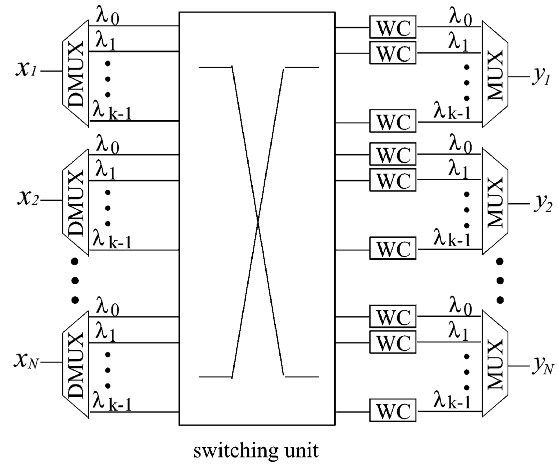

Fig. 1. $N \times N$ WDM switching network with k wavelengths.

Accordingly, a node at the input (output) side can be involved in up to k multicast connections simultaneously. However, the restrictions are that: 1) a wavelength at an output port cannot be used in more than one multicast connections at a time, or it leads to blocking inherently; and 2) no more than one wavelength at an output port can be used in the same multicast connection (this is referred to as *restriction 2*), because it is not necessary for an output port to use two or more wavelengths in the same multicast connection.

A set of multicast connections that do not involve the same source wavelength at the input side and the same destination wavelength at the output side is referred to as a *multicast assignment* in this paper. A multicast assignment is called a *full-multicast assignment* [4] if no new multicast connection can be added to this multicast assignment to form a new multicast assignment. A full-multicastassignment is in fact the maximal set of multicast connections which can be established simultaneously in a network without conflict. In an $N \times N$ k-wavelength network, this means that each wavelength on each output port needs to be connected to exactly one of the wavelengths at the input side.

We now further clarify the above definitions by using an example network, in which $N = 2$ and $k = 2$. In this case, we have two operating wavelengths $\{\lambda_0, \lambda_1\}$. Also, we assume full wavelength conversion is utilized, hence, λ_0 (λ_1) on the input side can be connected to λ_1 (λ_0) on the output side.

We list all 12 possible connections for output port 1 in the left half of Fig. 2. The sub-figures $(a1) \sim (a3)$ represent the case of λ_0 on output port 1 is connected to λ_0 on input port 1. Similarly, the sub-figures $(b1) \sim (b3), (c1) \sim (c3)$, and $(d1) \sim (d3)$ represent the cases of λ_0 on output port 1 is connected to λ_1 on input port 1, λ_0 on input port 2, and λ_1 on input port 2, respectively. Note that in such sub-figures, λ_1 on output port 1 may be connected to any wavelength on any input port except for the one that λ_0 has already been connected to; the multicast connection restriction 2 mentioned earlier applies here. Similar to output port 1, there are also 12 possible connections for output port 2, which are illustrated in the right half of Fig. 2.

A full-multicast assignment for this 2×2 2-wavelength network can be obtained by combining one of the connections for output port 1 and one of the connections for output port 2, hence, there are $12 \times 12 = 144$ full-multicast assignments in all.

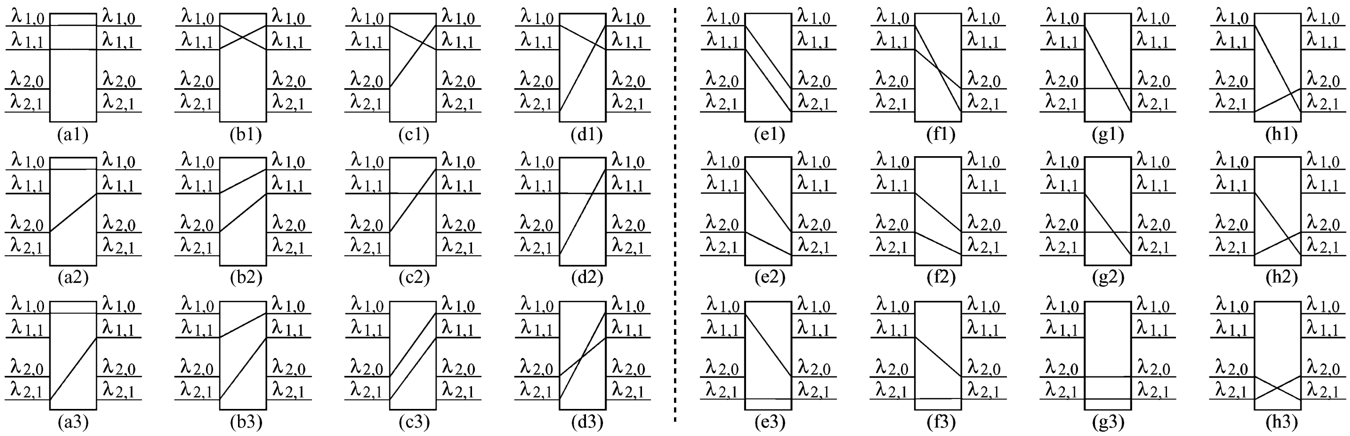

Fig. 2. Possible connections at the output side for a 2 × 2 2-wavelength network with full wavelength conversion. The left and right half of this figure list all 12 possible connections for output port 1 and output port 2, respectively.

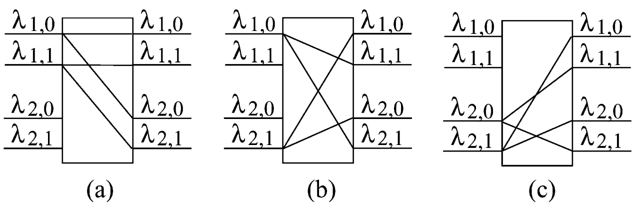

Fig. 3. Three examples of full-multicast assignments for a 2 × 2 2-wavelength network with full wavelength conversion.

We give three such examples in Fig. 3. The three full-multicast assignments shown in Fig. 3(a)–(c) are formed by combining the sub-figures $(a1)$ and $(e1)$, sub-figures $(d1)$ and $(h1)$, and sub-figures $(d3)$ and $(h3)$ from Fig. 2, respectively. As can be seen from Fig. 3, each of these three full-multicast assignments involves two multicast connections.

Switches with a larger number of full-multicast assignments offer more degrees of connecting freedom to the network, which may improve the network performance. We shall refer to the total number of the full-multicast assignments that a network can realize as *multicast connection capacity*, or *multicast capacity* for short, of the network and denote it as M_c in this paper. The functions of establishing lightpaths for all-optical wavelength-routed networks are implemented in the WDM switches. Therefore, the greater the multicast connection capacity (or the more functionality) these switches have, the more flexible is the network in reacting to fluctuating user demand, changing loads, and equipment problems, and the better it will perform under all types of conditions. Thus, multicast connection capacity can be used to quantify the multicast performance of a WDM switching network.

Yang *et al.* [4] considered the problem under no and full wavelength conversion, and showed that the network performance in terms of multicast connection capacity has been greatly improved under full wavelength conversion over no conversion. In this paper, we present a more practical and complex analysis of multicast connection capacity under limited wavelength conversion. Such a deterministic analysis provides the knowledge of the connectivity of a switching network and is useful for the network designers to determine the system parameters such as the number of wavelengths and wavelength conversion degree at the design stage based on different QoS requirements.

B. Limited Wavelength Conversion in WDM Networks

Realistic wavelength converters demonstrated in laboratories to date are only capable of limited conversion. That is, low degree wavelength conversion is likely to be far easier to realize in practice than higher degree conversions. A limited wavelength converter has *conversion degree* d (for some integer d, $1 \leq d \leq k$) if an input wavelength can be converted to $d-1$ output wavelengths in addition to the input wavelength itself. Clearly, the case $d=1$ is the no conversion, and the case $d=k$ is the full conversion. For a WDM network with limited wavelength conversion of degree two $(d=2)$, incoming wavelength λ_i $(0 \leq i \leq k-1)$ can be converted to outgoing wavelengths λ_i and $\lambda_{(i+1) \bmod k}$. Here, we assume that a limited wavelength converter converts a wavelength circularly. For notational convenience, we use λ_{i+1} instead of $\lambda_{(i+1) \bmod k}$ throughout the paper, with the understanding that $(i+1) \equiv (i+1) \bmod k$. For conversion degree $d > 2$, we assume that a wavelength can be converted to its adjacent wavelengths on either side of the input wavelength. For example, when $d=3$, incoming wavelength λ_i can be converted to outgoing wavelengths λ_{i-1}, λ_i, and λ_{i+1}.

In an $N \times N$ k-wavelength WDM switching network, each fiber link at both input and output sides carries k wavelengths, and there are a total of N channels operating on each λ_i at both input and output sides. We shall refer to each of these $N\lambda_i$'s as a *wavelength group*, which is denoted as u_{λ_i} for the input ports and v_{λ_i} for the output ports. Now, we can think of having a bipartite graph G with parts X and Y, where the set of vertices X and Y represent the set of wavelength groups on the input side (u_{λ_i}'s) and the output side (v_{λ_i}'s), respectively; hence, the cardinality of both sets is equal to the number of wavelengths in the WDM switching network, i.e., $|X| = |Y| = k$.

The edges in the bipartite graph represent the possible conversions from one wavelength to another wavelength. For example, when $d=2$ (see Fig. 4), each vertex $u_{\lambda_i} \in X$ has two edges incident to $v_{\lambda_i}, v_{\lambda_{i+1}} \in Y$. In general, the d vertices in set X (or Y) which are connected to vertices in set Y (or X) are called *reachable vertices* (or *reachable wavelength groups*), and the

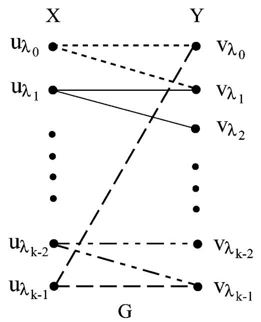

Fig. 4. Possible wavelength conversion at each wavelength group for $d = 2$.

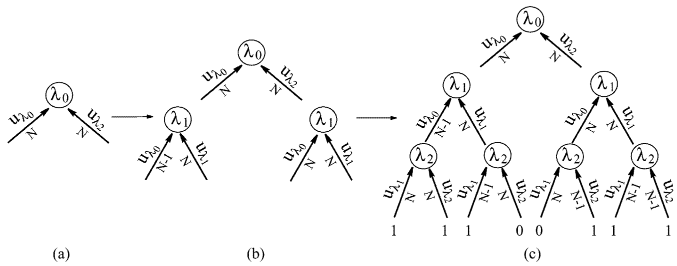

Fig. 5. Conversion tree construction when $k = 3$ and $d = 2$.

corresponding wavelengths of such vertices are called *reachable wavelengths*. Let $\Lambda_i \subset X$ denote those reachable wavelength groups for v_{λ_i}. For example, we have $\Lambda_i = \{u_{\lambda_{i-1}}, u_{\lambda_i}\}$ for $d = 2$.

Given the above preparations, we are now in a position to analyze the multicast connection capacity of a WDM network under limited wavelength conversion. We will present a systematic method for calculating the multicast connection capacity of an $N \times N$ k-wavelength WDM network with conversion degrees $(d = 2, 3)$ in the next two sections.

III. Multicast Connection Capacity With Limited Wavelength Conversion of Degree Two

In this section, we first study two examples for $k = 3$ and $k = 4$. In the meantime, we define some terms which facilitate the calculation of multicast connection capacity, and then establish two lemmas. Finally, we derive an explicit formula for the multicast connection capacity under conversion degree two.

A. Examples and Observations

Example 1: An $N \times N$ WDM network when $k = 3$. We have $\Lambda_0 = \{u_{\lambda_0}, u_{\lambda_2}\}, \Lambda_1 = \{u_{\lambda_0}, u_{\lambda_1}\}$, and $\Lambda_2 = \{u_{\lambda_1}, u_{\lambda_2}\}$. Consider the three wavelengths at an output port: λ_0 can be paired with any one of the wavelengths in u_{λ_0} and u_{λ_2}. We can think of building a tree-like directed graph where λ_0 is the root node and its left (right) arc represents the connection from $u_{\lambda_0}(u_{\lambda_2})$.

Fig. 5 shows the construction procedure of such a tree. Since each wavelength group (u_{λ_0} and u_{λ_2}) contains N wavelengths from different input ports, for λ_0 at an output port, there are N different ways to form a connection from u_{λ_0} and another N ways to form a connection from u_{λ_2}. We can continue building the tree by considering multicast connections for λ_1 at the same output port [see Fig. 5(b)]. Similar to λ_0, λ_1 has N choices to

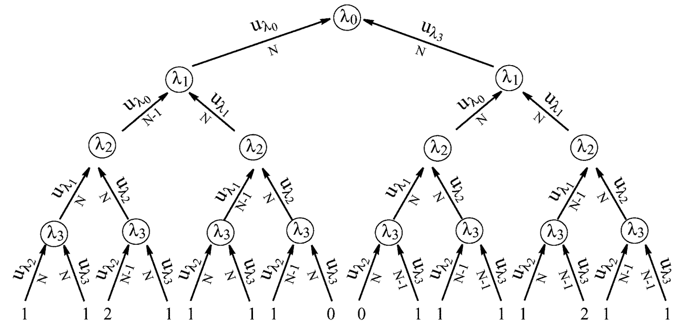

Fig. 6. Conversion tree for $k = 4$ and $d = 2$.

form a connection from both u_{λ_0} and u_{λ_1} under the right arc of λ_0; under the left arc of λ_0, there are also N ways to form a connection from u_{λ_1}, but $N - 1$ choices are left for λ_1 to be paired to u_{λ_0}. This is because one of the wavelengths in u_{λ_0} has already been used in a previous connection for λ_0 at the same output port. Similarly, we can complete the tree-like graph by considering multicast connections for λ_2 as shown in Fig. 5(c).

Note that the tree-like graph constructed by the above method explores every possible way for an output port to be connected to input wavelengths according to the wavelength conversion capability. We shall refer to such a graph as a *conversion tree*, and define each path from the "leaf node" to the root node in the conversion tree as a *connection pattern*, in which every output wavelength is assigned to a specific input wavelength group.

When conversion degree is two, each wavelength on an output port can be paired to wavelengths in its two reachable input wavelength groups. Therefore, in the above example $(k = 3)$, there are 2^3 different connection patterns. Under each connection pattern, the number of possible multicast connections that an output port can be involved in may be obtained by multiplying the numbers of choices (i.e., N or $N - 1$) for output wavelengths to be connected to input wavelength groups. Such choice of $(N - 1)$ occurs whenever an input wavelength group is repeated in a connection pattern, and is counted at the bottom of the conversion tree under each connection pattern. Note that these numbers are symmetric for the left subtree and right subtree, and we will discuss this property in detail later. Clearly, once we obtain the numbers of possible multicast connections in each of these 2^3 connection patterns, the sum of them is the total number of multicast connections that an output port can be involved in, which is

$$2N^3 + 6N^2(N-1) = 2\left[\binom{3}{0}N^3 + \binom{3}{2}N^2(N-1)\right].$$

Given that each of the N output ports can be involved in different multicast connections independently, the number of full-multicast assignments, or the multicast connection capacity for an $N \times N$ 3-wavelength WDM switching network, may be calculated as

$$M_c = 2^N\left[\binom{3}{0}N^3 + \binom{3}{2}N^2(N-1)\right]^N \quad (1)$$

Example 2: An $N \times N$ WDM network when $k = 4$. We have $\Lambda_0 = \{u_{\lambda_0}, u_{\lambda_3}\}, \Lambda_1 = \{u_{\lambda_0}, u_{\lambda_1}\}, \Lambda_2 = \{u_{\lambda_1}, u_{\lambda_2}\}$, and $\Lambda_3 = \{u_{\lambda_2}, u_{\lambda_3}\}$. Similar to $k = 3$, we build a conversion tree for $k = 4$ as shown in Fig. 6.

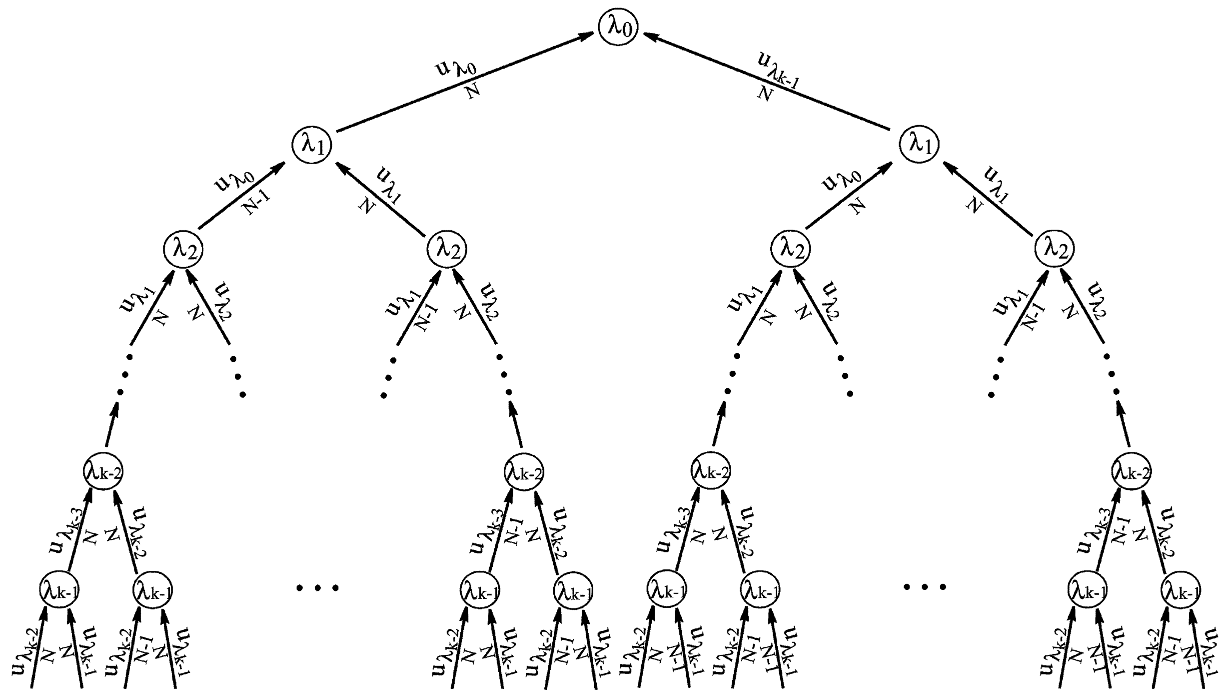

Fig. 7. General conversion tree for k wavelengths and $d = 2$.

The number of multicast connections that an output port can be involved in is

$$\begin{aligned}&2N^4 + 12N^3(N-1) + 2N^2(N-1)^2\\&= 2[N^4 + 6N^3(N-1) + N^2(N-1)^2]\\&= 2\left[\binom{4}{0}N^4 + \binom{4}{2}N^3(N-1)\right.\\&\quad\left.+\binom{4}{4}N^2(N-1)^2\right].\end{aligned}$$

Then the multicast connection capacity may be calculated as

$$M_c = 2^N\left[\binom{4}{0}N^4 + \binom{4}{2}N^3(N-1) + \binom{4}{4}N^2(N-1)^2\right]^N. \quad (2)$$

From the above examples, we have the following observations.

1) The numbers of repeated input wavelength groups under each connection pattern in the left and right subtrees are equal.
2) If the number of repeated input wavelength groups is $i(0 \le i \le \lfloor k/2 \rfloor)$ for a connection pattern, then the number of multicast connections for this connection pattern is $N^{k-i}(N-1)^i$.
3) The number of full-multicast assignments that an output port with k wavelengths can be involved in is in the form of

$$\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} 2\binom{k}{2i} N^{k-i}(N-1)^i$$

where $2\binom{k}{2i}$ is number of connection patterns with i repeated input wavelength groups.

In general, for an $N \times N$ k-wavelength WDM switching network with limited wavelength conversion of degree two, each of the k wavelengths on an output port, say, $\lambda_i (0 \le i \le k-1)$, can be paired to one of its reachable wavelength groups $\Lambda_i = \{u_{\lambda_{i-1}}, u_{\lambda_i}\}$. A binary conversion tree in Fig. 7, which is built in a similar way as we used in previous examples, illustrates all possible such connections, where the total number of connection patterns is 2^k.

Notice that in such a conversion tree, all connection patterns characterized by λ_0 being connected to u_{λ_0} form the *left subtree*, while all connection patterns characterized by λ_0 being connected to $u_{\lambda_{k-1}}$ form the *right subtree*. Clearly, the number of connection patterns in each of the subtrees is 2^{k-1}. Notice also that in each of these connection patterns any input wavelength group appears at most twice, since the conversion degree is two. However, different wavelength groups may repeat simultaneously in the same connection pattern. We denote the number of repeated wavelength groups in a connection pattern as m, where $0 \le m \le \lfloor k/2 \rfloor$ (note that the number of repeated wavelength groups (m) cannot exceed $\lfloor k/2 \rfloor$, since $k - 2m \ge 0$).

We now look at an interesting property on how repeated input wavelength groups can be present in a connection pattern in the left and right subtrees. When a connection pattern contains exactly m repeated wavelength groups, $k - 2m$ wavelengths groups are not repeated in such a connection pattern. Suppose these $k - 2m$ nonrepeated wavelength groups are taken out and the remaining $2m$ wavelength groups are: $u_{\lambda_{i_1}}, u_{\lambda_{i_2}}, \ldots, u_{\lambda_{i_{2m-1}}}$ and $u_{\lambda_{i_{2m}}}$. Without loss of generality, we assume $u_{\lambda_0} \le u_{\lambda_{i_1}} < u_{\lambda_{i_2}} < \cdots < u_{\lambda_{i_{2m-1}}} < u_{\lambda_{i_{2m}}} \le u_{\lambda_{k-1}}$, and define $\Phi = \{u_{\lambda_{i_1}}, u_{\lambda_{i_2}}, \ldots, u_{\lambda_{i_{2m-1}}}, u_{\lambda_{i_{2m}}}\}$ as the *set* *of possible repeating wavelength groups*, where $|\Phi| = 2m$. We also define m subsets of possible repeating wavelength groups as: $\Phi_1 = \{u_{\lambda_{i_1}}, u_{\lambda_{i_2}}\}, \Phi_2 = \{u_{\lambda_{i_3}}, u_{\lambda_{i_4}}\}, \ldots, \Phi_m = \{u_{\lambda_{i_{2m-1}}}, u_{\lambda_{i_{2m}}}\}$, where $\cup_{i=1}^{m} \Phi_i = \Phi$. Note that in the case of $m = 0, \Phi = \emptyset$. For $m \neq 0$, under the conversion degree $d = 2$, we observe that once the $k - 2m$ nonrepeated wavelength groups are chosen for a connection pattern, the m repeated wavelength groups in the same connection pattern cannot be arbitrarily selected from Φ. They must be those smaller ones in every subset Φ_i, i.e., $u_{\lambda_{i_1}}, u_{\lambda_{i_3}}, \ldots, u_{\lambda_{i_{2m-1}}}$ for the left subtree, while the m repeated wavelength groups must be those larger ones in every subset Φ_i for the right subtree.

B. Multicast Connection Capacity

We now formally calculate the multicast connection capacity for conversion degree two. From the discussions in previous subsection, we can obtain two lemmas.

Lemma 1: For every connection pattern in the left subtree, the repeated wavelength groups can be uniquely determined by selecting nonrepeated wavelength groups in the same connection pattern. Only the smaller wavelength group in each of the subsets of possible repeating wavelength groups $\Phi_i (1 \leq i \leq m)$ can be chosen for such a connection pattern. Also, these selected nonrepeated wavelength groups together with the determined repeated wavelength groups define a unique connection pattern.

Proof: By induction on the number of repeated wavelength groups in a connection pattern m.

Base case 1: $m = 0$, no wavelength group is repeated in a connection pattern.

In this case, all of the k wavelength groups $u_{\lambda_0}, u_{\lambda_1}, \ldots, u_{\lambda_{k-1}}$ have to be selected for connections to k wavelengths on an output port. Consider the connection pattern starting from output λ_0, λ_0 is to be connected to u_{λ_0} only in the left subtree. As for output wavelength λ_1, even though it could be connected to one of its reachable wavelength groups $\Lambda_1 = \{u_{\lambda_0}, u_{\lambda_1}\}, u_{\lambda_0}$ cannot be chosen since u_{λ_0} has already been used for λ_0. Otherwise u_{λ_0} repeats in such a connection pattern. Therefore, the only choice for output wavelength λ_1 is u_{λ_1}. Similarly, λ_2 can only be connected to $u_{\lambda_2}, \ldots$, and λ_{k-1} can only be connected to $u_{\lambda_{k-1}}$. This completes all connections for k output wavelengths on an output port, and the connection pattern is unique.

Base case 2: $m = 1$, one wavelength group is repeated in a connection pattern.

In this case, $k - 2$ wavelength groups are not repeated. Suppose these $k - 2$ nonrepeated wavelength groups are chosen, and the set of possible repeating wavelength groups is $\Phi = \{u_{\lambda_a}, u_{\lambda_b}\}$, where we assume $u_{\lambda_0} \leq u_{\lambda_a} < u_{\lambda_b} \leq u_{\lambda_{k-1}}$. Clearly, only one of the wavelength groups in Φ can be chosen. Suppose u_{λ_b}, the larger one in Φ, is the repeated wavelength group, then, λ_a can only be connected to $u_{\lambda_{a-1}}$, because $\Lambda_a = \{u_{\lambda_{a-1}}, u_{\lambda_a}\}$. As for output wavelength λ_{a-1}, even though $\Lambda_{a-1} = \{u_{\lambda_{a-2}}, u_{\lambda_{a-1}}\}, u_{\lambda_{a-1}}$ cannot be chosen since $u_{\lambda_{a-1}}$ has already been used for λ_a. Hence, the only choice for λ_{a-1} is $u_{\lambda_{a-2}}$. Similarly, λ_{a-2} can only be connected to $u_{\lambda_{a-3}}, \ldots$, and λ_0 can only be connected to $u_{\lambda_{k-1}}$. But this contradicts with our assumption that λ_0 is to be paired to u_{λ_0} in the left subtree. Therefore, u_{λ_b} cannot be selected as the repeated wavelength group. Only u_{λ_a}, the smaller wavelength group in $\Phi = \{u_{\lambda_a}, u_{\lambda_b}\}$, can be repeated. The above argument also holds when $u_{\lambda_a} = u_{\lambda_0}$.

TABLE I
UNIQUE CONNECTION PATTERN WHEN u_{λ_a} IS REPEATED FOR $m = 1$

Output λ_i	λ_0	λ_1	$\ldots$	λ_{a-1}	λ_a	λ_{a+1}	λ_{a+2} $\ldots$
Λ_i	$\left\{\begin{matrix} u_{\lambda_0} \\ u_{\lambda_{k-1}} \end{matrix}\right\}$	$\left\{\begin{matrix} u_{\lambda_0} \\ u_{\lambda_1} \end{matrix}\right\}$	$\ldots$	$\left\{\begin{matrix} u_{\lambda_{a-2}} \\ u_{\lambda_{a-1}} \end{matrix}\right\}$	$\left\{\begin{matrix} u_{\lambda_{a-1}} \\ u_{\lambda_a} \end{matrix}\right\}$	$\left\{\begin{matrix} u_{\lambda_a} \\ u_{\lambda_{a+1}} \end{matrix}\right\}$	$\left\{\begin{matrix} u_{\lambda_{a+1}} \\ u_{\lambda_{a+2}} \end{matrix}\right\}$ $\ldots$
Connect.	u_{λ_0}	u_{λ_1}	$\ldots$	$u_{\lambda_{a-1}}$	u_{λ_a}	u_{λ_a}	$u_{\lambda_{a+1}}$ $\ldots$
Output λ_i	$\ldots$	λ_{b-1}	λ_b	λ_{b+1}	λ_{b+2}	$\ldots$	λ_{k-1}
Λ_i	$\ldots$	$\left\{\begin{matrix} u_{\lambda_{b-2}} \\ u_{\lambda_{b-1}} \end{matrix}\right\}$	$\left\{\begin{matrix} u_{\lambda_{b-1}} \\ u_{\lambda_b} \end{matrix}\right\}$	$\left\{\begin{matrix} u_{\lambda_b} \\ u_{\lambda_{b+1}} \end{matrix}\right\}$	$\left\{\begin{matrix} u_{\lambda_{b+1}} \\ u_{\lambda_{b+2}} \end{matrix}\right\}$	$\ldots$	$\left\{\begin{matrix} u_{\lambda_{k-2}} \\ u_{\lambda_{k-1}} \end{matrix}\right\}$
Connect.	$\ldots$	$u_{\lambda_{b-2}}$	$u_{\lambda_{b-1}}$	$u_{\lambda_{b+1}}$	$u_{\lambda_{b+2}}$	$\ldots$	$u_{\lambda_{k-1}}$

Next, we will check the connection pattern when u_{λ_a} is repeated to see if it is unique. In such a connection pattern, u_{λ_b} is not available, and output λ_b and λ_{b+1} can only be connected to $u_{\lambda_{b-1}}$, and $u_{\lambda_{b+1}}$, respectively (see Table I). For output wavelength λ_{b+2} and $\lambda_{b-1}, \Lambda_{b+2} = \{u_{\lambda_{b+1}}, u_{\lambda_{b+2}}\}$ and $\Lambda_{b-1} = \{u_{\lambda_{b-2}}, u_{\lambda_{b-1}}\}$, but $u_{\lambda_{b+1}}$ and $u_{\lambda_{b-1}}$ cannot be chosen since $u_{\lambda_{b+1}}$ and $u_{\lambda_{b-1}}$ has already been used for λ_{b+1} and λ_b, respectively. Therefore, the only choice for λ_{b+2} and λ_{b-1} is $u_{\lambda_{b+2}}$ and $u_{\lambda_{b-2}}$, respectively.

The same argument holds as we continue connections from λ_{b+3} to λ_{k-1} and from λ_{b-2} to λ_{a+2}. Thus, for $b+1 \leq l_1 \leq k-1, \lambda_{l_1}$ can only be connected to $u_{\lambda_{l_1}}$; and for $a+2 \leq l_2 \leq b, \lambda_{l_2}$ can only be connected to $u_{\lambda_{l_2-1}}$.

Now, we consider connections for the remaining wavelengths, λ_0 to λ_{a+1}, on the same output port. Note that the repeated wavelength group, u_{λ_a}, can only be reached by two output wavelengths, namely, λ_a and λ_{a+1}. Thus, u_{λ_a} must be connected to λ_a and λ_{a+1}. If $u_{\lambda_a} = u_{\lambda_0}$, the repeated wavelength group connects to the first output wavelength, which is a special case, we finish all the connections here. Otherwise, the connections from λ_0 to λ_{a-1} can be reduced to the previous case $(m = 0)$, and for $0 \leq l_3 \leq a-1, \lambda_{l_3}$ must be connected to $u_{\lambda_{l_3}}$ as shown in Table I. In summary, such a connection pattern with u_{λ_a} being repeated is unique.

Now suppose Lemma 1 holds for m repeated wavelength groups, where $1 \leq m < \lfloor k/2 \rfloor$, we prove Lemma 1 also holds for $m + 1$ repeated wavelength groups.

In a connection pattern with $m + 1$ repeated wavelength groups, assume those $k - 2(m + 1)$ nonrepeated wavelength groups are taken out, and the subsets of possible repeating wavelength groups are: $\Phi_1 = \{u_{\lambda_{i_1}}, u_{\lambda_{i_2}}\}, \Phi_2 = \{u_{\lambda_{i_3}}, u_{\lambda_{i_4}}\}, \ldots, \Phi_m = \{u_{\lambda_{i_{2m-1}}}, u_{\lambda_{i_{2m}}}\}$, and $\Phi_{m+1} = \{u_{\lambda_{i_{2m+1}}}, u_{\lambda_{i_{2m+2}}}\}$, where $u_{\lambda_0} \leq u_{\lambda_{i_1}} < u_{\lambda_{i_2}} < \cdots < u_{\lambda_{i_{2m-1}}} < u_{\lambda_{i_{2m}}} < u_{\lambda_{i_{2m+1}}} < u_{\lambda_{i_{2m+2}}} \leq u_{\lambda_{k-1}}$, and $\cup_{i=1}^{m+1} \Phi_i = \Phi$; clearly, only $m + 1$ of the wavelength groups in Φ can be chosen. Suppose none of the wavelength group in subset Φ_1 is selected for such a connection pattern, i.e., $u_{\lambda_{i_1}}$ and $u_{\lambda_{i_2}}$ are not available; from Table II, we have that output wavelength λ_{i_1+1} can only be connected to $u_{\lambda_{i_1+1}}$, and λ_{i_1+2} can only be connected to $u_{\lambda_{i_1+2}}$. Similarly, as we continue the connections up to λ_{i_2}, we have that λ_{i_2} must be connected to $u_{\lambda_{i_2}}$ as shown in Table II. Therefore, our assumption cannot be true. In other words, at least one of the wavelength groups in $\Phi_1 = \{u_{\lambda_{i_1}}, u_{\lambda_{i_2}}\}$ must be chosen.

TABLE II
CONNECTION PATTERN WHEN NONE OF THE WAVELENGTH GROUP IN $\Phi_1 = \{u_{\lambda_{i_1}}, u_{\lambda_{i_2}}\}$ IS REPEATED

Output λ_i	$\cdots$	λ_{i_1+1}	λ_{i_1+2}	$\cdots$	λ_{i_2}	$\cdots$
Λ_i	$\cdots$	$\{u_{\lambda_{i_1}}, u_{\lambda_{i_1+1}}\}$	$\{u_{\lambda_{i_1+1}}, u_{\lambda_{i_1+2}}\}$	$\cdots$	$\{u_{\lambda_{i_2-1}}, u_{\lambda_{i_2}}\}$	$\cdots$
Connect.	$\cdots$	$u_{\lambda_{i_1+1}}$	$u_{\lambda_{i_1+2}}$	$\cdots$	$u_{\lambda_{i_2}}$	$\cdots$

TABLE III
UNIQUE CONNECTION PATTERN FOR PART 1 WHEN m WAVELENGTH GROUPS ARE REPEATED

Output λ_i	λ_0	$\cdots$	$\lambda_{i_{2m}+1}$	$\lambda_{i_{2m}+2}$	$\cdots$	$\lambda_{i_{2m+1}-1}$
Λ_i	$\{u_{\lambda_0}, u_{\lambda_{k-1}}\}$	$\cdots$	$\{u_{\lambda_{i_{2m}}}, u_{\lambda_{i_{2m}+1}}\}$	$\{u_{\lambda_{i_{2m}+1}}, u_{\lambda_{i_{2m}+2}}\}$	$\cdots$	$\{u_{\lambda_{i_{2m+1}-2}}, u_{\lambda_{i_{2m+1}-1}}\}$
Connect.	u_{λ_0}	$\cdots$	$u_{\lambda_{i_{2m}+1}}$	$u_{\lambda_{i_{2m}+2}}$	$\cdots$	$u_{\lambda_{i_{2m+1}-1}}$

TABLE IV
UNIQUE CONNECTION PATTERN FOR PART 2 WHEN ONE WAVELENGTH GROUP IS REPEATED

Output λ_i	$\lambda_{i_{2m+1}}$	$\lambda_{i_{2m+1}+1}$	$\lambda_{i_{2m+1}+2}$	$\cdots$	λ_{k-1}
Λ_i	$\{u_{\lambda_{i_{2m+1}-1}}, u_{\lambda_{i_{2m+1}}}\}$	$\{u_{\lambda_{i_{2m+1}}}, u_{\lambda_{i_{2m+1}+1}}\}$	$\{u_{\lambda_{i_{2m+1}+1}}, u_{\lambda_{i_{2m+1}+2}}\}$	$\cdots$	$\{u_{\lambda_{k-2}}, u_{\lambda_{k-1}}\}$
Connect.	$u_{\lambda_{i_{2m+1}}}$	$u_{\lambda_{i_{2m+1}}}$	$u_{\lambda_{i_{2m+1}+1}}$	$\cdots$	$u_{\lambda_{k-1}}$

In fact, if we assume none of wavelength group in any subset, $\Phi_i (1 \le i \le m+1)$, is selected for the connection pattern, we may conclude that at least one of the wavelength groups in Φ_i must be chosen. Since we could only select a total of $m+1$ wavelength groups in $\Phi (\Phi = \cup_{i=1}^{m+1} \Phi_i)$ for such a connection pattern, by the pigeon hole principle [20], there must be a unique wavelength group chosen from every subset $\Phi_i (1 \le i \le m+1)$.

Now, we can divide the connection pattern for k wavelengths on an output port into two parts as follows.

1) Connection pattern from output wavelength λ_0 to $\lambda_{i_{2m+1}-1}$ with m repeated wavelength groups in $\cup_{i=1}^{m} \Phi_i$.
2) Connection pattern from $\lambda_{i_{2m+1}}$ to λ_{k-1} with one repeated wavelength group in subset Φ_{m+1}.

According to our hypothesis on m repeated wavelength groups, for connections in part one, only the smaller one in every subset $\Phi_i (1 \le i \le m)$ can be chosen as the repeated wavelength group, and there exists a unique connection pattern as shown in Table III.

For connections in part two, it can be treated as a special case for one repeated wavelength group, because $\lambda_{i_{2m+1}}$, the first wavelength in part two, connects to the repeated wavelength group $u_{\lambda_{i_{2m+1}}}$, which is the smaller one in Φ_{m+1}. Thus, a unique connection pattern could be set up for part two as shown in Table IV.

Note that the wavelength groups connected to part one and part two are in a nondecreasing order. Also, $\lambda_{i_{2m+1}-1}$, the largest output wavelength in part one, connects to $u_{\lambda_{i_{2m+1}-1}}$; and $\lambda_{i_{2m+1}}$, the smallest output wavelength in part two, connects to $u_{\lambda_{i_{2m+1}}}$. Therefore, none of the wavelength groups used in part one occurs in part two, and *vice versa*. Combining part one and part two gives us a unique connection pattern for $m+1$ repeated wavelength groups.

Thus, by the principle of mathematical induction, Lemma 1 holds. ∎

TABLE V
UNIQUE CONNECTION PATTERN WHEN $m = 0$

Output λ_i	λ_0	λ_1	$\cdots$	λ_{k-2}	λ_{k-1}
Λ_i	$\{u_{\lambda_0}, u_{\lambda_{k-1}}\}$	$\{u_{\lambda_0}, u_{\lambda_1}\}$	$\cdots$	$\{u_{\lambda_{k-3}}, u_{\lambda_{k-2}}\}$	$\{u_{\lambda_{k-2}}, u_{\lambda_{k-1}}\}$
Connect.	$u_{\lambda_{k-1}}$	u_{λ_0}	$\cdots$	$u_{\lambda_{k-3}}$	$u_{\lambda_{k-2}}$

TABLE VI
UNIQUE CONNECTION PATTERN WHEN u_{λ_b} IS REPEATED FOR $m = 1$

Output λ_i	λ_0	$\cdots$	λ_{a-1}	λ_a	λ_{a+1}	λ_{a+2}	$\cdots$
Λ_i	$\{u_{\lambda_0}, u_{\lambda_{k-1}}\}$	$\cdots$	$\{u_{\lambda_{a-2}}, u_{\lambda_{a-1}}\}$	$\{u_{\lambda_{a-1}}, u_{\lambda_a}\}$	$\{u_{\lambda_a}, u_{\lambda_{a+1}}\}$	$\{u_{\lambda_{a+1}}, u_{\lambda_{a+2}}\}$	$\cdots$
Connect.	$u_{\lambda_{k-1}}$	$\cdots$	$u_{\lambda_{a-2}}$	$u_{\lambda_{a-1}}$	$u_{\lambda_{a+1}}$	$u_{\lambda_{a+2}}$	$\cdots$

Output λ_i	λ_{b-1}	λ_b	λ_{b+1}	λ_{b+2}	$\cdots$	λ_{k-2}	λ_{k-1}
Λ_i	$\{u_{\lambda_{b-2}}, u_{\lambda_{b-1}}\}$	$\{u_{\lambda_{b-1}}, u_{\lambda_b}\}$	$\{u_{\lambda_b}, u_{\lambda_{b+1}}\}$	$\{u_{\lambda_{b+1}}, u_{\lambda_{b+2}}\}$	$\cdots$	$\{u_{\lambda_{k-3}}, u_{\lambda_{k-2}}\}$	$\{u_{\lambda_{k-2}}, u_{\lambda_{k-1}}\}$
Connect.	$u_{\lambda_{b-1}}$	u_{λ_b}	u_{λ_b}	$u_{\lambda_{b+1}}$	$\cdots$	$u_{\lambda_{k-3}}$	$u_{\lambda_{k-2}}$

Lemma 2: For every connection pattern in the right subtree, the repeated wavelength groups can also be uniquely determined by selecting nonrepeated wavelength groups in the same connection pattern as in Lemma 1, but the repeated wavelength groups must be the larger ones in every subset $\Phi_i (1 \le i \le m)$. Also, these selected nonrepeated wavelength groups together with the determined repeated wavelength groups define a unique connection pattern.

Proof: By induction on the number of repeated wavelength groups in a connection pattern m.

Base case 1: $m = 0$, no wavelength group is repeated in a connection pattern.

In the right subtree, λ_0 is to be connected to $u_{\lambda_{k-1}}$. Therefore, λ_{k-1} could only be connected to $u_{\lambda_{k-2}}$. Similarly, for $1 \le l \le k-2, \lambda_l$ must be paired to $u_{\lambda_{l-1}}$. The complete connection pattern is shown in Table V, and it is unique.

Base case 2: $m = 1$, one wavelength group is repeated in a connection pattern.

Assume $k-2$ nonrepeated wavelength groups are chosen, and the set of possible repeating wavelength groups is $\Phi = \{u_{\lambda_a}, u_{\lambda_b}\}$, where $u_{\lambda_0} \le u_{\lambda_a} < u_{\lambda_b} \le u_{\lambda_{k-1}}$. Suppose u_{λ_b} is not the repeated wavelength group, then, λ_{b+1} must be connected to $u_{\lambda_{b+1}} (\Lambda_{b+1} = \{u_{\lambda_b}, u_{\lambda_{b+1}}\})$. Moreover, $u_{\lambda_{b+2}}$ is the only choice for λ_{b+2}. Similarly, we can show that output wavelength λ_{k-1} must be connected to $u_{\lambda_{k-1}}$. Note that λ_0 has already been connected to $u_{\lambda_{k-1}}$ in the right subtree. Therefore, the above assumption is not true, and, u_{λ_b}, the larger one in Φ, must be the repeated wavelength group.

Now, we set up the connection pattern with u_{λ_b} repeated as shown in Table VI. We may start from λ_a and λ_{a+1}. Because u_{λ_a} is not repeated, λ_a and λ_{a+1} must be connected to $u_{\lambda_{a-1}}$, and $u_{\lambda_{a+1}}$, respectively. By doing so, the only choice left for λ_{a+2} is $u_{\lambda_{a+2}}$ and for λ_{a-1} is $u_{\lambda_{a-2}}$. Similarly, we can show that for $1 \le l_1 \le a, \lambda_{l_1}$ connects to $u_{\lambda_{l_1-1}}$; and for $a+1 \le l_2 \le b-1, \lambda_{l_2}$ connects to $u_{\lambda_{l_2}}$.

Now, we consider connections for remaining wavelengths from λ_b to λ_{k-1}. Note that u_{λ_b} must be repeated on λ_b and λ_{b+1}. If $u_{\lambda_b} = u_{\lambda_{k-1}}$, we complete all the connections here. Otherwise, the connections from λ_{b+2} to λ_{k-1} can be done in a similar way as that for the case of no wavelength group repeated ($m = 0$), and $\lambda_{l_3} (b+2 \le l_3 \le k-1)$ connects to $u_{\lambda_{l_3}-1}$. Thus, the connection pattern in Table VI is unique.

Suppose Lemma 2 holds for m repeated wavelength groups, where $1 \leq m < \lfloor k/2 \rfloor$. We can show that Lemma 2 also holds for $m+1$ repeated wavelength groups by dividing the $m+1$ repeated wavelength groups into m repeated wavelength groups in one part and one repeated wavelength group in another part. The method is similar to the inductive step in the proof of Lemma 1; hence, it is omitted. ∎

We are now in a position to present the main result of this section.

Theorem 1: For an $N \times N$ k-wavelength WDM multicast switching network with limited wavelength conversion of degree two $(d = 2)$, the number of full-multicast assignments, or multicast connection capacity, is

$$M_c = \left[2 \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2m} N^{k-m}(N-1)^m \right]^N. \quad (3)$$

Proof: Consider k wavelengths on an output port, with conversion degree two, $\lambda_i (0 \leq i \leq k-1)$ may be connected to any one of its two reachable wavelength groups, $\Lambda_i = \{u_{\lambda_{i-1}}, u_{\lambda_i}\}$. Hence, we have a total of 2^k different connection patterns (see Fig. 7), among which, 2^{k-1} connection patterns are with λ_0 being connected to wavelength group u_{λ_0} (the left subtree); and another 2^{k-1} connection patterns are with λ_0 being connected to wavelength group $u_{\lambda_{k-1}}$ (the right subtree).

Note that for any connection pattern in the conversion tree in Fig. 7, an input wavelength group may appear at most twice due to conversion degree two, but the number of repeated wavelength groups m in a connection pattern may vary from 0 to $\lfloor k/2 \rfloor$.

For any of the 2^k connection patterns, each of the k output wavelengths is specified to be connected to one of its two reachable input wavelength groups, yet, an output wavelength still has as many as N choices for its multicast connection, because a wavelength group is the collection of the same wavelength but from N different input ports. In fact, the number of different multicast connections that an output wavelength, say λ_i, may have is N if the specified wavelength group for λ_i is not repeated. On the other hand, whenever a wavelength group, say u_{λ_l}, repeats in a connection pattern, output wavelengths λ_l and λ_{l+1} that are assigned to such a repeated wavelength group cannot have N choices of multicast connections simultaneously. More specifically, consider the possible connections for λ_l first: λ_l can pair with any one of the N wavelengths in u_{λ_l} to form a multicast connection; λ_{l+1} can pair with any one of the remaining $N-1$ wavelengths in u_{λ_l} to form another multicast connection. Consequently, the two wavelengths assigned to a repeated wavelength group together have $N(N-1)$ different multicast connections. In general, a connection pattern with m repeated wavelength group includes $N^{k-m}(N-1)^m$ multicast connections.

In Lemma 1, we have proven that for any connection pattern in the left subtree with m repeated input wavelength groups, once $k-2m$ nonrepeated wavelength groups are chosen, it forms a unique connection pattern. In other words, any different combination of these $k-2m$ nonrepeated input wavelength groups defines a unique connection pattern but has the same property of having exactly m repeated input wavelength groups. It is equivalent to saying that the total number of connection patterns with such a property in the left subtree is $\binom{k}{k-2m}$.

Similarly, from Lemma 2, we can have that the total number of connection patterns with the property of having exactly m repeated input wavelength groups is $\binom{k}{k-2m}$ for the right subtree. Therefore, there are $2\binom{k}{k-2m}$ connection patterns with m repeated wavelength groups in the entire conversion tree.

Now, the number of full-multicast assignments that an output port can be involved in may be calculated as

$$\sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} 2 \binom{k}{k-2m} N^{k-m}(N-1)^m = 2 \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2m} N^{k-m}(N-1)^m.$$

Given that each of the N output ports can be involved in different multicast connections independently, the number of full-multicast assignments, or the multicast connection capacity, is

$$M_c = \left[2 \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2m} N^{k-m}(N-1)^m \right]^N.$$

∎

IV. Multicast Connection Capacity With Limited Wavelength Conversion of Degree Three

When conversion degree is three $(d = 3)$, we assume limited wavelength converters operate symmetrically, i.e., incoming wavelength λ_i can be converted to outgoing wavelength λ_{i+1} and λ_{i-1} in addition to λ_i. In this case, Λ_i, the set of reachable wavelength groups for output wavelength λ_i, has cardinality of three, and $\Lambda_i = \{u_{\lambda_{i-1}}, u_{\lambda_i}, u_{\lambda_{i+1}}\}$.

A conversion tree, which explores every possible multicast connection between an output port and input wavelengths, can be utilized to compute multicast connection capacity for $d = 3$. Unlike the $d = 2$ case, the conversion tree for $d = 3$ is a *ternary tree*, because each of the k wavelengths on an output port can be paired to one of its three reachable wavelength groups. Therefore, such a ternary conversion tree has a total of 3^k paths from the leaves to the root, or connection patterns. A ternary conversion tree for a WDM network when $k = 3$ is illustrated in Fig. 8.

The ternary tree constructed by the above procedure consists of three *subtrees*: 1) the *left subtree*, in which output λ_0 is connected to $u_{\lambda_{k-1}}$; 2) the *middle subtree*, in which output λ_0 is connected to u_{λ_0}; and 3) the *right subtree*, in which output λ_0 is connected to u_{λ_1}. Clearly, the number of connection patterns in each of the subtrees is 3^{k-1}. Recall that a wavelength group can repeat at most twice in a connection pattern for conversion degree two case; any input wavelength group can be used up to three times for a connection pattern in a ternary conversion tree due to the increased conversion degree. Let m_2 and m_3 denote the numbers of twice repeated wavelength groups and three times repeated wavelength groups for a connection pattern

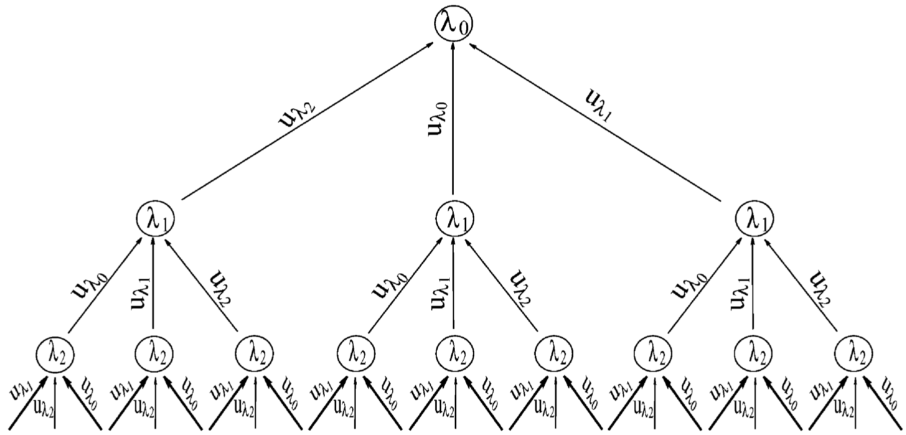

Fig. 8. Ternary conversion tree for $k = 3$ and $d = 3$.

in the ternary conversion tree, respectively. For a k-wavelength WDM network, a connection pattern can have any combinations of m_2 twice repeated wavelength groups and m_3 three times repeated wavelength groups, as long as the following condition is satisfied

$$3m_3 + 2m_2 \leq k. \tag{4}$$

For example, seven different types of connection patterns, which are characterized by m_2 and m_3, exist for a 6-wavelength WDM network: 1) $m_3 = 0, m_2 = 0$; 2) $m_3 = 0, m_2 = 1$; 3) $m_3 = 0, m_2 = 2$; 4) $m_3 = 0, m_2 = 3$; 5) $m_3 = 1, m_2 = 0$; 6) $m_3 = 1, m_2 = 1$; and 7) $m_3 = 2, m_2 = 0$. Such numbers of different connection patterns goes up very quickly as k increases. This potentially increases the complexity of analyzing multicast connection capacity with conversion degree three.

In the above discussion, we have categorized connection patterns into the same type if they have same numbers of m_2 and m_3, because once the numbers of twice and three times repeated wavelength groups are known, the number of multicast connections included in a connection pattern can be determined. Consider a connection pattern in the ternary tree: for those output wavelengths that are assigned to nonrepeated wavelength groups, each of them may have N different connections; for any two output wavelengths assigned to a twice repeated input wavelength group, they together have $P(N,2) = N(N-1)$ different connections; similarly, for any three output wavelengths that are assigned to a three times repeated input wavelength group, they together have $P(N,3) = N(N-1)(N-2)$ different connections. Therefore, the number of multicast connections that a connection pattern with m_2 twice and m_3 three times repeated wavelength groups may have is

$$N^{k-3m_3-2m_2}[P(N,2)]^{m_2}[P(N,3)]^{m_3} = N^{k-2m_3-m_2}(N-1)^{m_3+m_2}(N-2)^{m_3}.$$

Let $C_{m_3m_2}$ denote the total number of connection patterns with m_2 twice repeated and m_3 three times repeated wavelength groups. Now, the number of multicast assignments that an output port can be involved in may be calculated as

$$\sum_{\substack{m_2\geq 0, m_3\geq 0\\ 3m_3+2m_2\leq k}} C_{m_3m_2}N^{k-2m_3-m_2}(N-1)^{m_3+m_2}(N-2)^{m_3}. \tag{5}$$

Given that each of the N output ports can be involved in different multicast connections independently, the multicast connection capacity for conversion degree three may be expressed by the following formula:

$$M_c = \left[\sum_{\substack{m_2\geq 0, m_3\geq 0\\ 3m_3+2m_2\leq k}} C_{m_3m_2}N^{k-2m_3-m_2}(N-1)^{m_3+m_2} \times (N-2)^{m_3}\right]^N. \tag{6}$$

Now, a critical issue for calculating multicast connection capacity is to determine the values of $C_{m_3m_2}$. Unlike the conversion degree two case as we discussed in previous section, the numbers of same type connection patterns in the left, middle, and right subtrees are not equal for such a ternary tree. Also, those m_2 and m_3 repeated wavelength groups cannot be uniquely determined by simply selecting nonrepeated wavelength groups in a connection pattern. This greatly differs from that of $d = 2$ case. Therefore, it is difficult to determine the values of $C_{m_3m_2}$ theoretically. Instead, we use a computer to calculate these numbers. The algorithm we use consists of the following three steps:

Step 1) Generate the ternary conversion tree.

Step 2) For every connection pattern (path), count the numbers of twice and three times repeated wavelength groups, i.e., m_2 and m_3, and then categorize such a connection pattern into different types according to its values of m_2 and m_3.

Step 3) Sum up the numbers for every possible connection types and obtain the values of $C_{m_3m_2}$.

We list some of the results generated by the above algorithm in Table VII. For example, when $k = 4$ and $k = 6$, the multicast connection capacity for such an $N \times N$ WDM network can be derived according to Table VII as follows.

For $k = 4$, we have

$$M_c = [9N^4 + 48N^3(N-1) + 12N^2(N-1)^2 + 12N^2(N-1)(N-2)]^N. \tag{7}$$

TABLE VII
DIFFERENT CONNECTION TYPES AND THEIR NUMBERS ($C_{m_3 m_2}$)

Connection Types	$k=4$	$k=6$	$k=8$	$k=12$
$m_3=0, m_2=0$	9	20	49	324
$m_3=0, m_2=1$	48	228	848	9000
$m_3=0, m_2=2$	12	294	2448	62820
$m_3=0, m_2=3$	–	28	1312	137784
$m_3=0, m_2=4$	–	–	68	96534
$m_3=0, m_2=5$	–	–	–	17928
$m_3=0, m_2=6$	–	–	–	396
$m_3=1, m_2=0$	12	84	368	4368
$m_3=1, m_2=1$	–	72	1024	40032
$m_3=1, m_2=2$	–	–	336	83664
$m_3=1, m_2=3$	–	–	–	45408
$m_3=1, m_2=4$	–	–	–	4716
$m_3=2, m_2=0$	–	3	92	5964
$m_3=2, m_2=1$	–	–	16	14832
$m_3=2, m_2=2$	–	–	–	6336
$m_3=2, m_2=3$	–	–	–	264
$m_3=3, m_2=0$	–	–	–	780
$m_3=3, m_2=1$	–	–	–	288
$m_3=4, m_2=0$	–	–	–	3

For $k = 6$, we have

$$\begin{aligned} M_c = [&20N^6 + 228N^5(N-1) + 294N^4(N-1)^2 \\ &+ 28N^3(N-1)^3 + 84N^4(N-1)(N-2) \\ &+ 72N^3(N-1)^2(N-2) + 3N^2(N-1)^2(N-2)^2]^N. \end{aligned} \tag{8}$$

It should be noticed that the above approach can be extended to analyze the multicast connection capacity of higher conversion degrees. For example, when $d = 4$, a wavelength group can repeat up to four times in a connection pattern. Similar to the $d = 3$ case, let m_4 denote the number of four times repeated wavelength groups in addition to m_2 and m_3 representing the twice and three times repeated wavelength groups. The number of full-multicast assignments that an output port can be involved in may be calculated as

$$\sum_{\substack{m_2 \geq 0, m_3 \geq 0, m_4 \geq 0 \\ 4m_4 + 3m_3 + 2m_2 \leq k}} C_{m_4 m_3 m_2} N^{k-3m_4-2m_3-m_2} (N-1)^{m_4+m_3+m_2} \times (N-2)^{m_4+m_3} (N-3)^{m_4}$$

where $C_{m_4 m_3 m_2}$ is the total number of connection patterns with m_2 twice repeated, m_3 three times repeated, and m_4 four times repeated wavelength groups. The algorithm used to determine the values of $C_{m_3 m_2}$ for $d = 3$ case can be modified to calculate $C_{m_4 m_3 m_2}$.

For a general conversion degree d, the size of the conversion tree to be generated in the above algorithm is d^k. Since we need to transverse the entire conversion tree to determine the connection patterns for each path, the complexity of the algorithm is $O(d^k)$. For a given number of wavelengths k and conversion degree d, an explicit formula of multicast connection capacity [similar to (7) and (8)] can be obtained using the above algorithm off-line. For different sizes of networks, we can simply plug in the value of N into the formula and compute the multicast connection capacity.

V. NUMERICAL RESULTS AND COMPARISONS

In this section, we present numerical results for WDM multicast switching networks for three different cases: the case of no wavelength conversion; the case of limited wavelength conversion with conversion degrees $d = 2, 3$ for $k = 4$, and $d = 2, 3, 4$ for $k = 6$; and the case of full wavelength conversion.

In Fig. 9, we plot the results comparing our analysis for multicast connection capacity with no, full, and limited wavelength conversion for network sizes from 4×4 to 12×12 and $k = 4, 6$. From Fig. 9, we can see that full wavelength conversion provides the best achievable performance in terms of realizable multicast connection capacity for a given number of wavelengths per fiber. We observe that the network performance with limited wavelength conversion is greatly improved over no wavelength conversion. As is evident from the plots, limited wavelength conversion with small conversion degrees, such as $d = 2$ or $d = 3$, provides a considerable fraction of the improvement that full wavelength conversion provides over no wavelength conversion. By comparing Fig. 9(a) with Fig. 9(b), the benefits of wavelength conversion increase as the number of wavelengths k increases. The graphs also show that the network performance improves as conversion degree d increases. Hence, a favorable tradeoff exists between the conversion degree d and the fraction of the performance of full wavelength conversion. Furthermore, Fig. 9 shows that the rate of network improvement obtained by wavelength conversion diminishes as conversion degree d increases. Therefore, we view that utilizing realistic limited wavelength conversion with small conversion degrees in WDM networks is more cost-effective. This observation agrees with the results obtained by Sharma and Varvarigos [16], who found that for torus and hypercube WDM networks limited wavelength conversion with fairly small degrees is sufficient to obtain benefits comparable to those obtained by full wavelength conversion. Note that [16] focused on the throughput performance under unicast traffic and their results are probabilistic, while we investigate multicast traffic and the results are deterministic. Moreover, by comparing with the results in [17], we can see that the network performance improvement obtained by limited wavelength conversion in a multicast switching network is greater than that in a permutation switching network. For example, for a given network size $N = 10, k = 4$, and with limited wavelength conversion degree $d = 2$, the permutation capacity is 1.24×10^{10} times as much as that of no wavelength conversion; while the multicast capacity is 3.89×10^{11} times as much as that of no wavelength conversion.

VI. MULTISTAGE WDM MULTICAST SWITCHING NETWORKS

WDM switching networks may be implemented by gate switches such as semiconductor optical amplifiers (SOAs) in a broadcast-and-select way. In addition, the limited wavelength converters can be placed at the output side to switch a different source wavelength to its destination wavelength. Like a crossbar switch, such gate switches are capable of multicast operation. In general, an $N \times N$ k-wavelength WDM network with limited wavelength conversion of degree d requires $O(kdN^2)$ crosspoints (or SOAs), which grow very quickly as network

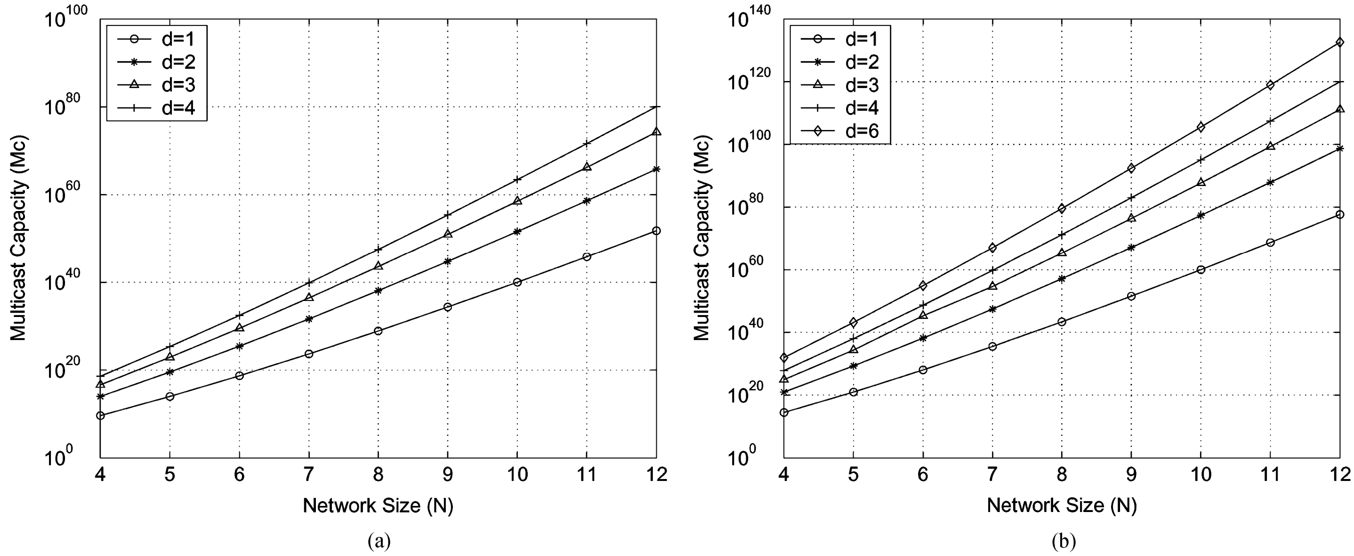

Fig. 9. Multicast connection capacity for WDM networks with four and six wavelengths for network sizes from 4×4 to 12×12. The plot shows the analytically calculated values for no $(d = 1)$, full $(d = k)$, and limited wavelength conversion with conversion degrees $d = 2, 3$ for $k = 4$, and $d = 2, 3, 4$ for $k = 6$. (a) $k = 4$. (b) $k = 6$.

size N increases. Because optical switching devices are costly, switch fabric realization with a minimum number of these devices are desirable.

In this section, we adopt the well-known Clos network [21] to provide a cost-effective solution to WDM multicasting with limited wavelength conversion. The Clos-type network has adjustable network parameters and can provide different types of connecting capabilities by choosing different values of the parameters. A three-stage Clos network with N input ports and N output ports, shown in Fig. 10, has r switch modules of $n \times m$ in the input stage, m switch modules of size $r \times r$ in the middle stage, and r switch modules of size $m \times n$ in the output stage with $N = nr$ and $m \geq n$. The network has exactly one link between every two switch modules in its consecutive stages. Such a three-stage network is conventionally denoted as a $v(m, n, r)$.

It is easy to see that the network cost of a $v(m, n, r)$ network is proportional to the number of middle stage switches m for a fixed N and r. A critical issue in designing such a network is how to ensure that the network is nonblocking, and at the meantime, minimize the number of middle stage switches m, hence reduce the number of crosspoints. Yang and Masson [22] gave a sufficient condition for a nonblocking multicast Clos network, $m \geq 3(n-1)(\log r / \log\log r)$, which yields the best available design for this type of multicast networks.

Similar to the notation of $v(m, n, r)$ for an electronic three-stage network, we denote a k-wavelength WDM three-stage network as $v_k(m, n, r)$, in which the links between switch modules are fiber links with k channels operated on them. To build a $v_k(m, n, r)$ network with the capability of wavelength conversion, it is not necessarily to provide wavelength converters at every stage, we can place the wavelength converters only at the output stage. This concept was presented by the MSW-dominant (Multicast with Same Wavelength) construction method in [4] to realize a $v_k(m, n, r)$ network under full wavelength conversion, where the output stage

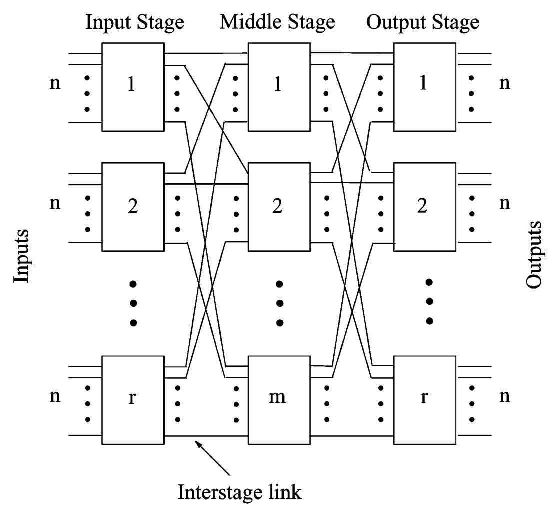

Fig. 10. Three-stage switching network.

switches are capable of wavelength conversions and the input and middle stage switches employ no wavelength conversion. Also, [4] demonstrated that the nonblocking condition under such construction method can be reduced to the electronic three-stage network case. In the following, we extend the above result to $v_k(m, n, r)$ networks with limited wavelength conversion.

Let (i, λ_{l_i}) denote an input wavelength λ_{l_i} at an input port i, where $i \in \{1, 2, \ldots, nr\}$ and $1 \leq l_i \leq k$. In a $v_k(m, n, r)$ network equipped with limited wavelength converters only at the output stage, a multicast connection with source input wavelength (i, λ_{l_i}) can be realized using wavelength λ_{l_i} in the first two stages and then realized in the third stage with the capability of limited wavelength conversion. Since each output stage switch is nonblocking itself, such a $v_k(m, n, r)$ network is nonblocking if we have sufficient middle stage switches to provide connection paths for any connection request from an idle input

TABLE VIII
COST COMPARISON OF THREE-STAGE AND CROSSBAR WDM NETWORKS UNDER NO, LIMITED, AND FULL WAVELENGTH CONVERSION (CB: CROSSBAR; TS: THREE-STAGE)

Network Model	# Crosspoints
No Conversion/CB	kN^2
No Conversion/TS	$O\left(kN^{\frac{3}{2}}\frac{\log N}{\log\log N}\right)$
Limited Conversion/CB	kdN^2
Limited Conversion/TS	$O\left(kd^{\frac{1}{2}}N^{\frac{3}{2}}\frac{\log N}{\log\log N}\right)$
Full Conversion/CB	k^2N^2
Full Conversion/TS	$O\left(k^{\frac{3}{2}}N^{\frac{3}{2}}\frac{\log N}{\log\log N}\right)$

port to some set of idle output ports. Therefore, the multicast routing in such a $v_k(m,n,r)$ network is equivalent to that of a traditional electronic $v(m,n,r)$ network. That is, a $v_k(m,n,r)$ multicast network with limited wavelength conversion is nonblocking if $m \geq 3(n-1)(\log r/\log\log r)$.

Based on this result, we can obtain that the number of crosspoints for a $v_k(m,n,r)$ network is in the order of $O(kd^{(1/2)}N^{(3/2)}(\log N)/(\log\log N))$, where d is the conversion degree. The optimal values of n and r were used to minimize the cost in the above calculation. We refer interested readers to [25], in which the details of such calculation is provided. Finally, we summarize network cost in terms of crosspoints under either crossbar or three-stage construction in Table VIII for limited wavelength conversion along with the previously obtained results for no and full wavelength conversion [4].

We observe that a $v_k(m,n,r)$ network has significantly fewer crosspoints than that of a crossbar-like fabric. Therefore, the multistage switching architecture is a cost-effective choice for a WDM multicast network with both limited and full wavelength conversion.

VII. CONCLUSION

In this paper, we have proposed a systematic approach to analyzing the multicast performance of WDM switching networks with realistic limited wavelength conversion of small degrees. In particular, we have derived an explicit formula for calculating the multicast connection capacity of conversion degree $d = 2$. We have also given a method and provided an algorithm to compute multicast connection capacity for conversion degree $d = 3$. This approach can be extended to study multicast connection capacity for higher conversion degrees. Our results have shown that the network performance in terms of multicast connection capacity has been significantly improved by limited wavelength conversion over that of no wavelength conversion. We have also found that the network performance improves as conversion degree d increases, but as d increases further, the rate of improvement decreases. Furthermore, our results have revealed that the performance improvement obtained by limited wavelength conversion with small conversion degrees (e.g., $d = 2, 3$) is comparable to that obtained by full wavelength conversion. Finally, we have presented an economical multistage switching architecture for limited wavelength conversion. Our results indicate that the multistage switching architecture along with limited wavelength conversion of small degrees is a cost-effective design for WDM multicast switching networks.

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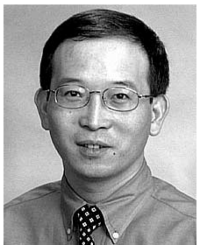

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