PERMUTATIONS

- **Basic definitions**

  – **Permutation:**

  Consider an $n \times n$ multistage interconnection network with n inputs and n outputs where $n = 2^m$. A *permutation* is a full one-to-one mapping between the network inputs and outputs. For an $n \times n$ network, suppose input x_i is mapped to output y_i, where $x_i = i$ and $y_i \in \{0, 1, \ldots, n-1\}$ for $i = 0, 1, \ldots, n-1$. This permutation is denoted as

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_{n-1} \\ y_0 & y_1 & \ldots & y_{n-1} \end{pmatrix}.$$

  – **Partial permutation:**

  A one-to-one mapping between n' network inputs and n' network outputs ($n' < n$) is called a *partial permutation*.

– **Semi-permutation:**

A partial permutation

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{\frac{n}{2}-1} \\ y_0 & y_1 & \dots & y_{\frac{n}{2}-1} \end{pmatrix}$$

of an n-element set $\{0, 1, \dots, n-1\}$, where n is an even integer, $x_i, y_i \in \{0, 1, \dots, n-1\}$ and $x_0 < x_1 < \cdots < x_{\frac{n}{2}-1}$, is referred to as a *semi-permutation* **of the n-element set, if**

$$\left\{\left\lfloor\frac{x_0}{2}\right\rfloor, \left\lfloor\frac{x_1}{2}\right\rfloor, \dots, \left\lfloor\frac{x_{\frac{n}{2}-1}}{2}\right\rfloor\right\} = \left\{0, 1, \dots, \frac{n}{2}-1\right\},$$

and

$$\left\{\left\lfloor\frac{y_0}{2}\right\rfloor, \left\lfloor\frac{y_1}{2}\right\rfloor, \dots, \left\lfloor\frac{y_{\frac{n}{2}-1}}{2}\right\rfloor\right\} = \left\{0, 1, \dots, \frac{n}{2}-1\right\}.$$

– **Example:**

For $n = 8$, partial permutation

$$\begin{pmatrix} 0 & 3 & 4 & 6 \\ 1 & 5 & 3 & 7 \end{pmatrix}$$

is a semi-permutation, since we have

$$\left\{ \left\lfloor \frac{0}{2} \right\rfloor, \left\lfloor \frac{3}{2} \right\rfloor, \left\lfloor \frac{4}{2} \right\rfloor, \left\lfloor \frac{6}{2} \right\rfloor \right\} = \{0, 1, 2, 3\},$$

and

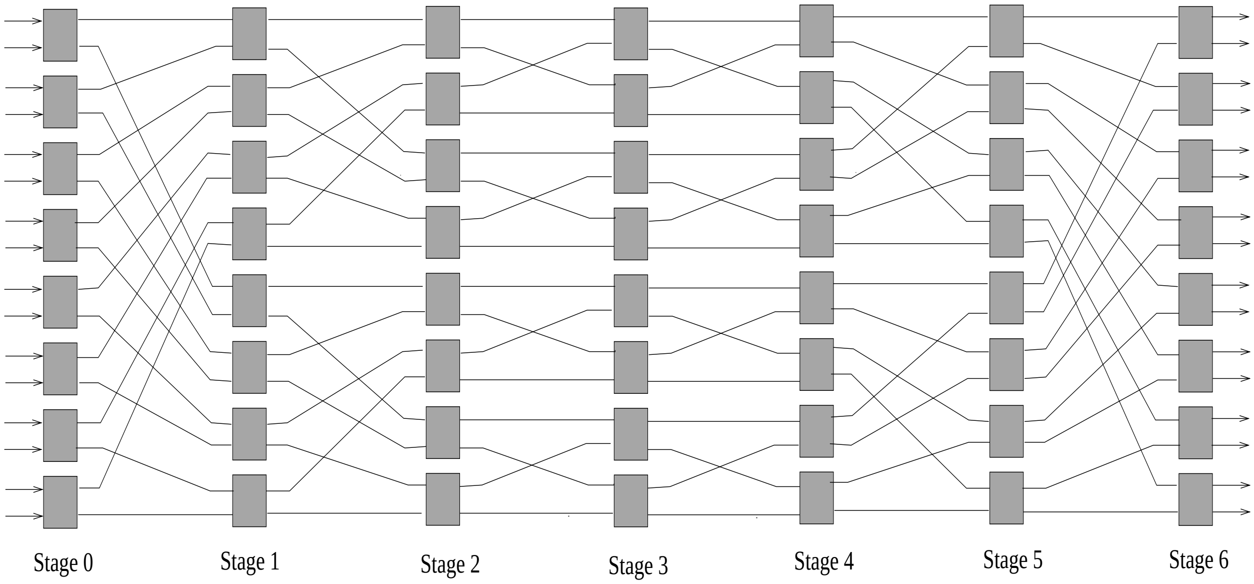
$$\left\{ \left\lfloor \frac{1}{2} \right\rfloor, \left\lfloor \frac{5}{2} \right\rfloor, \left\lfloor \frac{3}{2} \right\rfloor, \left\lfloor \frac{7}{2} \right\rfloor \right\} = \{0, 2, 1, 3\} = \{0, 1, 2, 3\}.$$

– **Any permutation can be decomposed into two semi-permutations.**

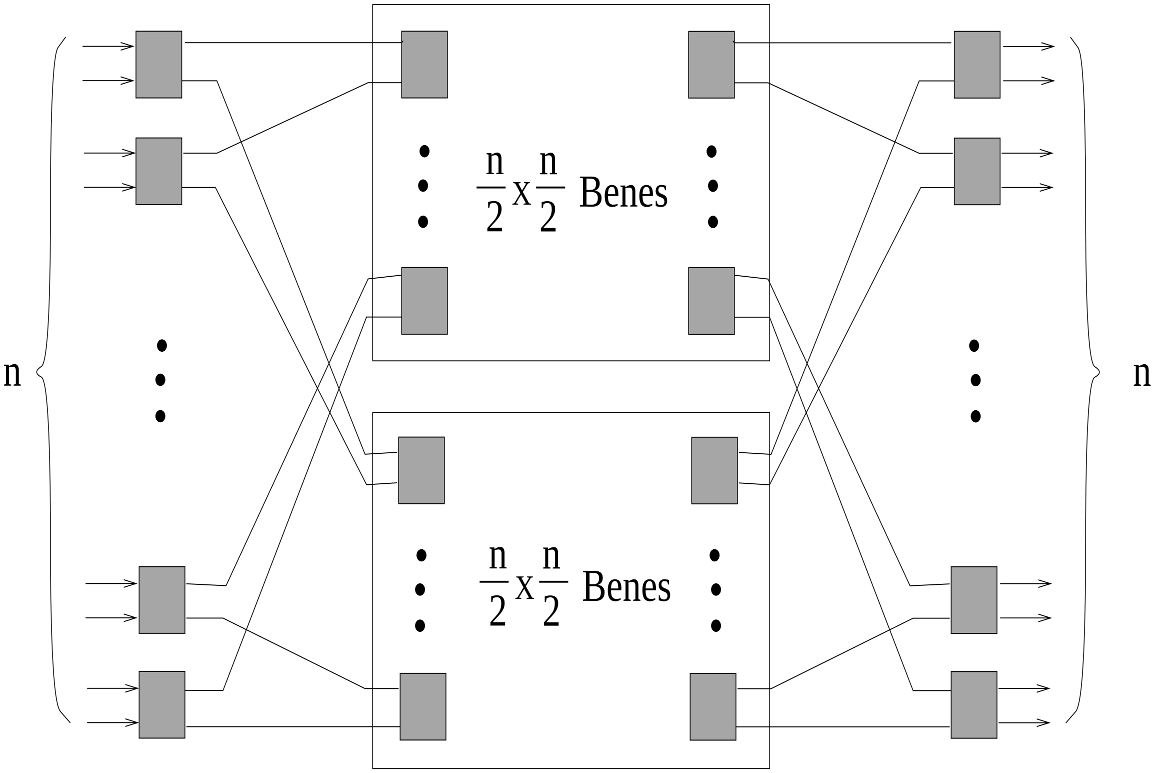
Proved by using Hall's theorem.

- **Rearrangeable network: the Benes network**
  - **Construction:**

    Consists of two back-to-back baseline networks:



Or let $m = n = 2$ in a Clos network and perform a recursive decomposition.


n
$\frac{n}{2} \times \frac{n}{2}$ Benes
$\frac{n}{2} \times \frac{n}{2}$ Benes
n

- **Connecting capability:**

  A rearrangeable network for arbitrary permutations.

  Proof. From the Clos network.
- **Edge-disjoint paths in a Benes network (realizing a permutation)**
- **Routing algorithm (called looping algorithm)**
- **Node-disjoint paths in a Benes network (realizing a semi-permutation)**
- **There exist node-disjoint paths for any semi-permutation in a Benes network.**

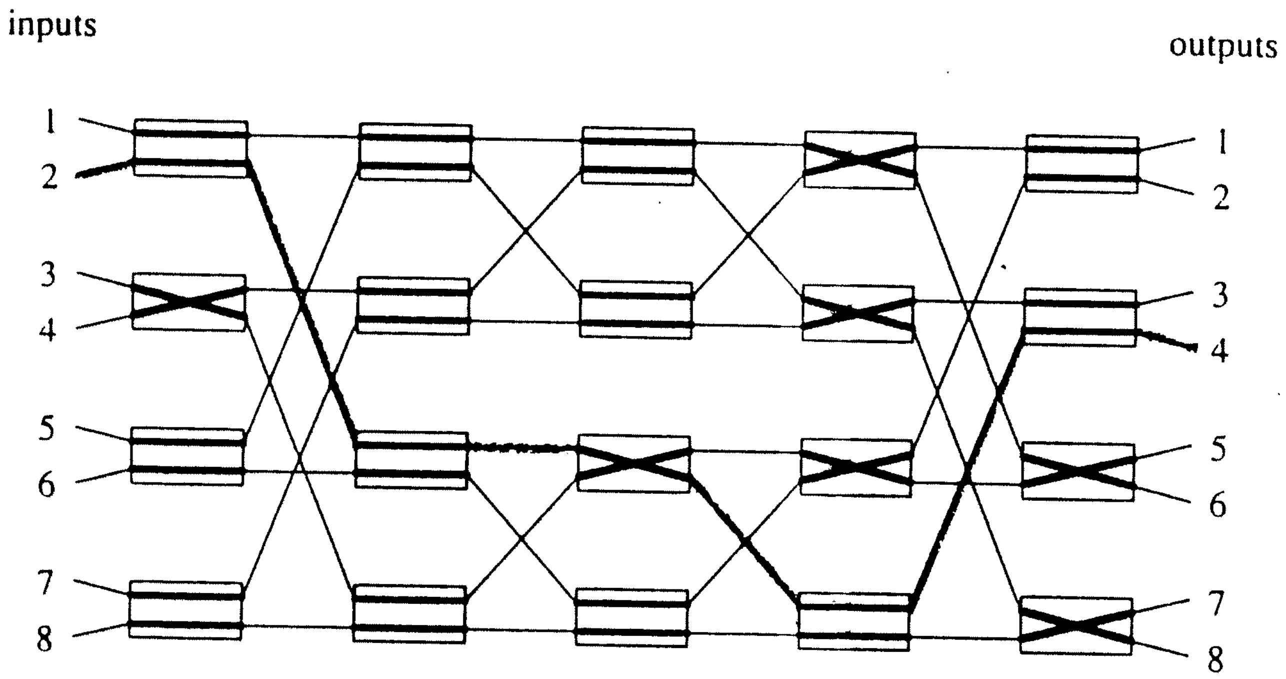


Figure 3-28 *Edge-disjoint paths in a two-dimensional Beneš network connecting input i to output $\pi(i)$ for $1 \leq i \leq 8$, where π is the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 4 & 5 & 8 & 1 & 2 & 3 & 7 \end{pmatrix}$ (i.e., $\pi(1) = 6$, $\pi(2) = 4$, and so on). For clarity, each node of the Beneš network is drawn as a 2×2 switch connecting the incoming pair of edges on the left to the outgoing edges on the right in one of two ways (crossing or straight through). In addition, the path from input 2 to output 4 is highlighted in order to illustrate what a particular path through the network looks like.*

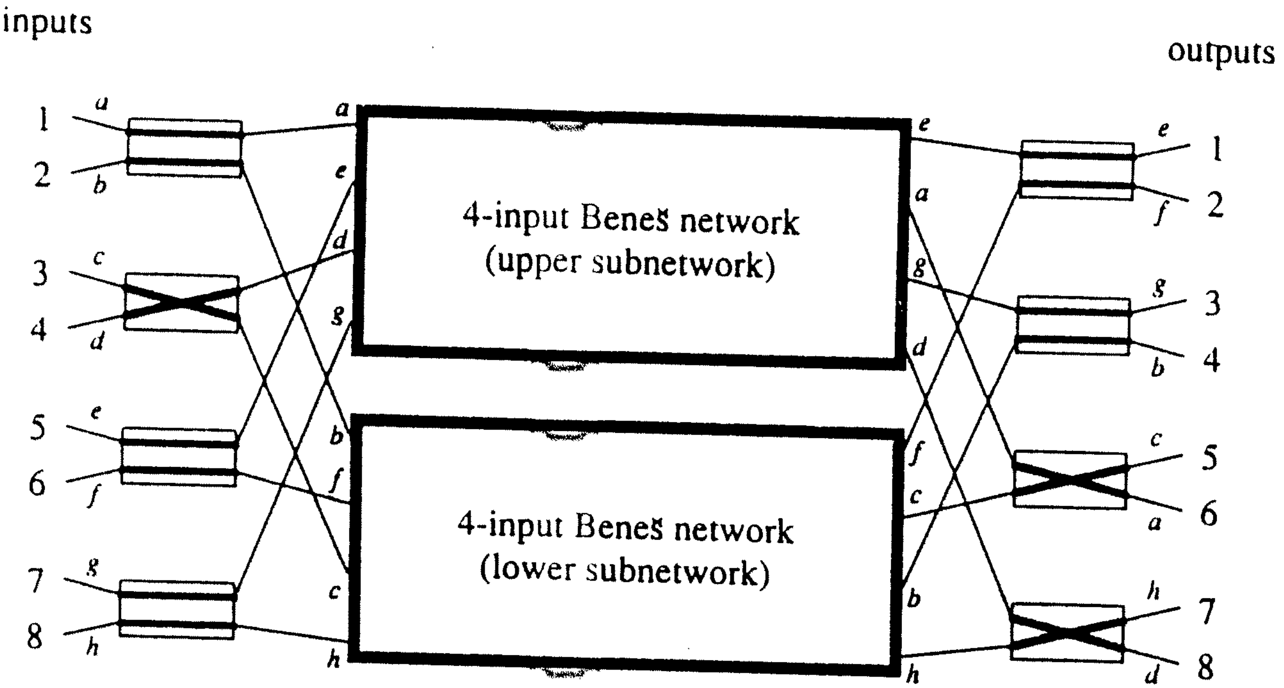


Figure 3-29 *The first step in finding the edge-disjoint paths for the permutation* $\binom{1\,2\,3\,4\,5\,6\,7\,8}{6\,4\,5\,8\,1\,2\,3\,7}$. *The paths* $1 \rightarrow 6$, $4 \rightarrow 8$, $5 \rightarrow 1$, *and* $7 \rightarrow 3$ *(labelled a, d, e, and g, above) are routed through the upper subnetwork, while the paths* $2 \rightarrow 4$, $3 \rightarrow 5$, $6 \rightarrow 2$, *and* $8 \rightarrow 7$ *(labelled b, c, f, and h, above) are routed through the lower subnetwork. At the recursive stage, the upper subnetwork routes the permutation* $\binom{1\,2\,3\,4}{2\,1\,4\,3}$ *and the lower subnetwork routes the permutation* $\binom{1\,2\,3\,4}{3\,1\,2\,4}$.

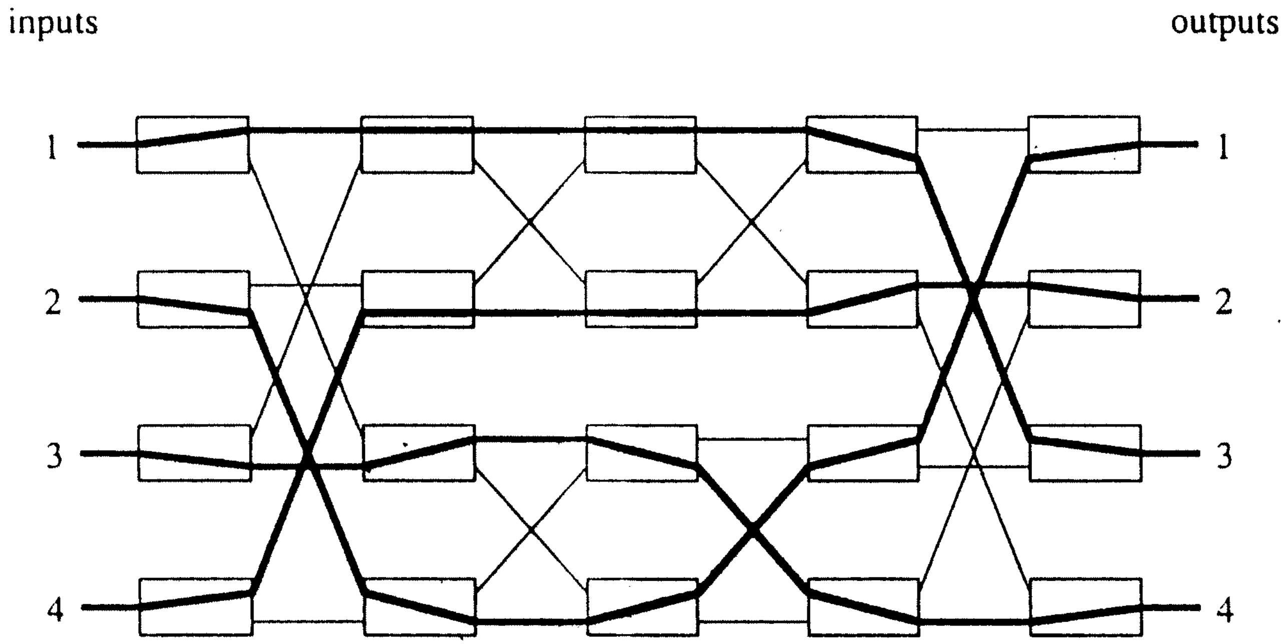


Figure 3-30 *Node-disjoint paths in a two-dimensional Beneš network connecting input i to output $\pi(i)$ for $1 \leq i \leq 4$ where π is the permutation $\binom{1\,2\,3\,4}{3\,1\,4\,2}$. Edges contained in the paths are shaded. Because of the node-disjoint property of the paths, precisely one path passes through each switch in the network.*

- **Wide-sense nonblocking network: the Clos network**
  - *Wide-sense nonblocking* **capability: a compromise between strictly nonblocking capability and rearrangeability.**

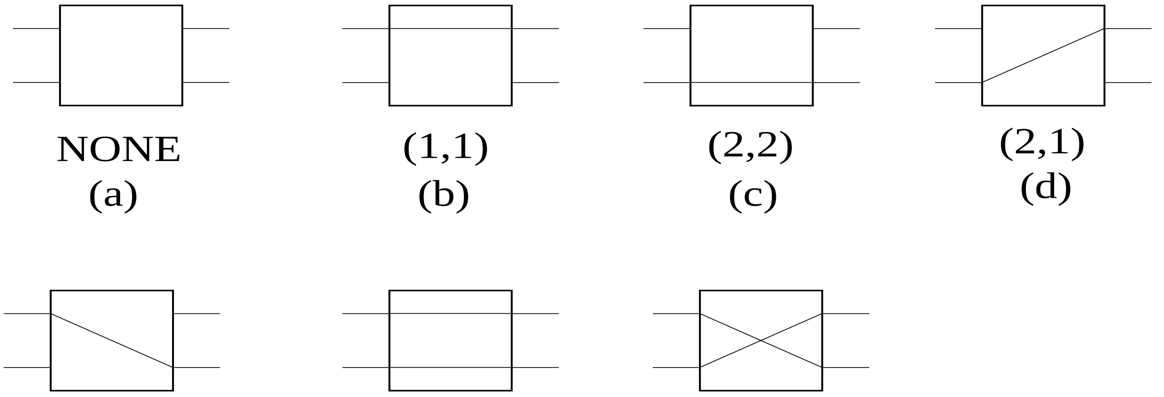
    For a wide-sense nonblocking network, an "intelligent" routing control strategy must be employed to govern the process of path routing. Through carefully selecting the paths used to satisfy the current connection request, the nonblocking capability for future connection requests can be maintained, and at the same time lower network cost can be achieved.

– **A commonly used routing control strategy for wide-sense nonblocking Clos networks:** *packing* **strategy.**

Under packing strategy, a connection is realized on a path found by trying the most used part of the network first and the least used part last.

For a Clos network, when choosing a middle stage switch for satisfying a connection request, an empty middle stage switch is not used unless there is not any partially filled middle stage switch that can satisfy this connection request.

- **For $r = 2$, when $m \geq \left\lfloor \frac{3}{2}n \right\rfloor$, the Clos network is wide-sense nonblocking for permutations under packing strategy.**
  - ∗ **Let x be a state of middle switches which is reachable from zero state (empty network).**
  - ∗ $S(x) =$ **number of middle switches in use in state x.**
  - ∗ **7 possible states of a middle switch**



NONE (a)

(1,1) (b)

(2,2) (c)

(2,1) (d)

(1,2) (e)

(1,1)(2,2) (f)

(1,2)(2,1) (g)

  - ∗ **A state can be represented by 7 integers:** $a(x), b(x), c(x), d(x), e(x), f(x)$ **and** $g(x)$**:**

$a(x)$ =**number of middle switches of type** a **when network is in state** x**.**

$\vdots$

$g(x)$ =**number of middle switches of type** g **when network is in state** x**.**

∗ **For any state** x**,**

$$a(x) + b(x) + \cdots + g(x) = 2n - 1,$$

$$b(x) + c(x) + \cdots + g(x) = S(x)$$

∗ **Let** p **denote the rule: Do not use an empty middle switch unless necessary. Let** x **be a state reachable under rule** p**. Then for** $n \geq 2$**,**

$$S(x) \leq \left\lfloor \frac{3}{2} n \right\rfloor, \qquad (1)$$

$$b(x) + c(x) + f(x) \leq n,$$

$$d(x) + e(x) + g(x) \leq n. \qquad (2)$$

∗ **Proof. By induction on** k **(steps from zero state).**

Assume at step k, network is in state y and (1) and (2) hold.

At step $k+1$, the network is in state x.

For a new connection, there are two possible types

Type 1:

$$a(y) \to a(y) - 1$$

and one of

$$(1,1) : b(y) \to b(y) + 1 \textbf{ with } c(y) = 0,$$

$$(2,2) : c(y) \to c(y) + 1 \textbf{ with } b(y) = 0,$$

$$(2,1) : d(y) \to d(y) + 1 \textbf{ with } e(y) = 0,$$

$$(1,2) : e(y) \to e(y) + 1 \textbf{ with } d(y) = 0,$$

Type 2:

$a(y)$ **remains fixed and one of**

$$(1,1) : f(y) \to f(y){+}1, c(y) \to c(y){-}1 \textbf{ with } c(y) > 0$$

$$(2,2) : f(y) \to f(y){+}1, b(y) \to b(y){-}1 \textbf{ with } b(y) > 0$$

$(2,1) : g(y) \rightarrow g(y){+}1, e(y) \rightarrow e(y){-}1$ **with** $e(y) > 0$

$(1,2) : g(y) \rightarrow g(y){+}1, d(y) \rightarrow d(y){-}1$ **with** $d(y) > 0$

Any state satisfies:

$$b(y) + e(y) + f(y) + g(y) \leq n$$
$$c(y) + d(y) + f(y) + g(y) \leq n$$
$$b(y) + d(y) + f(y) + g(y) \leq n$$
$$c(y) + e(y) + f(y) + g(y) \leq n$$

If the connection is type 2, the result holds in stage x**.**
If the connection is type 1, WLOG, let the new connection be $(1,1)$**. Then**

$$c(y) = 0$$

Also, since a $(1,1)$ connection is possible in state y, we must have

$$b(y) + d(y) + f(y) + g(y) \leq n - 1$$

$$b(y) + e(y) + f(y) + g(y) \leq n - 1$$

From the induction hypothesis

$$d(y) + e(y) + g(y) \leq n$$

Hence,

$$2(b(y) + d(y) + e(y) + f(y) + g(y)) \leq 3n - 2$$

Notice that $c(y) = 0$,

$$S(y) \leq \frac{3n}{2} - 1$$

Since $S(x) = S(y) + 1$,

$$S(x) \leq \left\lfloor \frac{3}{2} n \right\rfloor$$

To show (2) holds in state x, consider

$b(y) + e(y) + f(y) + g(y) \leq n - 1$ **and** $c(y) = 0$

It follows that

$$b(y) + c(y) + f(y) \leq n - 1.$$

Since x is obtained from y by putting up a $(1,1)$ connection of type 1, we have

$$b(x) = b(y) + 1, \;\; e(x) = e(y),$$

$$c(x) = c(y) = 0, \;\; f(x) = f(y),$$

$$d(x) = d(y), \;\; g(x) = g(y).$$

Thus, (2) holds for x.

*** For a general r, the condition becomes**

$$m \geq \left\lfloor \left(2 - \frac{1}{F_{2r-1}}\right) n \right\rfloor$$

where F_{2r-1} is the Fibonacci number.

The Fibonacci numbers are defined by the recurrence

$F_0 = 0, \; F_1 = 1,$ **and** $F_k = F_{k-1} + F_{k-2}, \quad$ **for** $k \geq 2$**.**

Proof: By a systematic approach based on linear programming.

r	2	3	4	5	6	7	8	9	10
F_{2r-1}	2	5	13	34	89	233	610	1597	4181
m	$\frac{3}{2}n$	$\frac{9}{5}n$	$\frac{25}{13}n$	$\frac{67}{34}n$	$\frac{177}{89}n$	$\frac{465}{233}n$	$\frac{1219}{610}n$	$\frac{3193}{1597}n$	$\frac{8361}{4181}n$

‖

$$m=\left(2-\frac{1}{F_{2r-1}}\right)n \quad \to \quad 2n-1$$