COLLECTIVE COMMUNICATION

- **Collective communication involves global data movement and global control among a group of processes in a parallel/distributed computing system.**
- **Frequently used collective communication operations include broadcast (one-to-all), multicast (one-to-many), all-to-all, gather, scatter, reduction and scan.**
- **Many scientific applications exhibit the need of such communication patterns. E.g. matrix multiplication, LU-factorization, matrix transposition and fast Fourier transform (FFT).**
- **Efficient support for collective communication can significantly reduce the communication latency and simplify the programming of parallel computers.**

- **Basic types of collective communication**
  - **One-to-all communication: one process is identified as the sender and all processes in the group are receivers.**
    - **Broadcast: the same message is sent to all receivers.**
    - **Scatter (personalized broadcast): the sender sends different messages to different receivers.**
  - **All-to-one communication: all processes in a process group are senders and one process is identified as the sole receiver.**
    - **Reduce: different message from different senders are combined together to form a single message for the receiver.**
      Combining operator: addition, multiplication, maximum, minimum, and logical OR, AND, exclusive OR operators.

    - **Gather: different messages from different senders are concatenated together for the receiver.**
  - **All-to-all communication: every process in a process group sends a message to all other processes in the group.**
    - **All-to-all broadcast: every process sends the same message to all other processes.**
    - **All-to-all personalized exchange (all scatter): every process sends a distinct message to every other process.**
- **An application of basic collective operations: Barrier synchronization.**
  - **Many numerical problems can be solved using iterative algorithms that successively compute better approximations to an answer, terminating when either the final answer has been computed or the final answer**

has converged. These algorithms normally require all the iterative processes to be synchronized at the end of each iteration.

do not_converged $\rightarrow$

code to implement process i

barrier (wait for all n processes to complete)

od

– **Barrier represents a barrier synchronization point which waits for all n processes to complete. This type of synchronization is called** *barrier synchronization* **because the delay point at the end of each iteration represents a barrier that all processes have to arrive at before any of them is allowed to pass.**

SYSTEM SUPPORT FOR COLLECTIVE COMMUNICATION

- **Implementing the basic operation: multicast (one-to-many) communication**
  - **Unicast approach**
  - **Software multicast**
  - **Hardware multicast**
- **Performance of multicast communication:**
  - **Mainly measured in terms of its latency in delivering a message to all destinations.**
  - **Other considerations: total traffic generated, deadlock-free routing**

- **A wide-sense nonblocking multicast network**
  - **Terminology**
    - **Interconnection network: A (space division) switching system which provides connection paths between input ports and output ports.**
    - **One-to-one connection: A connection of an input port to one output port. A maximal set of one-to-one connections is called a permutation assignment.**
    - **Multicast connection: A connection where an input port can be simultaneously connected to more than one output port (but an output port can be connected to at most one input port at a time). A maximal set of multicast connections is called a multicast assignment.**

- **∗ Connection request: Idle input port requests connection path(s) to idle output port(s).**
- **∗ Multicast network: A network which can realize all possible multicast assignments.**
- **∗ Nonblocking network: Satisfies all connection requests and rearrangement is never required.**

– Motivation:

- **∗ Multicast (one-to-many) communication is a fundamental communication pattern in both telecommunication networks and scalable parallel computers.**
- **∗ Examples:**
  - **· In telecommunication:**

    Teleconference calls and video-on-demand services.

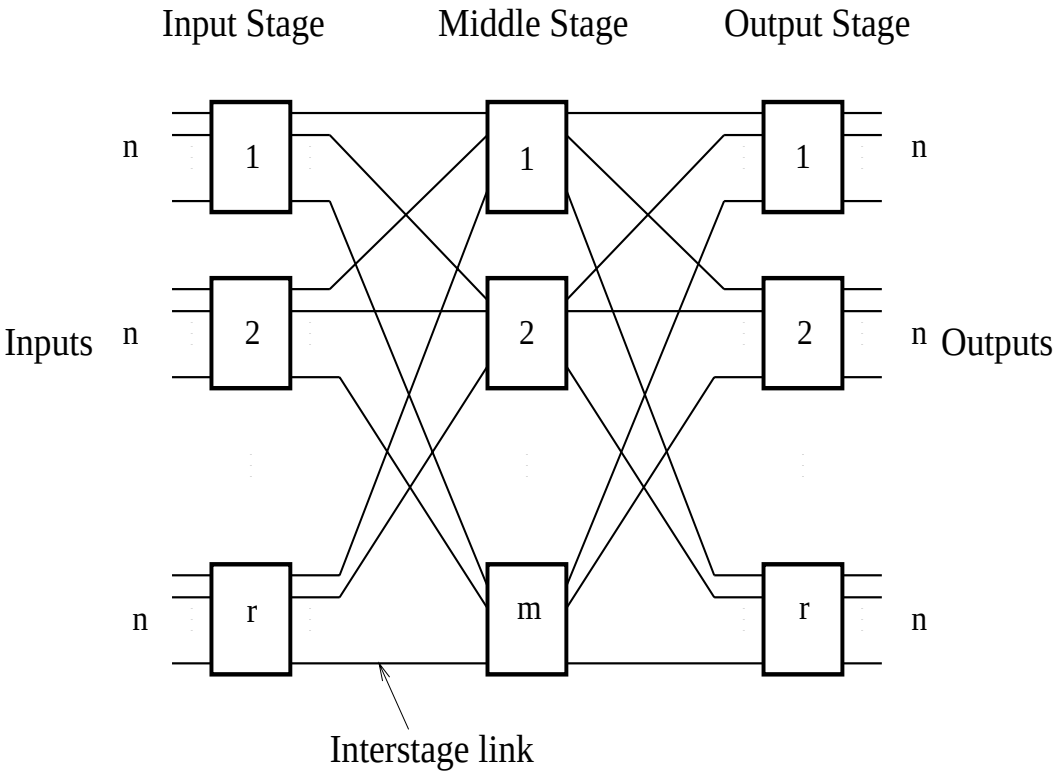
- **In parallel/distributed processing: Barrier synchronization, write update/ invalidate in cache coherence protocols, and updates in a distributed database.**

* **A permutation network in general cannot support arbitrary multicast.**
  For an $N \times N$ network,
  # Permutation patterns: $N!$
  # Multicast patterns: N^N
* **Efficient implementation for multicast is critical to system performance.**
* **Support multicast at interconnection network level.**
* **Many applications require nonblocking capability.**
  - **Reduce overhead associated with rearrangements**
  - **Avoid disturbances of existing connec-**

tions in the network

- **Especially important in real-time applications.**

– **Definition of the Clos network or $v(m, n, r)$ network**



Every switch module is assumed to have multicast capability.

- **Previous results on nonblocking conditions of the Clos network**
  - **A $v(m, n, r)$ network is nonblocking for permutation assignments [Clos, *BSTJ*, '53], if the number of middle stage switches**

$$m \geq 2n - 1$$

  - **A $v(m, n, r)$ network is nonblocking for multicast assignments [Masson, *Networks*, '72; Hwang and Jajszczyk, *IEEE Trans. Comm.*, '86], if the number of middle stage switches**

$$m \geq c(nr)$$

– **More terminology:**

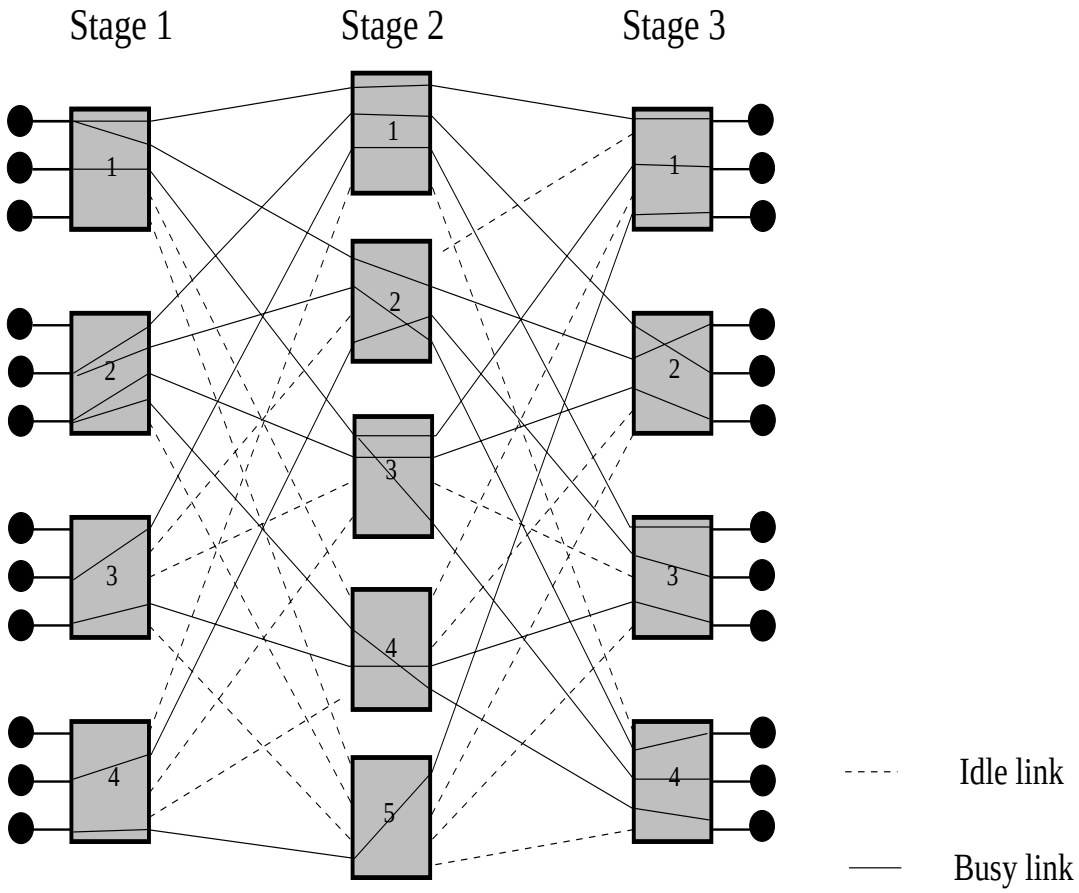
Fanout: **A multicast connection from an input port to output ports on r' output switches is said to have fanout r'.**

Input connection request: **An input connection request I_j is the set of output switches to which input port j is to be connected.**

Destination sets: **Destination set M_j is the set of output switches to which the middle switch j is providing connection paths from the input ports.**

Available middle switches: **The available middle switches of input port j is the set of middle switches with currently unused links to the input switch associated with input j.**

– **Example:**



A $(5, 3, 4)$ **network realizing a multicast assignment characterized as** $I_1 = \{1, 2\}$, $I_2 = \{1, 4\}$, $I_5 = \{2, 4\}$, $I_6 = \{2, 4\}$, $I_8 = \{3\}$, $I_9 = \{3\}$, $I_{11} = \{3\}$, $I_{12} = \{1\}$, **all other** $I_j = \phi$. **The destination sets for the middle switches for the shown network state are** $M_1 = \{1, 2, 3\}$, $M_2 = \{2, 3, 4\}$, $M_3 = \{1, 2, 4\}$, $M_4 = \{3, 4\}$, $M_4 = \{1\}$.

- **Design of multicast networks**
  - **Specify an "intelligent" control strategy for satisfying each multicast connection request: choose no more than a certain number of middle switches, say, x, whose destination set intersections are empty from available middle switches.**
  - **Determine how many available middle switches can guarantee that these x middle switches can always be chosen. We are interested in finding as few as possible available middle switches with this property.**
  - **Find the optimal value of x to minimize the number of middle switches.**
  - **Develop an efficient control algorithm to actually find these x middle switches.**
  - **Hardware implementation of the control algorithm to further speed up the routing**

process.

– **Main results**

Theorem 1 ***We can satisfy a new connection request*** I_i, $i \in \{1, 2, \ldots, nr\}$, ***in a*** $v(m, n, r)$ ***network using some*** x $(x \geq 1)$ ***middle switches, say,*** $j_1, \ldots, j_x$, ***from among the available middle switches if and only if*** I_i ***and the current destination sets of these*** x ***middle switches satisfy***

$$I_i \cap (\bigcap_{k=1}^{x} M_{j_k}) = \phi. \qquad (1)$$

Theorem 2 ***For all*** n', $1 \leq n' \leq n$, ***and for all*** x, $1 \leq x \leq \min\{n', r\}$, ***let*** m' ***be the maximum number of middle switches whose destination sets have the following properties:***

1. ***there are at most*** n' ***1's,*** n' ***2's, . . . ,*** n' r ***'s distributed among the destination sets;***
2. ***the intersection of any*** x ***of the destination sets is nonempty.***

Then

$$m' \leq n' r^{\frac{1}{x}}$$

Proof sketch of Theorem 2:

WLOG, suppose these m' middle switches are $1, 2, \ldots, m'$ with destination sets $M_1, M_2, \ldots, M_{m'}$. Clearly,

$$\sum_{i=1}^{m'} |M_i| \leq n' r$$

Let

$$c_1 = \min_i \{|M_i|\}.$$

Then, we obtain that

$$m' \leq \frac{n' r}{c_1} \tag{2}$$

WLOG, suppose that the destination set of middle switch 1 has cardinality c_1, and $M_1 = \{1, 2, \ldots, c_1\}$.

Intersect each of the destination sets $M_1, M_2, \ldots, M_{m'}$ with M_1 and obtain m' sets $M_1^1, M_2^1,$

..., $M^1_{m'}$ which consist of only elements in $\{1, 2, \ldots, c_1\}$, and distributed among the M^1_i 's are at most n' 1's, n' 2's, ..., n' c_1's.

Let

$$c_2 = \min_i\{|M^1_i|\}$$

WLOG, suppose that M^1_2 has cardinality c_2, and $M^1_2 = \{1, 2, \ldots, c_2\}$. We then get that

$$m' \leq \frac{n'c_1}{c_2} \tag{3}$$

Then intersect each of $M^1_1, M^1_2, \ldots, M^1_{m'}$ with M^1_2 and obtain m' sets $M^2_1, M^2_2, \ldots, M^2_{m'}$, which consist of only elements in $\{1, 2, \ldots, c_2\}$.

After repeating the above process $x-1$ times, we have

$$m' \leq \mathbf{min}\{\frac{n'r}{c_1}, \frac{n'c_1}{c_2}, \ldots, \frac{n'c_{x-2}}{c_{x-1}}\} \tag{4}$$

We now have a set of m' intersected destination sets M^{x-1}_1, M^{x-1}_2, ..., $M^{x-1}_{m'}$. Moreover, each M^{x-1}_k consists of only elements in

$\{1, 2, \ldots, c_{x-1}\}$, and there are at most n' 1's, n' 2's, ..., and n' c_{x-1}'s distributed among these m' sets. Thus, we have

$$m' \leq n' c_{x-1} \tag{5}$$

Therefore, from (3) and (4), we get that

$$m' \leq \textbf{min}\{\frac{n'r}{c_1}, \frac{n'c_1}{c_2}, \ldots, \frac{n'c_{x-2}}{c_{x-1}}, n'c_{x-1}\}$$

It can be shown that

$$\max_{c_1, c_2, \ldots, c_{x-1}} \textbf{min}\{\frac{n'r}{c_1}, \frac{n'c_1}{c_2}, \ldots, \frac{n'c_{x-2}}{c_{x-1}}, n'c_{x-1}\} = n'r^{\frac{1}{x}}$$

Therefore,

$$m' \leq n'r^{\frac{1}{x}}$$

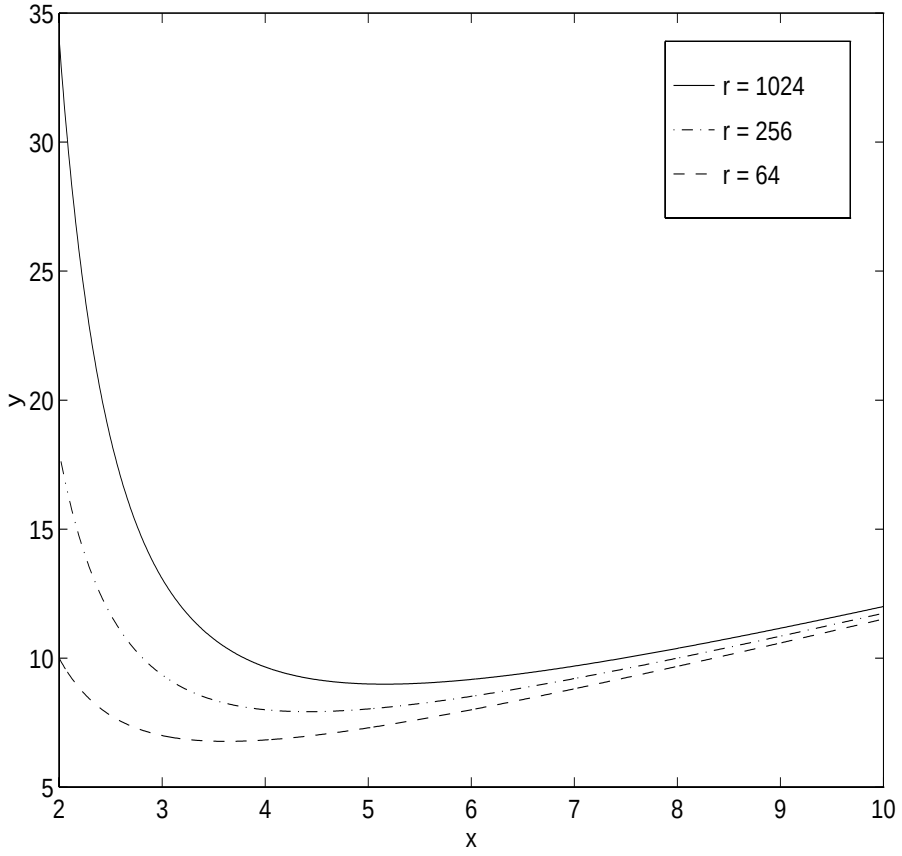
Sufficient condition on realizing multicast assignment with nonblocking capability:

Theorem 3 ***A*** $v(m,n,r)$ ***network is nonblocking for multicast assignments if***

$$m > \min_{1 \le x \le \min\{n-1,r\}} \{(n-1)(x + r^{\frac{1}{x}})\}$$

Curve of function

$$y = x + r^{\frac{1}{x}}$$



Relationship between r and the coefficient of m: $\min_{1 \le x \le \min\{n-1, r\}} \left(x + r^{\frac{1}{x}}\right)$

r	x	$x + r^{\frac{1}{x}}$
1	**1**	**2**
2	**1**	**3**
$4 = 2^2$	**2**	**4**
$9 = 3^2$	**2**	**5**
$27 = 3^3$	**3**	**6**
$81 = 3^4$	**4**	**7**
$256 = 4^4$	**4**	**8**
$1024 = 4^5$	**5**	**9**
$4096 = 4^6$	**6**	**10**
$16384 = 4^7$	**7**	**11**
$78125 = 5^7$	**7**	**12**
$390625 = 5^8$	**8**	**13**
$1953125 = 5^9$	**9**	**14**
$10077696 = 6^9$	**9**	**15**

A bound on m as a function of n and r

Theorem 4 ***A*** $v(m,n,r)$ ***network is nonblocking for multicast assignments if***

$$m \geq 3(n-1)\frac{\log r}{\log\log r}$$

Generalization to restricted multicast assignments:

Corollary 1 ***A*** $v(m,n,r)$ ***network is nonblocking for restricted multicast assignments, in which each input port can be connected to at most*** d $(1 \leq d < r)$ ***output switches, if***

$$m > \min_{1 \leq x \leq \min\{n-1,d\}} \{(n-1)(x + d^{\frac{1}{x}})\}$$

In particular, we have

$$m > (n-1)\left(\frac{2\log d}{\log\log d} + (\log d)^{\frac{1}{2}}\right)$$

Permutation is a special case:

Corollary 2 ***Setting*** $d = 1$ ***in Corollary 1 yields*** $m \geq 2n - 1$***, which is the bound on*** m ***associated with the classical Clos non-blocking permutation networks.***

Generalization to $(2k+1)$**-stage networks for** $k > 1$**:**

Recursively applying the design criteria on each middle stage switch.

Crosspoint complexity:

Theorem 5 ***For each fixed integer*** $k \geq 1$***, the minimum number of crosspoints of our*** $(2k+1)$ ***stage*** $(N \times N)$ ***multicast network is***

$$O\left(N^{1+\frac{1}{k+1}}(\log N/\log\log N)^{\frac{k+2}{2}-\frac{1}{k+1}}\right)$$

Crosspoints comparisons:

* **Constructive multicast networks:**
  - **Masson's three-stage network:** $O(N^{\frac{5}{3}})$
  - **Hwang and Jajszczyk's three-stage network:** $O(N^{\frac{5}{3}})$
  - **Feldman, Friedman and Pippenger's two-stage network:** $O(N^{\frac{5}{3}})$;
    three-stage network: $O(N^{\frac{11}{7}})$
  - **Three-stage version of this network:** $O(N^{\frac{3}{2}}(\frac{\log N}{\log \log N}))$
* **Nonconstructive multicast networks:**
  Feldman, Friedman and Pippenger k-stage network:
  $O\left(N^{1+\frac{1}{k}}(\log N)^{1-\frac{1}{k}}\right)$.
  The control algorithm is co-NP-complete.

- **A linear routing control algorithm**
  - **Given a $v(m, n, r)$ network satisfying the nonblocking condition on m in Theorem 3.**
  - **Some x, $1 \leq x \leq min\{n-1, r\}$**
  - **A connection request I_i with $|I_i| = r' \leq r$**
  - **$m' = (n-1)r'^{\frac{1}{x}}+1$ available middle switches with destination sets $M_1, M_2, \ldots, M_{m'}$.**

Algorithm:

Step 1: $mid\_switch \leftarrow \phi$**;**

for $j = 1$ **to** m' **do**

$S_j \leftarrow M_j \cap I_i$**;**

Step 2: **repeat**

find S_k $(1 \leq k \leq m')$ **such that**

$|S_k| =$ **min**$\{|S_1|, |S_2|, \ldots, |S_{m'}|\}$;

$min\_set \leftarrow S_k$**;**

$mid\_switch \leftarrow mid\_switch \cup \{k\}$**;**

if $min\_set \neq \phi$ **then**

for $j = 1$ **to** m' **do**

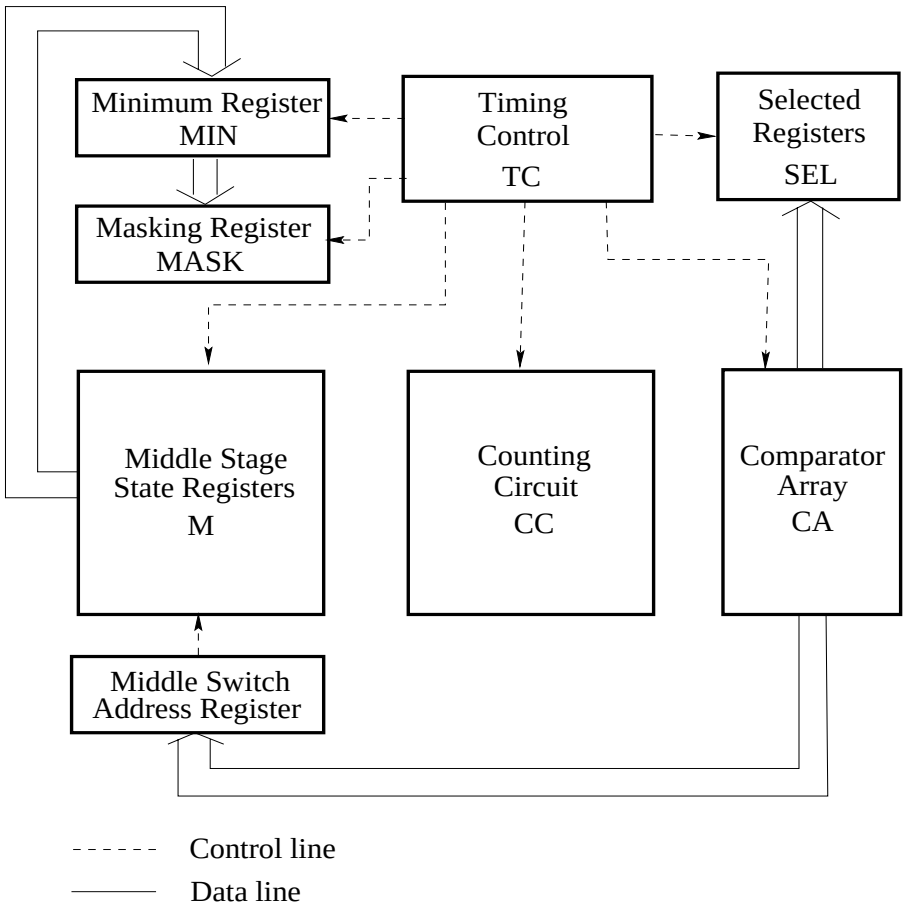
$S_j \leftarrow S_j \cap min\_set$**;**

until $min\_set = \phi$**;**

Step 3: **connect** I_i **through the middle switches in** mid_switch **and update the destination sets of these middle switches.**

– **More efficient implementation of control algorithm in hardware**

∗ **Overview structure**



- **Sequential implementation**
  - **Sequentially evaluate the cardinality of each destination set of middle switches and find the minimum cardinality set.**
- **Parallel implementation**
  - **Parallel evaluation of the cardinalities of all destination sets of middle switches by a sequential circuit or a combinational circuit.**

* Summary of the various designs of the controller

Design	**Gates**	**Clocks**	**Gate Delays**
Seq./ Counter	$O(\log r)$	$O\left(\frac{mr \log r}{\log \log r}\right)$	–
Seq./ Adder	$O(r)$	$O\left(\frac{m \log r}{\log \log r}\right)$	–
Parall./ Counter	$O(m \log r)$	$O\left(\frac{r \log r}{\log \log r}\right)$	–
Parall./ Adder	$O(mr)$	$O\left(\frac{\log r}{\log \log r}\right)$	$O\left(\frac{(\log r)^2}{\log \log r}\right)$

- **Necessary nonblocking condition**
  - **Can the sufficient condition we obtained be further reduced?**
  - **What is the optimal design for this type of multicast network?**
  - **Derive necessary conditions for supporting arbitrary multicast assignment under different control strategies by constructing worst case network states which force us to use a certain number of middle stage switches.**
  - **Employ these necessary conditions to guide the design process of multicast networks.**

– **Routing control strategies**

Strategy 1 ***For each input connection request,*** I_i, $i \in \{1, 2, \ldots, nr\}$, ***in the network, always choose the middle switch with the minimum cardinality of destination sets with regard to the unsatisfied portion of*** I_i ***from available middle switches, until*** I_i ***is satisfied, that is, until all middle switches chosen satisfy condition (1).***

Note: This is the strategy used by the routing control algorithm we discussed earlier.

Strategy 2 ***For each input connection request*** I_i, $i \in \{1, 2, \ldots, nr\}$, ***choose the minimum number of middle switches that satisfy condition (1) for the current network state from the available middle switches.***

Strategy 3 ***For each input connection request*** I_i, $i \in \{1, 2, \ldots, nr\}$, ***use an empty avail-***

able middle switch (i.e., middle switch with no connections) only when there is no any subset of non-empty available middle switches can satisfy condition (1).

– **A fundamental lemma for constructing worst case network states:**

Lemma 1 ***For sufficiently large*** n, r ***and*** $\frac{m}{n}(m > n)$, ***there exist*** $m+n$ ***subsets of set*** $\{1, 2, \ldots, r\}$, $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n, \mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_m$, ***which satisfy the following conditions:***

1. ***the flattened set of*** $\{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n, \mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_m\}$ ***is a multiset chosen from set*** $\{1, 2, \ldots, r\}$ ***with multiplicity of each element no more than*** n***;***
2. ***for some*** $x = \Theta\left(\frac{\log r}{\log m - \log n}\right)$ ***and for any*** $\mathcal{I}_i$ $(1 \leq i \leq n)$ ***and any*** $\mathcal{M}_{j_1}, \mathcal{M}_{j_2}, \ldots, \mathcal{M}_{j_x}$ $(1 \leq j_1 < j_2 < \cdots < j_x \leq m)$ $\mathcal{I}_i \cap (\cap_{k=1}^{x} \mathcal{M}_{j_k}) \neq \phi$.

– **Lemma 1 can be used to construct a sequence of connection/disconnection requests under these strategies and obtain the following necessary conditions:**

Theorem 6 ***The necessary condition for a*** $v(m,n,r)$ ***network to be strictly nonblocking for multicast assignments is*** $m \geq \Theta\left(n\frac{\log r}{\log\log r}\right)$.

Theorem 7 ***The necessary condition for a*** $v(m,n,r)$ ***multicast network to be nonblocking under Strategy 1 is*** $m \geq \Theta\left(n\frac{\log r}{\log\log r}\right)$.

Theorem 8 ***The necessary condition for a*** $v(m,n,r)$ ***multicast network to be nonblocking under Strategy 2 is*** $m \geq \Theta\left(n\frac{\log r}{\log\log r}\right)$.

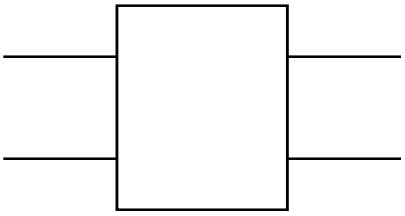
Theorem 9 ***The necessary condition for a*** $v(m,n,r)$ ***multicast network to be nonblocking under Strategy 3 is*** $m \geq \Theta\left(n\frac{\log r}{\log\log r}\right)$.

Conjecture 1 ***The necessary condition for a*** $v(m,n,r)$ ***multicast network to be nonblocking under any strategy is*** $m \geq \Theta\left(n\frac{\log r}{\log\log r}\right)$**.**

Based on this conjecture, $m = O\left(n\frac{\log r}{\log\log r}\right)$ **is optimal for nonblocking** $v(m,n,r)$ **multicast network.**

- **Generalization to multirate network**
  - **Single rate connection: A connection which uses up the entire bandwidth of the link carrying it.**
  - **Multirate connection: A connection which can consume an arbitrary fraction of the bandwidth of the link carrying it.**
  - **Why design multirate networks?**
    More general than single rate network. Can be operated in packet switching manner.

– Multirate switch



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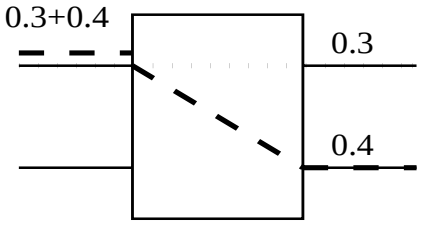

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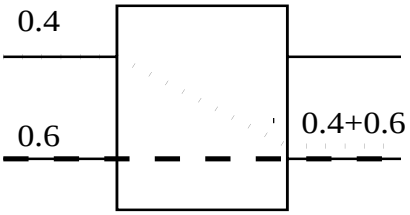

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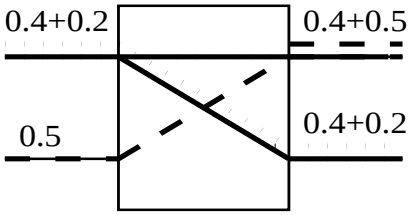


(d)

- **Previous work**
  - **Single rate network**
    - **A $v(m, n, r)$ network [Clos '53] is nonblocking for permutation assignments if**

$$m \geq 2n - 1$$

    - **A $v(m, n, r)$ network [Yang and Masson '92] is nonblocking for multicast assignments if**

$$m \geq 3(n-1)\frac{\log r}{\log \log r}$$

  - **Multirate network**
    - **A $v(m, n, r)$ multirate network [Melen and Turner '89; Chung and Ross '91] is nonblocking for permutation assignment if**

$$m > 2\lfloor 1/b \rfloor (n-1)$$

where b $(0 < b \leq 1)$ is the fraction of

bandwidth a minimum size packet occupies.

- **More terminology:**

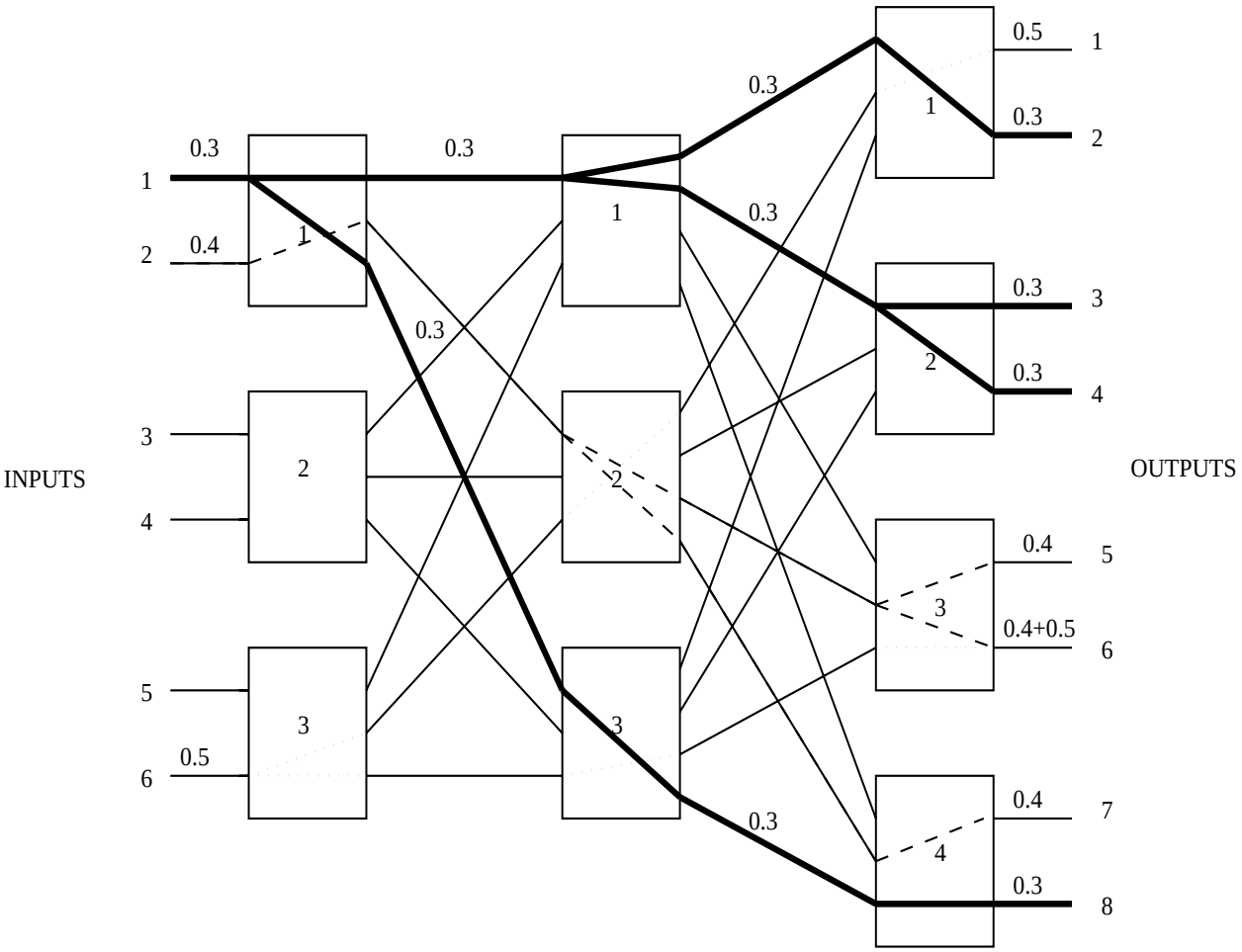
  ω-idle input port or output port: an input port or an output port is ω-idle, $b \leq \omega \leq 1$, if the fraction of available bandwidth of that port is at least ω.

  Multicast connection with weight ω: A connection from an ω-idle input port j to a set of ω-idle output ports I_i, denoted as (I_j, ω), uses ω fraction of bandwidth on all links carrying it.

  Multirate multicast assignment: A maximum set of multirate multicast connections where the weight of all input ports and output ports involved is no more than 1.

Available middle switches: **The available middle switches of an input connection (I_j, ω) is the set of middle switches with link weight no more than $1 - \omega$ to the input switch associated with input port j.**

Destination sets: **Destination set $M_{j,\omega}$ is the set of output switches to which the middle switch j is providing connection paths with weights greater than $1 - \omega$ from the input ports.**



A multirate multicast assignment in a $v(3, 2, 3, 2, 4)$ network where $(I_1, 0.3) = \{1, 2, 4\}$.

- **Main results**

Let $T(n_1, \omega, x)$ denote the number of output links with weights great than $1 - \omega$ in the input switch associated with the new connection request (I_i, ω), that is, the number of middle switches not available to the new connection request.

Lemma 2 ***For*** $T(n_1, \omega, x)$***, we have three cases:***

Case 1. ***If*** $b > 1 - \omega$***, then***

$$T(n_1, \omega, x) = \lfloor 1/b \rfloor (n_1 - 1)x.$$

Case 2. ***If*** $b \leq 1-\omega$ ***and*** $1-\left\lfloor \frac{1}{1-\omega} \right\rfloor (1-\omega) < b$***,*** ***then***

$$T(n_1,\omega,x) \leq \left\lfloor \frac{1}{1-\omega} \right\rfloor (n_1-1)x + \left(x - \left\lceil x / \left\lfloor \frac{1-\omega}{b} \right\rfloor \right\rceil\right) U(n_1),$$

where $U(n_1) = \begin{cases} 1 & \textbf{\textit{if}}\ n_1 > 1 \\ 0 & \textbf{\textit{otherwise}} \end{cases}$

Case 3. ***If*** $b \leq 1-\omega$ ***and*** $1-\left\lfloor \frac{1}{1-\omega} \right\rfloor (1-\omega) \geq b$***,*** ***then***

$$T(n_1,\omega,x) \leq \left\lfloor \frac{n_1-1}{1-\omega} \right\rfloor x + x.$$

Lemma 3

$$\max_{b \leq \omega \leq 1} T(n_1,\omega,x) = \lfloor 1/b \rfloor (n_1-1)x.$$

Lemma 4 ***If a new connection request*** (I_i,ω) ***is valid, then for each output switch*** k***,*** $1 \leq k \leq r_2$***, there are at most*** $T(n_2,\omega,1)$ ***links with the weights greater than*** $1-\omega$ ***connecting to output switch*** k ***from the middle stage.***

Lemma 5 ***We can satisfy a new connection request*** (I_i, ω), $|I_i| = r_2$, ***using some*** x $(x \geq 1)$ ***middle switches, say,*** $i_1, \ldots, i_x$, ***from among the available middle switches of a*** $v(m, n_1, r_1, n_2, r_2)$ ***network if and only if the current destination sets of these*** x ***middle switches are such that***

$$\bigcap_{j=1}^{x} M_{i_j,\omega} = \phi$$

Lemma 6 ***Assume that a*** $v(m, n_1, r_1, n_2, r_2)$ ***network is in a state in which there exist at most*** n' ***connection paths with the weights greater than*** $1 - \omega$, $1 \leq n' \leq T(n_2, \omega, 1)$ ***to each of the output switches. Then the intersection of more than*** n' $M_{i,\omega}$***'s is empty.***

Theorem 10 ***A*** $v(m, n_1, r_1, n_2, r_2)$ ***network is nonblocking for multirate multicast assignments if***

$$m > \max_{b \le \omega \le 1} \min_{1 \le x \le \min\{T(n_2,\omega,1), r_2\}} \{T(n_1, \omega, x) + T(n_2, \omega, 1) r_2^{1/x}\}$$

Corollary 3 ***For the symmetrical*** $v(m, n, r)$ ***multicast network, the nonblocking condition under multirate traffic model becomes***

$$m > \max_{b \le \omega \le 1} \min_{1 \le x \le \min\{T(n,\omega,1), r\}} \left\{T(n, \omega, x) + T(n, \omega, 1) r^{1/x}\right\}.$$

In particular, this condition can be written as

$$m > 3\lfloor 1/b \rfloor (n-1) \frac{\log r}{\log \log r}$$

Corollary 4 ***Setting*** $d = 1$**,** $n_1 = n_2 = n$ ***and*** $r_1 = r_2 = r$ ***in Corollary 1 yields***

$$m > 2\lfloor 1/b \rfloor (n-1)$$

which is the nonblocking bound on m ***associated with the one-to-one connection multirate*** $v(m, n, r)$ ***network.***

Corollary 5 ***Setting*** $b = 1$ ***in Corollary 3 yields*** $m \geq 2n-1$**,** ***which is the nonblocking bound on*** m ***associated with the circuit switching Clos one-to-one connection network.***

Finally, the proofs for the theorems also implied an efficient routing algorithm for the multirate network.