Hole Identification and Filling in k-times Extended Co-prime Arrays for Highly-Efficient DOA Estimation

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***Abstract*—Among various sparse array techniques, co-prime array is found to be more attractive because of its higher DoF with a smaller number of sensing elements. Generally, a co-prime array with $O(N)$ sensors can offer $O(N^2)$ number of DoFs by exploiting the difference co-array that can be obtained from the second-order statistics of received signals. However, the number of achievable DoFs of co-prime arrays is significantly smaller than expected due to the existence of "holes" in the difference co-array. In this paper, we make three major contributions to co-prime arrays. We first introduce a k-times extended co-prime configuration which can achieve lager number of DoFs and higher flexibility in array configurations, and the later helps better meet different application needs. Second, based on our k-times extended geometry, we uncover the mysterious veil of the holes in the difference co-arrays. We find several general rules of the locations of holes and derive the close-form expressions of the exact locations of all holes in the difference co-arrays of different co-prime array structures. Finally, we propose a specific array structure called complementary sub-array that can fill all of holes existing in the difference co-array. Compared with the traditional hole-existing co-prime arrays, our k-times complementary co-prime array has either similar number of sensors and DoFs with much smaller array aperture required, or much higher DoFs with the same aperture.**

***Index Terms*—Sparse arrays, co-prime arrays, difference co-arrays, DOA estimation.**

I. Introduction

In array signal processing, temporal and spatial data collected by an array of sensors are exploited to estimate unknown parameters of signal sources [2]. Particularly, finding the direction-of-arrival (DOA) of sources is of critical importance and is widely applied in the areas of radar, sonar, astronomy, and wireless communications [3]–[7]. A conventional uniform linear array (ULA) with $N+1$ elements can identify N sources at most, and has a degree of freedom (DoF) of N. Thus, it would need a large number of sensors to detect a large number of sources using ULA.

To reduce the cost, sparse arrays such as minimum redundancy arrays (MRAs) [8], co-prime arrays [9] and nested arrays [10] are proposed. With fewer physical elements, these sparse-array techniques exploit difference co-arrays [11] calculated from the correlation of the received signals to generate more elements in the virtual array. Generally, a sparse array with $O(N)$ sensors can offer $O(N^2)$ elements in the virtual array. However, MRA systems do not have closed-form expressions of the array geometries and the number of achievable DoFs, so they are complicated to design [12], [13]. With long dense ULA segment in the physical configurations, nested arrays suffer from higher mutual coupling [14]–[16]. As an important effort, super nested arrays [17], [18] alleviate the coupling problem, but at the cost of robustness and uncertainties of DoF. Allowing for the tradeoffs among different performance needs, co-prime arrays have attracted a lot of attentions in recent years. Various DOA estimation methods were applied to co-prime arrays based on subspace, frequency offsets and compressive sensing techniques [19]–[24].

Although a co-prime structure is promising, the major problem is that its difference co-array does not have completely consecutive elements but contains holes [9], which significantly reduces its number of achievable DoFs. Some recent efforts have been made to alleviate the hole problem. An extended co-prime array structure is introduced in Pal *et. al* [25] to lengthen the consecutive segment of the virtual array by expanding the physical array structure. With more sensors and larger aperture, the spatial efficiency is not improved but reduced to about half. The work in [26] uses proportional frequencies to fill some of the holes in the virtual array. Besides the need of additional frequencies, these frequencies may not be available at the sources. Authors in [27] propose to apply temporal signal coherence (TCP) in moving co-prime arrays to fill in part of holes, while the precise temporal coherence is difficult to achieve in the practical environment. In addition to the schemes that aim at filling holes, two modified co-prime structures, the co-prime array with reduced sensors (CARS) [28] and the thinned co-prime array (TCA) [29] propose to reduce the number of physical sensors while achieving the same or even larger number of DoFs.

Despite the importance of the initial efforts, only a small number of holes in the difference co-arrays are filled, with the need of additional sensors [25], frequencies [26] or the precise temporal coherence [27] respectively. A lot of deserved freedoms outside the consecutive segment are sacrificed because of the holes. The major challenge in fulfilling the DoFs is that no close-form expressions have been found for all the hole positions in the virtual array of an arbitrary co-prime array. With the focus on achieving the maximum number of consecutive lags and the minimum inter-element spacing, the work in [24] provides an expression of the hole positions in the negative range of one cross-difference co-array. The

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expression, however, cannot capture the final hole positions in the entire virtual array. In addition, the paper did not give the proof on the sufficiency of hole positions by showing that there exist no holes besides the locations given. The co-prime array with multi-period sub-arrays (CAMpS) is proposed recently in [30], where authors also make the attempt to analyse the hole positions in the new array infrastructure. The structure itself is not efficient by extending both of the sub-arrays and there is also a lack of detailed analyses on the characteristics of holes. The incomplete hole analysis in the above two papers further prevents the design of methods that can assuredly and efficiently fill all of the holes without resort to a heuristic search of holes.

In order to enhance the performance of the co-prime array, we consider the problem from two perspectives: 1) Lengthening the consecutive segment in the difference co-array, and 2) Identifying and eradicating the holes outside the consecutive segment. We first propose an advanced co-prime geometry that has an increased ratio of the consecutive segment in the virtual array. Then we derive a concise and explicit expression of all the hole positions in the difference co-array of our proposed co-prime geometry. Finally, based on our full knowledge of the hole positions, we further design a hole-free co-prime structure that can fully exploit all DoFs gained from the co-array method. Our main contributions in this paper are:

- We propose a new co-prime configuration, *k-times extended co-prime array*, which has an increased ratio of the consecutive segment in the virtual array. Our proposed k-times extended geometry can generalize most of the co-prime geometries and offer more flexibilities.
- We provide the exact expressions of all hole positions in the difference co-array of the k-times extended co-prime geometry, which can also represent hole positions of most existing co-prime arrays. We rigorously prove that there exist no holes besides the expressions we give, summarize some interesting hole characteristics, and introduce a new 2D representation of holes that offers a better understanding of the virtual array of co-prime arrays. Our quantitative hole analyses provide a base for the design of more efficient co-prime infrastructures.
- Based on our analyses on the difference co-array, we propose a sparse array geometry, *complementary co-prime array*, whose difference co-array is hole-free and has much higher DoF. We study the effect of the co-prime parameters selection and mathematically formulate the problem as an optimization problem. We provide the optimal solution to this problem.
- Taking into account the mutual coupling effect and the robustness, we provide numerical comparisons among existing co-prime structures, our proposed complementary co-prime arrays, nested arrays and super-nested arrays.
- We perform extensive simulations to investigate the impact of various factors on the performance of DOA estimation, and to evaluate the effectiveness of our proposed array design.

To our best knowledge, this is the first work that systematically studies various design options and tradeoffs of co-prime arrays, including the design of a general and flexible new array geometry to meet different user needs, the detailed analyses of hole positions and characteristics of co-array, and the providing of a complementary sub-array to fill the holes to significantly increase the degree of freedom without expanding the aperture. To help better understand the tradeoffs among different array structures, we compare the co-prime arrays with other sparse arrays, including nested arrays, super-nested arrays and arrays based on Wichmann Ruler. Furthermore, we provide detailed discussions and performance comparisons of different array structures from the perspectives of DoFs, array aperture, the effect of mutual coupling and the robustness. We reveal some interesting and vital indicators of different array geometries. We expect that this work can play a fundamental role in supporting further study and design on sparse arrays. Existing approaches on hole-filling such as using co-prime frequencies in [26] and moving co-prime arrays in [27] can be further enhanced by acquiring the exact expressions of all hole positions. We can leverage the TCA scheme [29] to reduce more physical sensors in our proposed k-times extended co-prime array and complementary co-prime array while keeping the same number of DoFs, and our complementary subarray can also help TCA to fill holes and increase DoFs.

The rest of the paper is organized as follows. In section II, we provide the background knowledge and our motivation for this work. Section III and section IV illustrate the k-times extended co-prime array structure and its hole positions in the difference co-array respectively. In section V, we present our proposed complementary co-prime array structure, which takes advantage of the knowledge of hole positions to achieve completely consecutive difference co-array for high performance DOA estimation. We provide performance studies with simulation results in Section VI and conclude the work in Section VII.

II. Preliminaries and Problems

We first introduce some backgrounds on sparse array signal processing, including the signal model and the difference co-array technique. We then make a quick review of the conventional co-prime arrays and the extended co-prime arrays. Finally, we present the limitations of existing co-prime array systems and the issues we will address in this work.

A. Signal Model

Assuming D narrowband sources with powers $[\sigma_1^2 \ \cdots \ \sigma_D^2]$ impinge on the array from directions $[\theta_1 \ \theta_2 \ \cdots \ \theta_D]$, the signals received at the array elements in the kth snapshot time can be expressed as

$$\mathbf{x}[k] = \mathbf{A}\mathbf{s}[k] + \mathbf{n}[k] \quad (1)$$

where $\mathbf{A}$ is the array manifold matrix of the form

$$\mathbf{A} = [\mathbf{a}(\theta_1) \quad \mathbf{a}(\theta_2) \quad \cdots \quad \mathbf{a}(\theta_D)] \quad (2)$$

and

$$\mathbf{a}(\theta_i) = [e^{j\frac{2\pi}{\lambda}l_1 sin\theta_i}, e^{j\frac{2\pi}{\lambda}l_2 sin\theta_i}, \cdots, e^{j\frac{2\pi}{\lambda}l_{N_0} sin\theta_i}]^T \quad (3)$$

where $[l_1, l_2, \cdots, l_{N_0}]$ represents the locations of all N_0 physical antennas. $\mathbf{s}[k] = [\mathbf{s}_1(k) \ \cdots \ \mathbf{s}_D(k)]^T$ denotes the kth snapshot of the source signal vector, and the noise vector $\mathbf{n}[k]$ is assumed to be temporally and spatially white and uncorrelated from the source.

B. Difference Co-array

In sparse array signal processing, to achieve a given number of DoFs with fewer physical sensors, a virtual array generated from the difference co-array is usually applied to substitute for the original physical array. For a sparse array with antennas located at the set $\mathbb{L}$, the difference co-array is defined as

$$\mathbb{D} = \{l_p - l_q \mid l_p, l_q \in \mathbb{L}\} \tag{4}$$

which includes location differences or known as lags of all pairs of antennas in the sparse array.

To obtain the information captured by a difference co-array, we get the covariance matrix $\mathbf{R}_{xx}$ of the received data $\mathbf{x}[k]$

$$\mathbf{R}_{xx} = E[\mathbf{x}(k)\mathbf{x}(k)^H] = \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H + \sigma^2\mathbf{I} \tag{5}$$

where $\mathbf{R}_{ss}$ is the source covariance matrix, with

$$\mathbf{R}_{ss} = diag([\sigma_1^2 \ \sigma_2^2 \ \cdots \ \sigma_D^2]), \tag{6}$$

σ_i^2 and σ^2 denote the power of the ith source and the power of noise respectively. Taking (2) and (6) into (5), we have

$$\mathbf{R}_{xx} = \sum_{i=1}^{D} \sigma_i^2 \mathbf{a}(\theta_i)\mathbf{a}^H(\theta_i) + \sigma^2\mathbf{I} \tag{7}$$

The (p,q)th entry of $\mathbf{a}(\theta_i)\mathbf{a}^H(\theta_i)$ has the form of $e^{j\frac{2\pi}{\lambda}(l_p - l_q)sin\theta_i}$, which can represent the manifold matrix of the difference co-array.

C. Co-prime Arrays

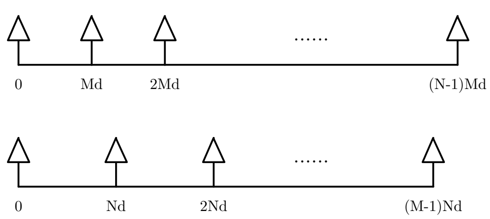


Fig. 1. Structure of the conventional co-prime array.

A conventional co-prime array [9] shown in Fig. 1 consists of two uniform linear sub-arrays with the separation Md and Nd respectively. There are N sensors in the first sub-array and M sensors in the second sub-array. M and N are co-prime integers, i.e., $gcd(M,N) = 1$, and d is the unit of inter-element spacing. To avoid spatial aliasing, d is typically set to $\lambda/2$, where λ is the wavelength of impinging narrowband signals. The locations of sensors in the conventional co-prime array can be described by the set

$$\mathbb{L}_C = \{pMd\} \cup \{qNd\}, \tag{8}$$

where $0 \leq p \leq N-1$ and $0 \leq q \leq M-1$. Since the first sensors of the two uniform linear sub-arrays are co-located, the total number of sensors in the conventional co-prime array is $M + N - 1$.

The basic objective of forming the conventional co-prime array is to enlarge the number of distinct elements in the difference co-array by utilizing the co-primality of M and N. The work in [9] shows that although an (M,N) conventional co-prime array has at least MN distinct elements in the difference co-array, these elements are not consecutive and there exist holes. This reduces the length of the ULA segment in the virtual array and thus the ultimate effective number of DOFs after applying the spatial smoothing [31], [32]. For example, the difference co-array of a $(3,4)$ conventional co-prime array shown in Fig. 2 can be represented by the set $[-9d, +9d]$ except $\pm 7d$. It has 17 distinct elements but only consecutive in the range $[-6d, 6d]$. Therefore, after the spatial smoothing, merely 6 DoFs can be obtained.

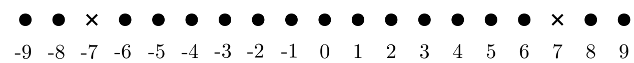


Fig. 2. The difference co-array of a $(3,4)$conventional co-prime array. Holes locate at $\pm 7d$.

To deal with the hole problem, the extended co-prime array was proposed in [25]. As shown in Fig. 3, the number of sensors in the second sub-array is doubled to enlarge the size of the consecutive ULA segment in the difference co-array. It has been proven in [25] that after the spatial smoothing, an (M,N) extended co-prime array can still offer at least MN DOFs.

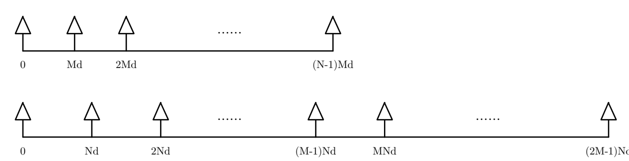


Fig. 3. Structure of the extended co-prime array.

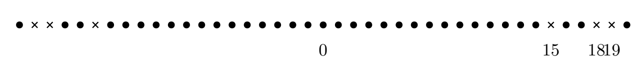


Fig. 4. The difference co-array of a $(3,4)$ extended co-prime array. Holes locate at $\pm 15d$, $\pm 18d$ and $\pm 19d$.

Fig. 4 shows the difference co-array of a $(3,4)$ extended co-prime array. It has consecutive ULA segment from $-14d$ to $14d$. After the spatial smoothing, it can still offer 14 DoFs.

D. Problems and Motivations

With its capability of increasing the number of consecutive virtual array elements, the extended co-prime array has been popularly used in existing co-prime related techniques. From the introduction and examples above, the existence of holes in the difference co-array significantly reduces the number of DoFs a virtual array can apply to effectively detect the signals. The use of the extended co-prime array helps to increase the number of DoFs, but at the cost of additional antenna elements and doubling the aperture of the physical array. The example of $(3,4)$ extended co-prime array can achieve 14 DoFs, but it has 6 elements lie outside the range $[-14, 14]$ in the difference co-array without contributing to the number of DoFs.

Generally speaking, the difference co-array of an (M, N) extended co-prime array has the range $[-(2M-1)N, (2M-1)N]$, but is only consecutive without holes in the range $[-MN-M+1, MN+M-1]$ [24]. The elements outside this range are wasted. From the spatial perspective, the array efficiency is $\frac{MN+M-1}{2MN-N}$, which is just about 50%. Too many elements in the difference co-array are wasted. If the spatial efficiency of the virtual array can be increased, the co-prime array technique will achieve significantly better performance in signal detection and estimation.

In order for our analyses to be general and applicable for all possible types of co-prime arrays, we first introduce a generalized k-times extended co-prime structure, which can achieve higher spatial efficiency by increasing k. Then we make comprehensive analyses on the difference co-array to provide precise expressions of all hole positions in the difference co-array. We also propose a new array structure, the *complementary co-prime array*, which introduces a complementary sub-array to fill all holes in the difference co-array based on our k-times extended structure. Our proposed complementary co-prime array can obtain a hole-free difference co-array, which means the spatial efficiency can reach 100%. Our proposed k-times structure can not only summarize all existing co-prime infrastructures but also provide users with the flexibility of configuration to meet different application needs. More specifically, with appropriate selection of (M, N) and k, the complementary co-prime array can either achieve larger number of DoFs or suffer less from mutual coupling as compared to the nested array.

III. k-TIMES EXTENDED CO-PRIME ARRAYS

Although the conventional co-prime array is introduced to increase the degree of freedom (DoF), the existence of holes in the difference co-array limited the actual number of DoFs it can achieve. The extended co-prime array is introduced to extend the consecutive part of the difference co-array, which allows its number of DoFs to reach $MN + M - 1$. In this section, we would like to introduce a *k-times extended co-prime array* to serve as the fundamental infrastructure that allows customers to flexibly configure the array infrastructure to meet different DoF requirements and array aperture constraints. Our k-times extended co-prime array is a generalized co-prime structure that can summarize all existing co-prime structures. Compared to existing designs, it allows for the increase of k to achieve higher spatial efficiency.

A. k-times extended co-prime configuration

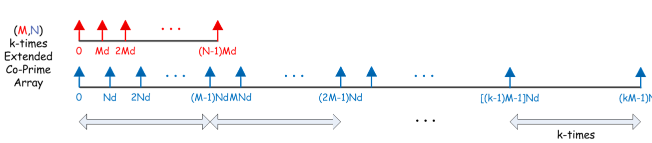


Fig. 5. Structure of the k-times extended co-prime array.

The basic configuration of an (M, N) k-times extended co-prime array is shown in Fig. 5. Instead of only having twice the number of sensors on the second sub-array in Fig. 3, the k-times extended co-prime array is expanded to have k times the number of sensors. The total number of sensors is $kM+N-1$ and the array aperture is $(kM-1)Nd$.

It is clear that the conventional co-prime array in Fig. 1 and the extended co-prime array in Fig. 3 are the special cases of the k-times extended co-prime array where $k = 1$ and $k = 2$ respectively. Furthermore, the compressed inter-element spacing (CACIS) configuration in [24] can be also considered as a special case and it forms a subset of our proposed k-times extended configuration. Each CACIS configuration with parameters p, $\breve{M}$ and N can be considered as a (M', N') k-times extended configuration with parameters $M' = \breve{M}$, $N' = N$ and $k = p$. However, as CACIS configuration compresses one of the sub-arrays, it is limited by the assumption that M can be expressed as a product of two positive integers p and $\breve{M}$, and $M = p\breve{M}$ and N are co-prime. It requires p and N to be co-prime, while our extending factor k does not have any additional limitations. k can be any positive integer regardless of the value of N. Therefore, compared to CACIS, our proposed k-times extended co-prime array is a more generalized configuration and can express a much broader range of co-prime structures. The flexibility of the co-prime configuration allows our infrastructure to better support different applications.

B. DoF analysis of k-times extended co-prime arrays

As introduced in Section III, the extended co-prime array has a consecutive ULA segment without holes in the range of $[-(MN+M-1), MN+M-1]$ and therefore has $MN+M-1$ number of DoFs. Following, we will analyze the number of DoFs of our proposed k-times extended co-prime arrays.

Proposition 1. *The difference co-array of an (M, N) k-times extended co-prime array ($k \geq 2$) has a consecutive ULA segment without holes in the range of $[-(k-1)MN - M + 1, (k-1)MN + M - 1]$.*

Proposition 2. *The difference co-array of an (M, N) 1-time extended co-prime array, which is the conventional co-prime array, has a consecutive ULA segment without holes in the range of $[0, M + N - 1]$.*

The proofs are similar to that in [9] and are omitted here.

Definition 1. ***(Spatial Efficiency)****. The spatial efficiency of a sparse array is the ratio of the number of DoFs to the physical array aperture which represents the ratio of the length of consecutive segment to the length of the virtual array in the positive part.*

The spatial efficiency of an (M, N) k-times extended co-prime array then can be expressed as

$$r_{se} = \frac{(k-1)MN + M - 1}{kMN - N} \tag{9}$$

For a fixed k, we have

$$\lim_{M,N \to +\infty} r_{se} = \frac{k-1}{k} \tag{10}$$

With the increase of k, the ratio increases and a higher spatial efficiency can be achieved.

IV. Holes in the difference co-array of k-times extended co-prime arrays

The existence of holes in virtual array formed by co-prime arrays significantly reduces the number of DoFs it may achieve. Although the benefit of hole filling is obvious and well acknowledged by the researchers in the field, it remains a big challenge to find the exact hole positions. In most past studies, for an (M, N) extended co-prime array ($M < N$), it has only been proven that its difference co-array has a consecutive ULA segment in the range of $[-(MN+M-1), MN+M-1]$. Although holes are known to appear outside this range, there are no close form equations to represent the hole locations. Various efforts have been made to search for holes for a specific (M, N) pair through some heuristics methods.

In this section, we first introduce our observed general rules related to the positions of elements in the difference co-array of an (M, N) k-times extended co-prime array. We then provide and prove the precise expressions of the locations of all holes in the difference co-array. Finally, we will discuss some special cases on the hole positions. Because of the symmetry of the difference co-array, we only present our analyses for the holes located on the non-negative part. We take the inter-element spacing d as the unit.

A. General Rules

Fig. 6 shows the general structure of the difference co-array of an (M, N) k-times extended co-prime array when $k \geq 2$. The difference co-array contains two segments, a consecutive ULA segment and an inconsecutive segment. Fig. 7 shows the difference co-array of a $(4, 5)$ 2-times extended co-prime array as an example. Based on the tests from a large number of (M, N, k) combinations, we have observed and summarized 4 general rules related to the element positions in the difference co-arrays:

When $k \geq 2$:

1) The difference co-array has a consecutive ULA segment without holes in the range of $[0, (k-1)MN + M - 1]$.
2) The first hole locates at $(k-1)MN + M$.
3) In the range $[(k-1)MN+M, kMN-N]$, the position P is a hole **if and only if** P is in the form of $(k-1)MN + aM + bN$ where $a \geq 1$ and $b \geq 0$.

When $k = 1$:

1) The difference co-array has a consecutive ULA segment without holes in the range of $[0, M + N - 1]$.
2) The first hole locates at $M + N$.
3) In the range $[M + N, MN - N]$, the position P is a hole **if and only if** P is in the form of $aM + bN$ where $a \geq 1$ and $b \geq 1$.

Following, we will give detailed descriptions and proofs for the case where $k \geq 2$. The other case where $k = 1$ has similar and simpler proof and is omitted here.

The consecutive ULA segment has already been discussed in the previous section when analysing the number of DoFs. We will start from the hole positions.

Proposition 3. *In the difference co-array of an (M, N) k-times extended co-prime array ($k \geq 2$), the position $P \in [(k-1)MN + M, kMN - N]$ is a hole **if and only if** P is in the form of $(k-1)MN + aM + bN$ where $a \geq 1$ and $b \geq 0$.*

Proof. Since this is a sufficient and necessary condition, we divide it into two parts. We first prove the necessity:

If $P = (k-1)MN+aM+bN \in [(k-1)MN+M, kMN-N]$ where $a \geq 1$ and $b \geq 0$, then P is a hole.

We give the proof by contradiction:

Suppose the position $P = (k-1)MN + aM + bN$ is not a hole, that is,
$\exists p \in [0, N-1]$ and $q \in [0, kM-1]$ such that

$$qN - pM = (k-1)MN + aM + bN \tag{11}$$

From $P \in [(k-1)MN + M, kMN - N]$, $a \geq 1$ and $b \geq 0$, we have

$$(k-1)MN + aM + bN \leq kMN - N. \tag{12}$$

and we can get

$$1 \leq a \leq N - \frac{N}{M} - \frac{bN}{M} \leq N - \frac{N}{M} \tag{13}$$

We can rewrite (11) as

$$(p + a)M = [q - b - (k-1)M]N \tag{14}$$

Since $p + a \neq 0$, we have

$$\frac{M}{N} = \frac{q - b - (k-1)M}{p + a} \tag{15}$$

From (13) and $p \in [0, N-1]$, we can get

$$1 \leq p + a \leq 2N - \frac{N}{M} - 1 < 2N \tag{16}$$

Since the co-primality of M and N, $p+a$ and $q-b-(k-1)M$ need to be exactly N and M respectively. We can get

$$q = kM + b \geq kM, \tag{17}$$

which contradicts to the fact that $q \in [0, kM-1]$ and the necessity has been proved.

Secondly, we will prove the sufficiency by proving its converse-negative proposition:

If $P \in [(k-1)MN + M, kMN - N]$ can not be expressed as $P = (k-1)MN + aM + bN$ where $a \geq 1$ and $b \geq 0$, then P is not a hole.

We represent all positions $P \in [(k-1)MN+M, kMN-N]$ as $(k-1)MN + I$ where I can be any integer in the range $[M, MN - N]$.
From the Euclidean theorem, we can always find two integers a_0 and b_0 such that

$$I = a_0 M + b_0 N \tag{18}$$

$$\text{Let} \quad a' \equiv a_0 \ (mod\ N) \tag{19}$$

$$\text{That is} \quad a_0 = xN + a' \tag{20}$$

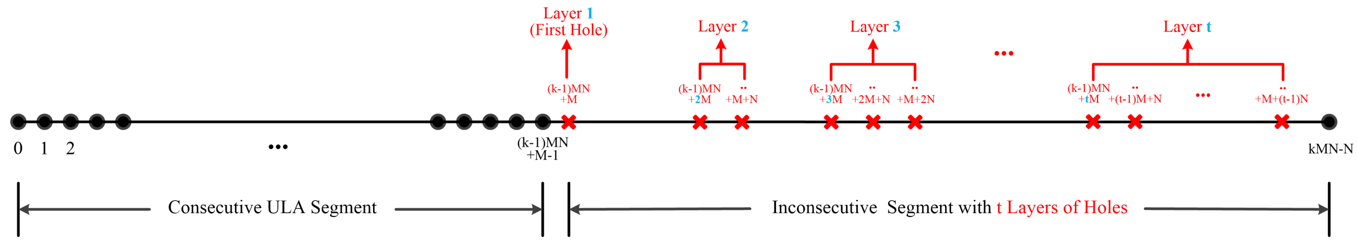


Fig. 6. The difference co-array of an (M, N) k-times extended co-prime array.

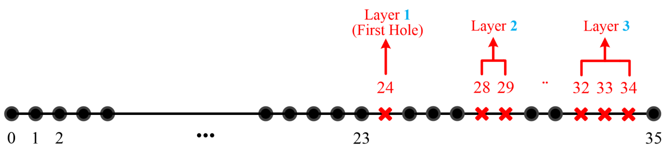


Fig. 7. The difference co-array of a $(4, 5)$ 2-times extended co-prime array.

where x is the quotient and $a' \in [0, N)$ is the remainder.

Then we can rewrite (18) as

$$\begin{aligned} I &= (a_0 - xN)M + b_0N + xNM \\ &= a'M + (b_0 + xM)N \\ &= a'M + b'N \end{aligned} \tag{21}$$

Since $a'M + b'N = I \in [M, MN - N]$ and $a' \in [0, N)$, we can get

$$\begin{aligned} -a'M &< b'N \leq MN - N - a'M \\ -a'\frac{M}{N} &< b' \leq M - 1 - a'\frac{M}{N} \\ -M &< b' < M \end{aligned} \tag{22}$$

We want to figure out if there exist appropriate p and q where $p \in [0, N-1], q \in [0, kM-1]$ such that

$$qN - pM = (k-1)MN + I = (k-1)MN + a'M + b'N \tag{23}$$

holds. Following we will split the problem into three cases according to the value of a' and b'.

(i) $a' = 0,\ -M < b' < M$

Equation (23) becomes

$$qN - pM = (k-1)MN + b'N \tag{24}$$

and can be rewritten as

$$\frac{M}{N} = \frac{q - b'}{(k-1)N + p} \tag{25}$$

Since $p \in [0, N-1]$ and the co-primality of M and N, we have

$$p = 0, \qquad q = (k-1)M + b' \tag{26}$$

Since $b' \in (-M, M)$, we can get $q \in ((k-2)M, kM)$ which is a subset of the range $[0, kM-1]$ when $k \geq 2$. Therefore, (26) satisfies the constraints of p and q. In this case, the position P is not a hole.

(ii) $0 < a' < N,\ -M < b' < 0$

Equation (23) can be rewritten as

$$\frac{M}{N} = \frac{q - b'}{(k-1)N + p + a'} \tag{27}$$

From $p \in [0, N-1]$ and $a' \in (0, N)$, we have

$$(k-1)N < (k-1)N + p + a' < (k+1)N - 1 \tag{28}$$

From (27) (28) and the co-primality of M and N, we have $q - b' = kM$ and $(k-1)N + p + a' = kN$, so

$$\begin{aligned} p &= N - a' \\ q &= kM + b' \end{aligned} \tag{29}$$

Since $a' \in (0, N)$ and $b' \in (-M, 0)$, (29) satisfies the constraints of p and q. In this case, the position P is not a hole.

(iii) $0 < a' < N,\ 0 \leq b' < M$

In this situation, the position P can be expressed as $P = (k-1)MN + a'M + b'N$ where $a' \geq 1$ and $b' \geq 0$. We have proved earlier that it is a hole.

After taking all three possible cases into consideration, we can conclude that in the range $[(k-1)MN + M, kMN - N]$, except for the positions that can be expressed as $P = (k-1)MN + aM + bN$ where $a \geq 1$ and $b \geq 0$, all other positions are not holes.

Therefore, the sufficiency has been proved and comes to the end of proof of **Proposition 3**.

□

B. Special Situations

In the previous subsection, we have presented and proven the general rules followed by positions of holes in the difference co-array of an (M, N) k-times extended co-prime array. However, in some situations, there may exist some minor differences between the structure shown in Fig. 6 and the real structure. Following we will provide additional discussions on these special cases.

1) Intersection of Layers: To better observe the pattern of holes, we divide the holes in Fig. 6 into several layers according to the parameter $a+b$ in the expression $(k-1)MN + aM + bN$. Different from the structure in Fig. 6, in some cases, the positions of holes in two adjacent layers may not be strictly in the increasing order. As shown in Fig. 8, the first hole of the latter layer is ahead of the last hole of the previous layer. We call it the intersection between two layers.

Theorem 1. *In the difference co-array of an (M, N) k-times extended co-prime array, there exists an intersection between layer-i and layer-$(i+1)$ if and only if $i > \frac{N}{N-M}$.*

Proof. From the definition of layer intersection, we have

$$(k-1)MN + (i+1)M < (k-1)MN + M + (i-1)N, \tag{30}$$

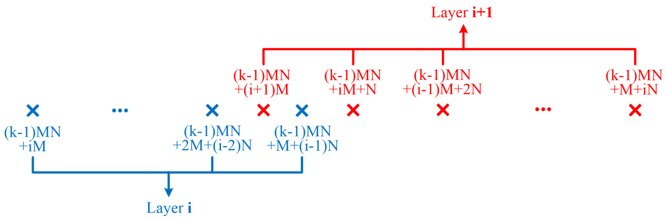


Fig. 8. Case 1: Intersection of two adjacent layers.

from which

$$i > \frac{N}{N-M} \tag{31}$$

□

Corollary 1. *In the difference co-array of an (M, N) k-times extended co-prime array, two different hole layers cannot have the same element.*

Proof. Suppose two different hole layers layer-i and layer-j have a common element H, that is

$$\begin{aligned} H &= (k-1)MN + a_1M + (i-a_1)N \\ H &= (k-1)MN + a_2M + (j-a_2)N \end{aligned} \tag{32}$$

We can get

$$(a_1 - a_2)M = (a_1 - a_2 + j - i)N. \tag{33}$$

Since $i \neq j$, we have $a_1 \neq a_2$. Then (33) becomes

$$\frac{M}{N} = \frac{a_1 - a_2 + j - i}{a_1 - a_2} \tag{34}$$

From (13), we have

$$1 \leq a_1, a_2 \leq N - N/M < N \tag{35}$$

Therefore

$$0 < |a_1 - a_2| < N - 1 \tag{36}$$

From (36), we know that (34) contradicts to the fact of the co-primality of M and N, as M/N cannot be reduced to a ratio of smaller integers. □

2) Incomplete Layers: Another potential situation that needs to be considered is that some of the hole layers may not be intact because of the limitation on the array aperture. In Fig. 9, some of the holes in layer-i exceed the array aperture and should be excluded from the difference co-array.

Theorem 2. *In the difference co-array of an (M, N) k-times extended co-prime array, the ith layer of holes is complete if and only if $i \leq M - 1$.*

Proof. The position of the last hole in layer-i can be expressed as $(k-1)MN + M + (i-1)N$. Then if a layer-i is complete, it means

$$(k-1)MN + M + (i-1)N < kMN - N \tag{37}$$

which is

$$i < M - \frac{M}{N} \leq M - 1 \tag{38}$$

□

Corollary 2. *If the layer-i is incomplete, then the layer-$(i+1)$ is incomplete.*

Proof. This follows directly from the fact that

$$(k-1)MN + M + iN > (k-1)MN + M + (i-1)N > kMN - N \tag{39}$$

□

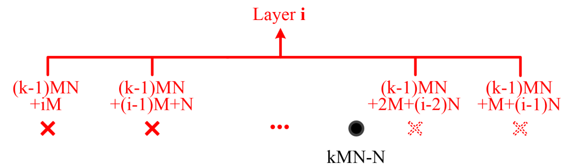


Fig. 9. Case 2: Incomplete layer-i.

Theorem 3. *The difference co-array of an (M, N) k-times extended co-prime array has $\lfloor N - N/M \rfloor$ layers of holes.*

Proof. From (12) and the fact $aM + bM < aM + bN$, we have

$$(k-1)MN + aM + bM < kMN - N \tag{40}$$

$$\text{and we can get} \quad a + b < N - \frac{N}{M} \tag{41}$$

□

C. Summary

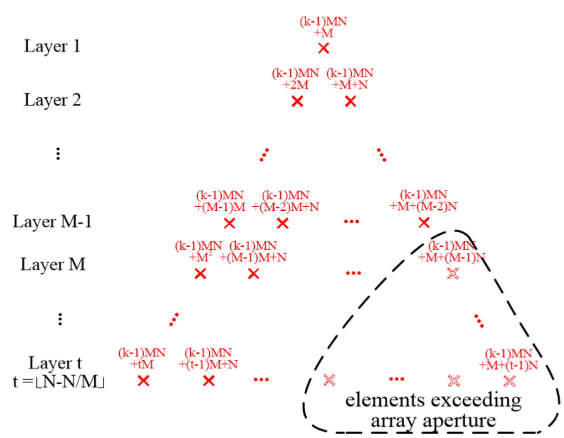


Fig. 10. 2D-Representation of holes in the difference co-arrays of k-times extended co-prime arrays.

To better understand the general rules and the special situations discussed above, we present a 2D-representation of all holes in the extended co-prime array in Fig. 10. All holes can be found at positions $(k-1)MN + aM + bN$ where $a \geq 1$ and $b \geq 0$. These positions can be divided into several different layers according to the value of $a + b$ and there are $\lfloor N - \frac{N}{M} \rfloor$ layers in total. Starting from the layer-M, layers become incomplete. Some elements exceed the array aperture and shouldn't be considered as holes and are excluded from the hole list. Layer-i has i holes if it is complete. In each layer, the positions of holes are in the ascending order and the distance between two adjacent holes is $N - M$. Between two layers, the elements in the latter layer may not be always larger than all elements in the former layer. We call this phenomenon the layer intersection. It happens after the layer-$\lceil \frac{N}{N-M} \rceil$. We define the distance between two layers as the distance between two elements with the same sequence number in the two layers, for example, the distance between the jth element of two layers. Then the distance between two adjacent layers is M. This constant inter-layer distance helps in filling the holes in our complementary co-prime array proposed in Section V.

V. Complementary co-prime array

In the last section, we have derived the expressions for the positions of all holes in the difference co-arrays, for k-times extended co-prime arrays. With the knowledge of the exact hole positions, we propose a new array structure to ensure the difference co-array hole-free. This will significantly increase the array spatial efficiency and DoF.

We call our new array structure *complementary co-prime array.* In order to create a hole-free difference co-array, the intuition behind our design is to add as few additional antennas as possible inside the original co-prime array aperture so that the differences of the newly added elements and the original elements can contain all hole positions. In addition, since all of the newly added elements are located within the original aperture, it can increase the number of DoFs without extending the array aperture.

We first introduce the basic configuration of our proposed array structure. We then present its difference co-array and show how exactly the complementary sub-array can fill all holes in the original virtual array. Furthermore, we provide a strategy for selecting appropriate co-prime parameters (M, N) and k for a given physical array aperture. Finally, we make a comparison between our proposed structure and other sparse arrays.

A. Array Configuration

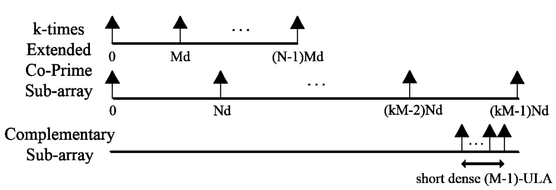


Fig. 11. The complementary co-prime array configuration.

As Fig. 11 shows, our proposed complementary co-prime array consists of three sub-arrays. Two sub-arrays form an (M, N) k-times extended co-prime array ($k \geq 2$, $M < N$). The third one is the complementary sub-array, which is a short ULA located at the tail of the co-prime array with the length $M-1$ and inter-element spacing d. The total number of sensors is $(k+1)M + N - 2$ and the array aperture is $kMN - N$. We will prove later that using this complementary sub-array can fill all holes in the difference co-array generated by the co-prime sub-array.

Fig. 12 shows an example of complementary co-prime array. It consists of a $(3, 5)$ 2-times extended co-prime array and a

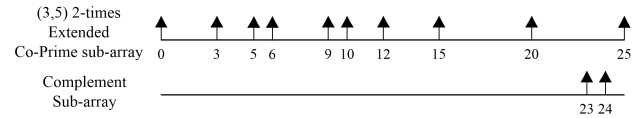


Fig. 12. An example of complementary co-prime array built upon $(3, 5)$ 2-times extended co-prime array.

complementary sub-array with 2 sensors. The total number of sensors is 12. The array aperture is $25d$. Its difference co-array, which is hole-free, is a ULA with unit spacing d from $-25d$ to $25d$. Therefore, the number of DoFs after the spatial smoothing is 25. The spatial efficiency, which is defined as the ratio of the number of DoFs and the array aperture, is 100%.

B. The Hole-free Difference Co-array

In this section, we will prove that the difference co-array of a complementary co-prime array is hole-free.

We use the set $\mathbb{H}$ to hold the positions of all holes in the difference co-array of the original (M, N) k-times extended co-prime array and the sets $\mathbb{K}$ and $\mathbb{C}$ to contain the positions of the elements in the original k-times extended co-prime array and the newly added complementary sub-array respectively. From our previous conclusion of hole positions and the definition of the complementary co-prime array, we have

$$\mathbb{H} = \{(k-1)MN + aM + bN \mid a \in \mathbf{N}^{+}, b \in \mathbf{N}^{0}, \\ (k-1)MN + aM + bN < kMN - N\} \tag{42}$$

$$\mathbb{K} = \{pM \mid p \in [0, N-1]\} \cup \{qN \mid q \in [0, kM-1]\} \tag{43}$$

$$\mathbb{C} = \{kMN-N-1, kMN-N-2, ..., kMN-N-(M-1)\} \tag{44}$$

Our proof on our hole-free can be carried out in 3 steps:

1) $\mathbb{C} \subseteq \mathbb{H}$, that is, all the elements in $\mathbb{C}$ are the hole positions in the original difference co-array.

Proof. We give proof by contradiction. Suppose $\mathbb{C} \nsubseteq \mathbb{H}$, that is $\exists i \in \mathbb{C}$ but $i \notin \mathbb{H}$ which means i is an element of the original difference co-array. Then $\exists p \in [0, N-1]$ and $q \in [0, kM-1]$ such that

$$qN - pM = i \tag{45}$$

Since $i \in \mathbb{C}$ and (44), we have

$$kMN - N - (M-1) \leq i \leq kMN - N - 1 \tag{46}$$

Take (45) into (46), we can get

$$kMN - N - (M-1) \leq qN - pM \leq kMN - N - 1 \tag{47}$$

Since $p \geq 0$, we have

$$qN \geq qN - pM \geq kMN - N - (M-1) \tag{48}$$

and get

$$q \geq kM - 1 - \frac{M-1}{N} \tag{49}$$

From (49) and $q \in [0, kM-1]$, we have

$$q = kM - 1 \tag{50}$$

Substitute (50) into (47) and simplify it we can get

$$\frac{1}{M} \leq p \leq \frac{M-1}{M} \tag{51}$$

which contradicts the fact that p is an integer. Therefore, by proof of contradiction, $\mathbb{C} \subseteq \mathbb{H}$.

□

2) The elements in the set $\mathbb{C}$ are at the bottom of all anti-diagonal directions (top right to bottom left) in the 2D-representation of holes.

Proof. From **Theorem 3** we know there are $M-1$ complete layers in the 2D-representation of holes and therefore there are $M-1$ anti-diagonal directions.
We have proven that all elements in $\mathbb{C}$ are holes. Since they are located at the end of the difference co-array, they are the $M-1$ greatest holes. Thus it is not possible for other holes to be below them in the anti-diagonal directions, and they are the bottom of all $M-1$ anti-diagonal directions in the 2D-representation of holes.

□

3) All elements in the set $\mathbb{H} \setminus \mathbb{C}$ can be found in the difference set $\{n_1 - n_2 \mid n_1 \in \mathbb{C}, n_2 \in \mathbb{K}\}$.

Proof. In the discussion part of Fig. 10, we have shown that the distance between two adjacent layers is constant value M. So any element in the 2D-representation of holes can be expressed as its bottom element in the anti-diagonal direction (i.e., in the set $\mathbb{C}$) minus its distance to the bottom element, where the distance is an integral multiple p' of the constant inter-layer distance M. From **Theorem 4**, we know there are $\lfloor N - \frac{N}{M} \rfloor$ layers, so p' meets the following condition:

$$p' \leq \lfloor N - \frac{N}{M} \rfloor - 1 < N - 1 \tag{52}$$

From (52) and (43), we know the set $\{p'M\} \subset \mathbb{K}$. Therefore, all elements in the set $\mathbb{H} \setminus \mathbb{C}$ can be expressed as $n_1 - n_2$, where $n_1 \in \mathbb{C}$ and $n_2 \in \mathbb{K}$.

□

From the three steps above, we have proven that the newly added complementary sub-array itself can fill a part of holes and the remaining holes can be filled by the difference between the complementary sub-array and the co-prime sub-array. Therefore, using our proposed complementary sub-array can fill all holes in the difference co-array of the original (M, N) k-times extended co-prime array and the new complementary co-prime array structure has a hole-free difference co-array.

C. Comparison of Different Co-prime Structures

Table. I shows a comparison of the specification of different co-prime array structures. We can see that compared with the k-times extended co-prime array, the complementary co-prime array can achieve about MN more DoFs without expanding the array aperture. On the other hand, when utilizing similar number of sensors, the complementary co-prime array can achieve similar number of DoFs with MN smaller array aperture. This makes the complementary co-prime array better fit for the use in the space constrained scenarios.

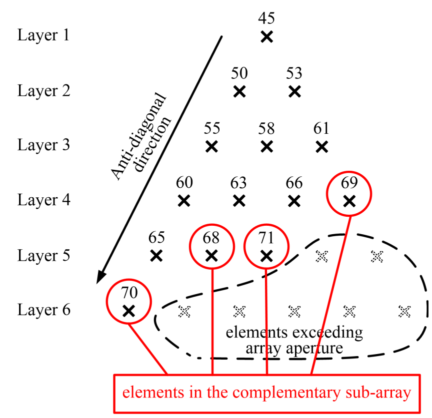


Fig. 13. An example of hole-filling in the difference co-array of a (5,8) extended co-prime array.

TABLE I
COMPARISON OF THE SPECIFICATION OF DIFFERENT CO-PRIME ARRAYS

	Extended Co-prime Array	k-times Extended Co-prime Array	Complementary Co-prime Arrray
No. of Sensors	$2M+N-1$	$kM+N-1$	$(k+1)M+N-2$
Array Aperture	$2MN-N$	$kMN-N$	$kMN-N$
No. of DoFs	$MN+M-1$	$(k-1)MN+M-1$	$kMN-N$

D. Strategy of Selecting (M, N) with fixed k

In all co-prime array techniques, selecting a pair of appropriate co-prime array factors (M, N) is crucial and a challenge. In this section, we provide a scheme to find the optimal (M, N) in our proposed complementary co-prime array for a given number of DoFs (N_0) requested.

Since there exist no holes in the difference co-array of the complementary co-prime, the number of DoFs equals to the physical array aperture. To achieve the required DoFs, we would like the array to have as few sensors as possible. For the complementary co-prime array built upon an (M, N) k-times extended co-prime array, the problem can be formulated as:

$$\begin{aligned} \min_{M,N \in N^*} \quad & |T| = (k+1)M + N - 2 \\ \text{subject to} \quad & kMN - M \geq N_0 \\ & gcd(M, N) = 1 \end{aligned} \tag{53}$$

We first do not take the integrality and the coprimality of (M, N) into consideration, then the original problem becomes:

$$\begin{aligned} \min_{M,N \in R^+} \quad & |T| = (k+1)M + N - 2 \\ \text{subject to} \quad & kMN - M = N_0 \end{aligned} \tag{54}$$

When k is fixed, its optimum solution can be achieved when

$$\begin{aligned} M &= \sqrt{\frac{N_0}{k(k+1)} + \frac{1}{k}} \\ N &= \sqrt{\frac{(k+1)N_0}{k}} \end{aligned} \tag{55}$$

Equations (55) shows that the ideal N to M ratio is close to $k+1$ in the complementary co-prime array.

We can then test the coprimality of the integers around the optimum M and N in (55) and find the optimal solution to the original problem (53).

E. Comparison with Other Sparse Arrays

Following, we will compare our proposed complementary co-prime array with the nested array, the super nested array and the Wichmann Ruler [33] on two aspects, the coupling effect and the robustness analysis.

1) Mutual Coupling Effect: In practical antenna arrays, electromagnetic interaction between the antenna elements always exists. The radiation pattern in each antenna element of an array depends on its own excitation and also the contributions from adjacent antenna elements. The effect of mutual coupling is inversely proportional to the spacing between the different antenna elements in an array. It has been studied in [3], [15] that the first weight function $w(1)$ which represents the number of pairs of physical antennas with unit element-spacing $d = \lambda/2$ dominates the effects of mutual coupling.

Proposition 4. *The first weight function $w(1)$ of an (M, N) k-times extended co-prime array equals to 2.*

Proof. The proof can be divided into three parts:

First, from the fact that the difference co-array of a co-prime array always contains the element $1d$, we know there exists at least one pair of physical antennas with unit element-spacing d, that is, $w(1) \geq 1$.

Second, we can prove that for any pair $(x_1 d, x_2 d) = (p_1 M d, q_1 N d)$ where $|x_2 - x_1| = 1$ in the physical array, there always exists another pair $(y_1 d, y_2 d) = ((N - p_1)Md, (M - q_1)Nd)$ in the physical array such that $|y_2 - y_1| = 1$, that is, $w(1)$ is even. The proof is as follows:

Suppose $x_2 - x_1 = 1$, that is, $q_1 N - p_1 M = 1$. Then $y_1 - y_2 = (N - p_1)M - (M - q_1)N = q_1 N - p_1 M = 1$. Similarly, when $x_1 - x_2 = 1$, we can get $y_2 - y_1 = 1$.

Finally, we need to prove that there are no more than 2 pairs, that is, $w(1) \leq 2$. The proof is provided by contradiction:

Suppose there are 2 more pairs,

$((p'_1 M d, q'_1 N d)$ and $(p'_2 M d, q'_2 N d)$.

We must have

$$p_1 M - q_1 N = p'_1 M - q'_1 N \tag{56}$$

or

$$p_1 M - q_1 N = p'_2 M - q'_2 N \tag{57}$$

Suppose (56) holds, then we can get

$$(p_1 - p'_1)M = (q_1 - q'_1)N \tag{58}$$

Since $p_1 \neq p'_1$ and $q_1 \neq q'_1$, (58) becomes

$$\frac{M}{N} = \frac{q_1 - q'_1}{p_1 - p'_1} \tag{59}$$

and it contradicts to the co-primality of M and N. □

Proposition 5. *The first weight function $w(1)$ of an (M, N) complementary co-prime array equals to $M + 1$.*

Proof. From **Proposition 4** we know the original co-prime sub-arrays has 2 pairs of physical antennas with unit element-spacing d. Since $N - M \geq 1$, the newly added complementary sub-array only leads to another $M - 1$ pairs. Therefore, the total number of pairs is $M + 1$. □

Proposition 6. *The first weight function $w(1)$ of an (N_1, N_2) nested array equals to N_1.*

Proof. This proposition directly follows the definition of the nested array. □

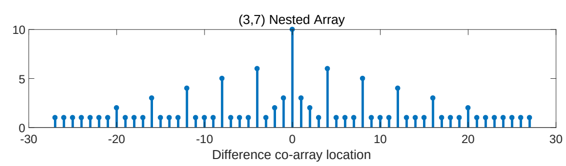
In our previous analysis, the optimal N to M ratio is close to $k + 1$. As a result, the length of the dense ULA in the complementary sub-array $(M - 1)$ is usually far smaller than that in the optimal nested array. Table II shows some numerical results. Our complementary co-prime array has $w(1) = 4$, which is much smaller than that of the optimal nested array $(w(1) = 17)$. On the other hand, when having the same first weight function $w(1) = 4$, our proposed complementary co-prime array has 266 DoFs, which is much larger than that of the nested array (154).

2) Robustness Analysis: In this part, we take the robustness of different array geometries into consideration. In practice, there may exist perturbations, that is, small horizontal shifts, in the antenna positions. Moreover, antenna failures may happen in some extreme conditions. Therefore, holding appropriate redundancy can help keep the system robust and stable.

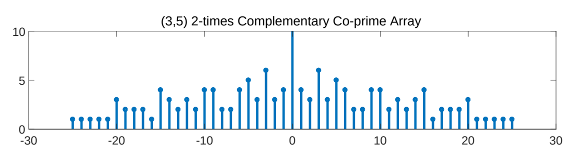
Definition 2. ***(Redundancy Rate)**. The redundancy rate of a sparse array is the ratio of the number of repetitive elements in the difference co-array to the total number of elements (including repetitions) in the difference co-array.*

$$r_{redun} = \frac{|\mathbb{L}|^2 - |\mathbb{D}|}{|\mathbb{L}|^2} \tag{60}$$

The sets $\mathbb{L}$ and $\mathbb{D}$ denote the locations of physical array and difference co-array receptively. Operator $|\cdot|$ denotes the cardinality of the set (number of elements in the set).



(a) (3,7) Nested array



(b) (3,5) 2-times Complementary co-prime array

Fig. 14. The co-array location of the nested array and the complementary co-prime array.

Although the redundancy rate can reflect the robustness to some degree, it has limitation. When the redundancy is restricted to be around some particular points, even if there

TABLE II
NUMERICAL RESULTS

	Array Specification	Number of Sensors	Array Aperture	Number of DoFs	Spatial Efficiency	$w(1)$	Redundancy Rate	Reliability
Extended Co-prime Array	(9,19)	36	323	179	55.42%	2	61.19%	42.35%
k-times Extended Co-prime Array	(3,19) 6-times	36	323	287	88.85%	2	52.85%	17.18%
Complementary Co-prime Array	(3,19) 5-times	35	266	266	100%	4	56.49%	33.58%
Nested Array	(4,31)	35	154	154	100%	4	74.78%	21.04%
	(17,18)	35	323	323	100%	17	47.18%	10.05%
Super Nested Array	(17,18) 2nd-order	35	323	323	100%	1	47.18%	19.01%
	(17,18) 5th-order	35	323	51	15.79%	1	47.51%	40.90%
Wichmann Ruler	Wichmann (5,12)	35	419	419	100%	10	31.51%	15.61%

is a high redundancy rate, we cannot achieve high robustness. Fig.14(a) shows the co-array location of a $(3,7)$ nested array, where most of the redundancy is concentrated at locations 0, ± 4, ± 8, ± 12, ± 16 and ± 20. Many locations in the co-array have no redundancy, and any perturbations or failures in these locations will compromise the detection performance.

Therefore, besides the redundancy rate, we define another more appropriate parameter to evaluate the robustness.

Definition 3. ***(Reliability)****. The reliability of a sparse array is the ratio of the number of elements that is not unique in the difference co-array to the number of distinct elements in the difference co-array.*

$$\textit{Reliability} = \frac{|\{i|w(i) \neq 1, i \in \mathbb{D}\}|}{|\mathbb{D}|} \tag{61}$$

The reliability is defined as the ratio of locations with redundancy in the virtual array. Compared to the redundancy rate, it can better reflect the robustness of the system.

Fig.14(b) shows the co-array location of a $(3,5)$ 2-times complementary co-prime array. Compared to the nested array shown in Fig.14(a), the redundancy of our proposed array geometry has more even distribution. This helps to increase the reliability to keep the system more robust and stable when unexpected perturbations or antenna failures happen.

Table II shows the trade-offs among different array structures. The traditional extended co-prime array has higher reliability but lower spatial efficiency and smaller number of DoFs. The k-times extended co-prime array sacrifices the reliability for a larger number of DoFs. As an enhancement of the k-times extended co-prime array, the complementary co-prime array can achieve comparable number of DoFs and higher spatial efficiency but twice the reliability with a smaller array aperture. The optimal nested array and the array based on Wichmann Ruler [33] can achieve a larger number of DoFs but suffer a lot from the mutual coupling and are sensitive to perturbations due to the limited reliability. Keeping $w(1)$ the same, the non-optimal nested array has much smaller number of DoFs than our proposed complementary co-prime array. Although super nested arrays reduce the mutual coupling, low order super nested arrays still have poor reliability while high order super nested arrays may have holes in the difference co-array which will significantly reduce the number of DoFs. In practice, customers should select appropriate structure according to different application scenarios.

VI. PERFORMANCE EVALUATION

We evaluate the performance of our proposed complementary co-prime array through simulations over matlab. We apply the MUSIC algorithm [34] to detect DOAs of a group of uniformly distributed sources. We first compare the performance of our complementary co-prime array built upon k-times extended co-prime array (CCPA-k) with two other co-prime geometries: the traditional extended co-prime array (ECPA) and the k-times extended co-prime array (kCPA). Then we take the mutual coupling and the robustness into consideration and compare our structures with three other sparse geometries: the nested array (NA), the super nested array (SNA) and the sparse array based on Wichmann Ruler (WR).

A. Comparison with Co-prime Arrays

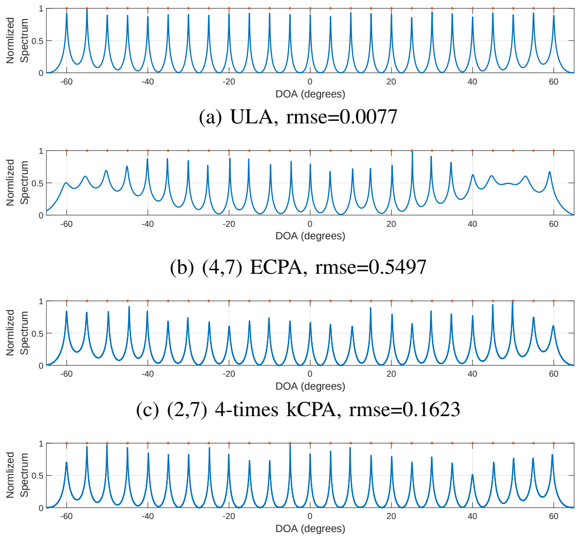
TABLE III
PARAMETERS OF DIFFERENT ARRAY STRUCTURES

	ULA	ECPA	kCPA	CCPA-k
Array Aperture	50	49	49	50
Array Specification	51-ULA	(4,7)	(2,7) 4-times	(3,10) 2-times
Number of Sensors	51	14	14	17
Number of DoFs	50	31	43	50

We first compare the performance of different arrays with the constraint of the physical array aperture ($A \leq 50d$).

Table III shows the specific parameters of different array structures. The co-prime parameters $M = 3$, $N = 10$ and $k = 2$ in CCPA-k are selected based on our proposed strategy. We can see that with the same aperture constraint, our proposed CCPA-k has the largest number of DoFs. On the other hand, when compared with ULA, it requires much fewer sensors.

Fig. 15 shows the MUSIC spectra of different array structures described above. 25 target sources are uniformly distributed within the range $-60°$ to $60°$. The covariance matrix is estimated by using 500 snapshots. SNR is set to 0dB. We can see that our proposed kCPA and CCPA-k have much clearer spectra compared with the traditional ECPA and the root mean squared error (RMSE) can decrease by over 70% and 75% respectively. When compared with ULA of 51 sensors, our proposed structures only need 14 and 17 sensors respectively with little decrease of the detection quality. They can still successfully identify all targets with RMSE $< 0.2°$. This demonstrates the feasibility and effectiveness of our proposed



(a) ULA, rmse=0.0077

(b) (4,7) ECPA, rmse=0.5497

(c) (2,7) 4-times kCPA, rmse=0.1623

(d) (3,10) 2-times CCPA-k, rmse=0.1304

Fig. 15. MUSIC spectra of different array structures, where SNR=0dB, 500 snapshots are applied, and 25 target sources are uniformly distributed within the range $-60°$ to $60°$.

TABLE IV
PARAMETERS OF DIFFERENT ARRAY STRUCTURES

	ULA	ECPA	kCPA	CCPA-k
Number of Sensors	30	30	30	30
Array Specification	30-ULA	(8,15)	(4,15) 4-times	(3,14) 5-times
Array Aperture	29	225	225	196
Number of DoFs	29	127	183	196

k-times extended co-prime structure and complementary co-prime array structure.

Fig. 16 shows the root mean squared errors of the DOA estimation of three co-prime structures with the increase of the number of target sources. SNR is set to 0dB and 2000 snapshots are applied. The number of target sources D varies from 3 to 38. For a $0.2°$ RMSE threshold, the maximum number of targets detected by our proposed CCPA-k approaches 38 instead of 24 in the traditional ECPA and 32 in kCPA. Our complementary co-prime array can detect about 50% and 20% more targets compared with the traditional extended co-prime array and the k-times extended co-prime array when utilizing the same array aperture. Therefore, our proposed complementary co-prime arrays have higher capacity.

We then study the impact of the number of snapshots on the root mean squared error of the DOA estimation of different co-prime structures. As Fig. 17 shows, our proposed CCPA-k has the lowest RMSE, which is consistent with the theoretical analysis in Table III that the complementary co-prime array has the largest number of DoFs under the same array aperture. Compared with ECPA and kCPA, the CCPA-k structure can save over 90% and 50% snapshots respectively to achieve the same RMSE threshold ($0.2°$) under the same SNR (0dB). The significant reduction of the number of snapshots will allow for faster scanning and detection as well as cost reduction.

Next, we consider the scenario where all the array structures have the same number of physical sensors (30). Table IV shows the specific parameters of different array structures. We can see that with the same number of sensors, our proposed CCPA-k has much larger number of DoFs compared with ECPA and ULA. On the other hand, when compared with kCPA, it has comparable DoFs with a much smaller array aperture.

Fig. 18 shows the root mean squared errors of the DOA estimation of three co-prime structures under different SNR. We can see that our proposed structures kCPA and CCPA-k have similar results. In the theoretical analysis of Table. IV, they have similar number of DoFs which is much larger than the traditional extended co-prime array. As a result, our proposed structures can reduce RMSE over 50% under the same SNR. In other words, if we would like to achieve the same RMSE threshold, our methods can bear much lower SNR. Furthermore, since CCPA-k has even smaller array aperture compared to kCPA, it can perform much better estimation in a space constrained and low SNR scenario such as underwater sonar detection.

B. Comparison with Other Sparse Arrays

We have shown above that our proposed complementary co-prime geometry can improve the performance of co-prime arrays. Following we will compare the performance of our proposed geometry with nested arrays (NA), super nested arrays (SNA) and the array based on Wichmann Ruler (WR) when taking mutual coupling and robustness into consideration.

We take the mutual coupling model in [35] and set the impedances of the element and load to 50. Table V shows the specific parameters of different array structures. All array structures have 14 physical sensors. Fig. 19 shows the RMSE of DOA estimation of different structures with/without mutual coupling. Under the impact of mutual coupling, even though the nested array has a larger number of DoFs, our CCPA-k performs better. Super nested array performs the best in this situation due to its smallest $w(1)$. Although WR has the same $w(1)$ as our CCPA-k, it has a larger number of DoFs and thus lower RMSE. However, compared to other sparse array structures, the array aperture of CCPA-k is much smaller.

Finally, we evaluate the robustness of different array structures from the perspective of redundancy rate and reliability in Fig. 21 and Fig. 22 respectively. We can see that the co-prime arrays (ECPA, kCPA and CCPA-k) have higher redundancy rate then the nested arrays (NA and SNA) and the array based on Wichmann Ruler (WR). Although NA has similar redundancy rate as SNA which is greater than that of WR, the reliability of NA is the worst. As discussed in SectionV, redundant virtual sensors of NA tend to center around a few locations. The reliability of high-order super nested array is similar to that of CCPA-k, however, a high-order super nested array does not have a determinate number of DoFs. As Fig. 23 shows, existing holes in the super nested array may significantly reduce its number of DoFs from $O(N^2)$ to $O(N)$. The uncertainty and the instability of the DoF will prevent the use of high-order super nested arrays in practice. Furthermore, Fig. 24 shows that our CCPA-k has the smallest array aperture among the referenced geometries.

Fig. 20 shows the RMSE of DOA estimation of different structures where all structures have a $0.25d$ horizontal shift on

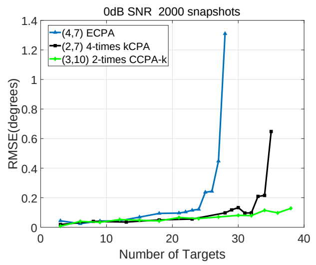


Fig. 16. RMSE versus the number of targets D, with 2000 snapshots and 0 dB SNR. D targets are uniformly distributed from $-60°$ to $60°$.

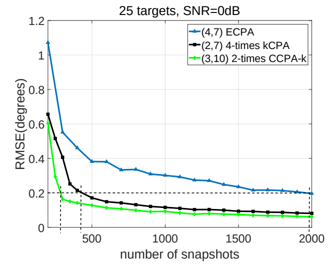


Fig. 17. RMSE under the impact of the number of snapshots, with SNR=0dB and 25 target sources uniformly distributed within the range $-60°$ to $60°$.

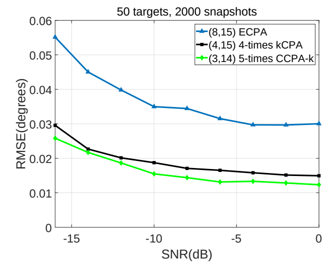


Fig. 18. RMSE under the impact of SNR, where 2000 snapshots are applied and 50 target sources are uniformly distributed from $-60°$ to $60°$.

TABLE V
PARAMETERS OF DIFFERENT ARRAY STRUCTURES

	ULA	ECPA	kCPA	CCPA-k	NA	SNA	WR
Number of Sensors	14	14	14	14	14	14	14
Array Specification	14-ULA	(4,7)	(2,7) 4-times	(3,7) 2-times	(7,7)	(7,7) 3rd-order	W(2,3)
Number of DoFs	13	31	43	35	55	55	68
Array Aperture	13	49	49	35	55	55	68
$w(1)$	13	2	2	4	7	1	4
RMSE (no coupling/perturbations)	0.0804	0.0717	0.0399	0.0468	0.0298	0.0263	0.0291
RMSE (with coupling)	0.6517	0.1059	0.0752	0.1386	0.2399	0.0634	0.0847
RMSE (with perturbations)	0.0960	0.1032	0.1546	0.1078	N/A	0.0616	N/A

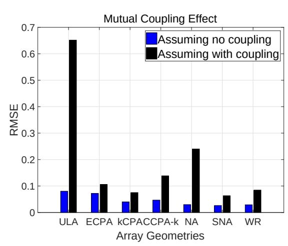


Fig. 19. RMSE under the impact of mutual coupling, where 2000 snapshots and 0dB SNR are applied. 10 target sources are uniformly distributed within the range $-60°$ to $60°$.

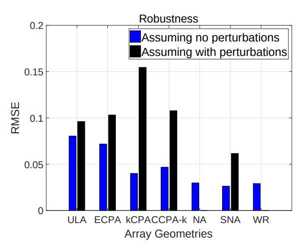


Fig. 20. RMSE under the impact of antenna perturbations, where 2000 snapshots and 0dB SNR are applied. 10 target sources are uniformly distributed within the range $-60°$ to $60°$.

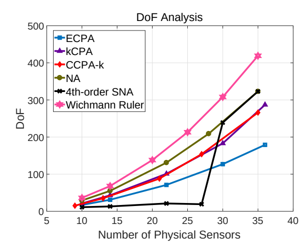


Fig. 23. DoF of different array geometries.

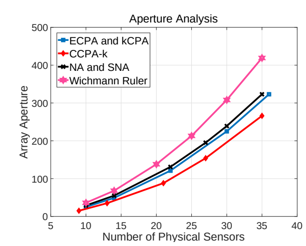


Fig. 24. Array aperture of different array geometries.

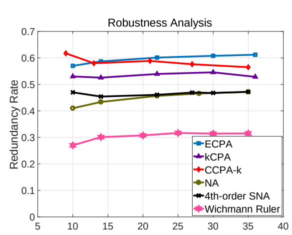


Fig. 21. Redundancy rate of different array geometries.

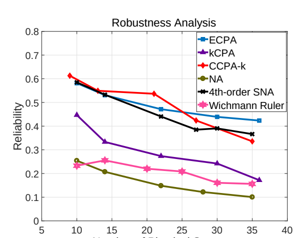


Fig. 22. Reliability of different array geometries.

the 4th physical antenna. As expected, most arrays have the higher detection errors, while NA and WR completely fail to detect the targets. kCPA has a larger RMSE compared with CCPA-k, ECPA and SNA. This result is in consistent with our analysis of reliability in Fig. 22.

VII. CONCLUSION

We first introduce the k-times extended co-prime geometry and illustrate the rules of the holes in its difference co-array, based on which we derive concise expressions for all hole positions and reveal the characteristics of holes. These analyses provide the fundamentals to facilitate advanced co-prime array designs. We then further propose a complementary co-prime array structure, which includes an extra sub-array that can fill all of the holes. Our performance results demonstrate that, the estimation quality of our complementary co-prime array outperforms that of other co-prime structures. Compared with other types of sparse arrays, complementary co-prime array has either equivalent or better performance when mutual coupling effect and antenna perturbations are taken into consideration. Furthermore, when using the same number of sensors, our proposed array has remarkable estimation quality and smaller aperture. This significant reduction of the array aperture will make it better fit for use in a space constrained scenario such as air-borne and underwater applications.

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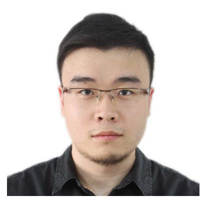

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