Optimal Spectrum Management in Multiuser Interference Channels

Yue Zhao, *Member, IEEE*, and Gregory J. Pottie, *Fellow, IEEE*

***Abstract*—In this paper, we study the problem of continuous frequency optimal spectrum management in multiuser frequency selective interference channels. We assume that interference is treated as noise by the decoders, and separate encoding is applied. First, a simple pair-wise channel condition for frequency division multiple access schemes to achieve all Pareto optimal points of the rate region is derived. It enables fully distributed global optimal decision making on whether any two users should use orthogonal channels. Next, we present an analytical solution to finding the maximum sum-rate in two-user symmetric frequency flat channels. Generalizing this solution to frequency selective channels, a convex optimization is established that yields the global optimum. Finally, we show that our method generalizes to K-user $(K \geq 2)$ weighted sum-rate maximization in asymmetric frequency selective channels, and we transform this classic nonconvex optimization to an equivalent convex optimization in the primal domain.**

***Index Terms*—Frequency-division multiple access (FDMA) optimality condition, multiuser interference channel, nonconvex optimization, optimal spectrum management.**

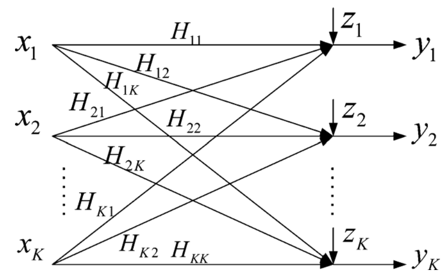

Fig. 1. Multiuser Gaussian interference channel.

I. Introduction

In multiuser communications systems, interference coupling between different users remains a major problem that limits the system performance. A general multiuser Gaussian interference channel is depicted in Fig. 1, in which each user consists of a transmitter and receiver pair, and there is cross interference coupling between every pair of users. In this paper, we consider the decoding assumption that *interference is treated as noise*. While treating interference as noise achieves the information theoretic capacity under certain *weak interference* conditions [1], [17], [18], in general, potentially higher system capacity can be achieved with more complex decoding techniques such as interference cancellation or joint decoding. However, finding the optimal schemes using such techniques to achieve the information theoretic capacity region remains an open problem particularly for interference channels with three or more users [11]. Furthermore, these techniques often incur higher implementation complexity in practice than treating interference as noise.

We consider the scenario of multiple multicarrier communication systems contending in a common frequency band. (There may sometimes be practical reasons to channelize the resources in some other fashion, e.g., in time. Here, we regard any such alternatives as equivalent to channelizing in the frequency domain.) We assume that separate encoding for each subcarrier is applied. We note that using joint encoding across subcarriers can sometimes achieve higher system capacity than using separate encoding [4]. We investigate the optimal *continuous frequency* spectrum and power allocation problem, for which the channel frequency responses and the users' power spectral density (PSD) can be any *bounded piecewise continuous functions* of frequency over a finite band. The continuous frequency problem is an infinite-dimensional optimization. However, in the special case of a *frequency flat* channel response, the problem has been shown to have a finite number of dimensions [13], [19] . Despite the infinite number of dimensions, significant insights can still be provided from solving the optimal solution in this continuous frequency form, as shown in the later sections of this paper.

In practical systems, a discrete frequency model with a finite number of sub-carriers is often assumed, and the PSD within each sub-carrier is required to be flat. For a variety of objective functions, the nonconvex optimization of spectrum and power allocation with the discrete frequency model has been shown to be NP complete in the number of users even for the single carrier case [16]. For the single carrier *sum-rate* maximization problem, two special cases have been solved: the two-user case with general channel parameters [10], [14], and the K*-user* $(K \geq 2)$ case of fully symmetric channels [2]. For the multicarrier weighted sum-rate maximization problem, also known as *spectrum management* or *spectrum balancing*, there

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has been considerable research addressing the nonconvexity of the problem and the NP completeness in both the number of users and the number of carriers. With sufficient primal objective relaxations, the problem can be approximated as convex optimization [8], [9] . For solving the original nonconvex optimization, dual decomposition methods have been widely applied to decompose the problem in frequency [6], [7], [20], [23] . While these methods effectively reduced the scale of the problem to solve, two remaining issues are as follows. 1) While the dual master problem is a convex optimization (which can be solved by, e.g., subgradient method [20],) the single carrier *sub-problem* is still a nonconvex optimization which is NP-complete in the number of users K. 2) The dual optimal solution does not necessarily give a primal optimal solution. Addressing the second issue, an important result is that the duality gap of the spectrum management problem goes to zero as the number of sub-channels goes to infinity, under mild technical conditions [16], [20]. Our results will be connected to this result at the end of this paper.

For the model studied here, there are essentially two strategies for multiple users to co-exist: *frequency-division multiple access* (FDMA) *and frequency sharing (overlapping)*. As the cross coupling varies from being extremely strong to extremely weak, the preferable co-existence strategies intuitively shift from complete avoidance (FDMA) to pure frequency sharing. We start from the strong coupling scenario, and investigate the weakest interference condition under which FDMA is still guaranteed to be optimal, regardless of the power constraints. In the literature, a relatively strong *pair-wise* coupling condition for FDMA to achieve all Pareto optimal points of the rate region is derived with the continuous frequency model [13]. By pair-wise we mean that whether two users should be orthogonalized in frequency only depends on the interference condition between these two users. For *sum-rate* maximization, the required coupling strengths for FDMA to be optimal are further lowered in [15], approaching roughly the weakest possible. However, this condition is derived in a *group-wise* form, requiring the couplings between all users to be sufficiently strong.

In this paper, by analyzing the continuous frequency model, the weakest possible *pair-wise* condition for FDMA to achieve *all* Pareto optimal points of the rate region is proved: for any two (among all of the K) users, as long as the product of the two normalized cross channel gains between them is greater than or equal to 1/4, an FDMA allocation between these two users benefits *every one of the K users*. In symmetric channels, when the cross channel gain is less than 1/2 (and thus the product of them is less than 1/4), we precisely characterize the nonempty power constraint region within which frequency sharing between two users leads to a higher rate than an FDMA allocation between them.

For the general nonconvex optimization of spectrum management, we develop a novel method that transforms the problem in the *primal domain into an equivalent convex optimization*. We begin with *sum-rate* maximization in two-user symmetric frequency flat channels. We show that the optimal spectrum management can be solved by computing a *convex hull function*. As a result, the optimal spectrum management always consists of one sub-band of flat frequency sharing and one sub-band of flat FDMA. This sets up our more general results. The optimal solution for the *sum-rate* maximization was also independently derived in [19] for *two-user asymmetric* frequency flat channels, and in [2] for K-*user* $(K \geq 2)$ *symmetric* frequency flat channels.

We first generalize our results to two-user symmetric *frequency selective* channels, and show that a convex relaxation of the original nonconcave objective actually leads to the *same* optimal value as the original problem. Next, we generalize our results to K-*user asymmetric* frequency flat channels for arbitrary *weighted sum-rate* maximization, and show that the optimal solution can be found by computing a convex hull function. Finally, we combine the ideas of these generalizations, and establish the equivalent primal domain convex optimization for the spectrum management problem in its general form, i.e., *arbitrary weighted sum-rate maximization for K-user $(K \geq 2)$ asymmetric frequency selective channels*.

The rest of this paper is organized as follows. The problem model is established in Section II. In Section III, we discuss the channel conditions under which FDMA schemes can achieve all Pareto optimal rate tuples. In Section IV, we solve the sum-rate maximization in two-user symmetric (potentially frequency selective) channels. In Section V, we extend our method to the general cases, and show that the continuous frequency optimal spectrum management scheme can be equivalently cast as a primal domain convex optimization. We then discuss the computational complexities of finding the optimal spectrum management scheme. Conclusions are drawn in Section VI.

II. Channel Model and Two Basic Co-Existence Strategies

A. Interference Channel Model and the Rate Density Function

As depicted in Fig. 1, a K-user Gaussian interference channel is modeled by

$$y_i = H_{ii}x_i + \sum_{j \neq i} x_j H_{ji} + Z_i, \quad i = 1, 2, \ldots, K$$

where x_i is the transmitted signal of user i, and y_i is the received signal of user i including additive Gaussian noise z_i, (a user corresponds to a transmitter and receiver pair). H_{ii} is the direct channel gain from the transmitter to the receiver of user i. H_{ji} is the cross channel gain from the transmitter of user j to the receiver of user i. For the purposes of the analysis in this paper, without loss of generality (WLOG), we assume that the channel is over a unit bandwidth frequency band $[0, 1]$. The results derived directly generalize to frequency bands with arbitrary bandwidths.

We denote the frequency selective channel gains H_{ii} and H_{ji} by $H_{ii}(f)$ and $H_{ji}(f)$, $f \in [0, 1]$. We denote the transmit PSD of user i by $p_i(f)$, and the noise PSD at receiver i by $\sigma_i(f)$. We assume that $H_{ji}(f)$, $\sigma_i(f)$, $p_i(f)$, $\forall i, j$ are all *bounded piecewise continuous functions* over the band $f \in [0, 1]$. Furthermore, we assume that all functions appearing in this paper have a finite number of discontinuities.

We assume that every user uses a random Gaussian codebook, and only decodes the signal from its own transmitter, treating interference from other transmitters as noise. Employing the

Shannon capacity formula for Gaussian channels, we have the following achievable rate for user $i(= 1, 2 \ldots, K)$:

$$R_i = \int_0^1 \log\left(1 + \frac{p_i(f)\left|H_{ii}(f)\right|^2}{\sigma_i(f) + \sum_{j\neq i} p_j(f)\left|H_{ji}(f)\right|^2}\right) df$$
$$= \int_0^1 \log\left(1 + \frac{p_i(f)}{n_i(f) + \sum_{j\neq i} p_j(f)\alpha_{ji}(f)}\right) df$$

where $\alpha_{ji}(f) \triangleq \frac{|H_{ji}(f)|^2}{|H_{ii}(f)|^2}$, $n_i(f) \triangleq \frac{\sigma_i(f)}{|H_{ii}(f)|^2}$ are the cross channel gains and the noise power normalized by the direct channel gains. We further make a technical assumption that

$$\exists n_\varepsilon > 0, \quad s.t \quad \forall f \in [0,1], \quad n_i(f) \geq n_\varepsilon, \forall i = 1, 2, \ldots, K \tag{1}$$

which naturally holds in all physical channels.

To reach any Pareto optimal point of the K-user rate region, we optimize the *spectrum management schemes* (i.e., the power and spectrum allocation functions)

$$\mathbf{p}(f) \triangleq [p_1(f), p_2(f), \ldots, p_K(f)]^T, \forall f \in [0,1].$$

As we consider the continuous frequency model, we make the following definition.

Definition 1: $\forall f \in [0,1]$,with $\mathbf{P} = [P_1, P_2, \ldots, P_K]^T$

$$r_i(\mathbf{P}, f) \triangleq \log\left(1 + \frac{P_i}{n_i(f) + \sum_{j\neq i} P_j\alpha_{ji}(f)}\right).$$

Now, we have the *rate density function* of user i at frequency f

$$r_i(\mathbf{p}(f), f) = \log\left(1 + \frac{p_i(f)}{n_i(f) + \sum_{j\neq i} p_j(f)\alpha_{ji}(f)}\right)$$

and $\mathbf{r}(\mathbf{p}(f), f) \triangleq [r_1(\mathbf{p}(f), f), r_2(\mathbf{p}(f), f), \ldots, r_K(\mathbf{p}(f), f)]^T$. Accordingly, $R_i = \int_0^1 r_i(\mathbf{p}(f), f)\, df, i = 1, 2, \ldots, K$.

B. Piecewise Continuous Functions as Limits of Piecewise Flat Functions

We consider the channel responses and power allocations as bounded piecewise continuous functions of frequency. Intuitively, one may approximate continuous functions by piecewise constant functions, by subdividing the support (frequency) to a sufficiently large number of small pieces. We make use of this idea in later sections, and provide a technical lemma for this purpose whose proof is relegated to Appendix A.

Lemma 1 (Approximation Lemma): Given $\mathbf{p}(f)$, $\{\alpha_{ji}(f)\}$, $\{n_i(f)\}$, $f \in [0,1]$, all bounded piecewise continuous, for any utility function $U(\mathbf{p}, \boldsymbol{\alpha}, \mathbf{n})$ that is a uniformly continuous function of $\{\{p_i\}, \{\alpha_{ji}\}, \{n_i\}\}$, $\forall \varepsilon > 0$, there exists a set of *piecewise flat* power allocation functions and channel responses

$$\bar{\mathbf{p}}(f) = [\bar{p}_1(f), \ldots, \bar{p}_K(f)]^T, \ \{\bar{\alpha}_{ji}(f)\}, \{\bar{n}_i(f)\}, \ f \in [0,1]$$

for which the band is divided into $M(< \infty)$ intervals $I_1, \ldots, I_M$, $I_m = [f_{m-1}, f_m]$, with $f_0 = 0$, $f_m = 1$, $f_{m-1} < f_m$, and $\forall_m = 1, 2, \ldots, M$

$$\begin{cases} \bar{\mathbf{p}}(f) = \mathbf{P}(m), \ \ \forall f \in I_m \\ \bar{\alpha}_{ji}(f) = \alpha_{ji}(m), \ \ \bar{n}_i(f) = n_i(m), \ \forall f \in I_m, \forall i, j. \end{cases}$$

where $\mathbf{P}(m) = [P_1(m, \ldots, P_k(m)]^T$, $\{\alpha_{ji}(m)\}$, $\{n_i(m)\}$ are *constants* that only depend on the interval index m, *such that the following three properties hold*:

P1. $\forall f \in [0,1]$, $\bar{P}_i(f) \leq P_i(f), \forall i = 1, 2, \ldots, K$.

P2. $\forall f \in [0,1]$, $\bar{\alpha}_{ji}(f) \geq \alpha_{ji}(f), \forall i \neq j, \bar{n}_i \geq n_i(f), \forall i$.

P3. $\forall f \in [0,1]$, $|U(\bar{\mathbf{P}}(f), \bar{\boldsymbol{\alpha}}(f), \mathbf{n}(\mathbf{f})) - U(\mathbf{p}(f), \boldsymbol{\alpha}(f), \mathbf{n}(\mathbf{f}))| < \varepsilon$.

From now on, we name the $\bar{\mathbf{p}}(f)$, $\{\bar{\alpha}_{ji}(f)\}$, and $\{\bar{n}_1(f)\}$ found in Lemma 1 a "*piecewise flat* ε*-approximation*."

Remark 1: Property P1 ensures that the approximate piecewise flat power allocations consume *less* power than the original ones. Property P2 ensures that the approximate piecewise frequency flat channel responses are *"worse"* than the original ones (as the cross channel gains and the noise power are all stronger, and interference is treated as noise.) Nonetheless, property P3 ensures that under these "adverse" conditions, these approximations can still achieve the original utility U arbitrarily closely.

With finite power constraints and nondegenerate channel parameters (1), most utility functions considered in practice (e.g., a weighted sum-rate) satisfy the uniform continuity condition of $U(\mathbf{p}, \boldsymbol{\alpha}, \mathbf{n})$.

C. Two Basic Co-Existence Strategies and one Basic Transformation

There are essentially two co-existence strategies for users to reside in a common band: frequency sharing and FDMA. We introduce two basic forms of these two strategies: *flat frequency sharing* and *flat* FDMA, both defined in frequency flat channels. We will see that these two basic strategies are the building blocks of general nonflat co-existence strategies in frequency selective channels.

Consider a two-user frequency flat channel: $\forall f \in [0,1]$

$$n_1(f) = n_1, \ \ n_2(f) = n_2, \ \ \alpha_{21}(f) = \alpha_{21}, \ \ \alpha_{12}(f) = \alpha_{12}. \tag{2}$$

Definition 2: A flat frequency sharing scheme of two users is any power allocation in the form of

$$p_1(f) = p_1, \ \ p_2(f) = p_2, \ \ \forall f \in [0, \ 1]. \tag{3}$$

Definition 3: A flat FDMA scheme of two users is any power allocation in the form of

$$\begin{cases} p_1(f)p_2(f) = 0, \ \forall f \in [0,1] \\ p_1(f) = p_1', \qquad \forall f, \ p_2(f) = 0 \\ p_2(f) = p_2', \qquad \forall f, \ p_1(f) = 0 \end{cases}$$

where $p_1' > 0$ and $p_2' > 0$ are given constants.

Definition 4: Given bandwidths $W_1', W_2' \ s.t. W_1' + W_2' \leq 1$, a *flat* FDMA *reallocation* is the following *power invariant* transform that reallocates the power of the two users from a flat frequency sharing scheme to a flat FDMA scheme.

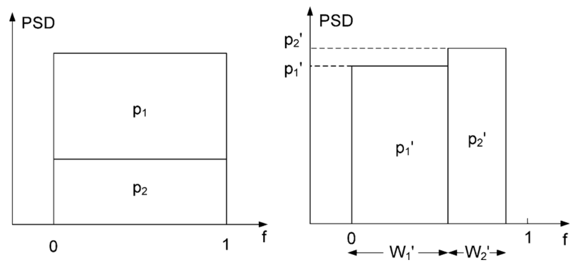

Fig. 2. Power allocations of flat frequency sharing and flat FDMA, and an illustration of flat FDMA reallocation.

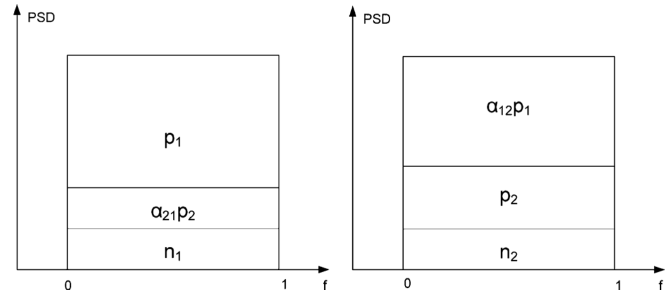

Fig. 3. The PSD composition at receivers 1 and 2.

1) User 1 reallocates all of its power within a sub-band W_1' with a flat PSD $p_1' \triangleq p_1/W_2'$.
2) User 2 reallocates all of its power within another disjoint sub-band W_2' with a flat PSD $p_2' \triangleq p_2/W_2'$.

Illustrations of the power allocations of the two basic co-existence strategies before and after a flat FDMA reallocation are depicted in Fig. 2.

Similarly, flat frequency sharing schemes, flat FDMA schemes, and flat FDMA reallocation can be defined for any $k(=1,2,3,\ldots)$ users. ($k=1$ is the degraded case in which flat frequency sharing is the same as flat FDMA.)

Remark 2: A flat FDMA scheme is mathematically the same as multiple disjoint bands each seeing a flat frequency sharing of only *one* user. Thus, it is actually sufficient to only define flat frequency sharing schemes of any $k(=1,2,3,\ldots)$ users, *without* introducing the definition of flat FDMA schemes. This alternative approach is used later in Section V for the general optimization in K-user frequency selective channels. Here, flat FDMA and flat FDMA reallocation are explicitly defined, because they offer clear intuitions for optimizing spectrum management as will be shown in Sections III and IV.

III. The Conditions for the Optimality of FDMA

In this section, we investigate the conditions under which the optimal spectrum and power allocation is FDMA. We show that our results apply to all Pareto optimal points of the achievable rate region. First, we provide a coupling condition under which FDMA schemes achieve all Pareto optimal rate tuples within a group of strongly coupled users. In real-world communication networks, however, there are usually users not strongly enough coupled with some other users. For these users outside the strongly coupled group, we show that they always benefit from an FDMA allocation within the strongly coupled group. These results lead to the following simple *pair-wise* condition: for any two of the K users, as long as the product of the normalized cross channel gains between them is greater than or equal to 1/4, *every one* of the K users will benefit from an FDMA allocation between these two users.

A. The Optimality of FDMA Within Strongly Coupled Users

In this section, we prove a sufficient condition in K-user interference channels under which FDMA among all users can achieve any Pareto optimal rate tuple. We begin with two-user frequency flat channels, and extend the results to K-user frequency selective channels.

Theorem 1: Consider a two-user frequency flat interference channel (2). Suppose the two users co-exist in a flat frequency sharing manner (3). If $\alpha_{12}\alpha_{21} \geq 1/4$, then there exists a flat FDMA power reallocation such that *both* users' rates will be higher (or unchanged.)

Before proving Theorem 1, we provide the following lemma whose proof is relegated to Appendix B.

Lemma 2: Let $f(x) = \frac{1}{x}\log(\frac{c+x}{c-x})$, $c>1$; then

$$f(1) \geq f(x),\ \forall x \in (0,\ 1].$$

Proof of Theorem 1: It is sufficient to prove for the case of $\alpha_{12}\alpha_{21} = 1/4$ since we are treating interference as noise. The received PSD of the desired signal, interference, and noise at both receivers are depicted in Fig. 3. The rates of users 1 and 2 are

$$R_1 = \log\left(1+\frac{p_1}{n_1+p_2\alpha_{21}}\right),\ R_2 = \log\left(1+\frac{p_2}{n_2+p_1\alpha_{12}}\right). \tag{4}$$

We apply a flat FDMA power reallocation (cf., Fig. 2) with the following specific bandwidths:

$$W_1' = \frac{p_1}{p_1+2\alpha_{21}p_2},\quad W_2' = \frac{p_2}{p_2+2\alpha_{12}p_1}. \tag{5}$$

Accordingly, $p_1' = p_1+2\alpha_{21}p_2, p_2' = p_2+2\alpha_{12}p_1$. It is straightforward to check that $\alpha_{12}\alpha_{21} = 1/4 \Rightarrow W_1' + W_2' = 1$, i.e., this reallocation is feasible.

Denote user 1's rate after this reallocation by

$$R_1' = W_1'\log\left(1+\frac{p_1'}{n_1}\right) = \frac{p_1}{p_1+2\alpha_{21}p_2}\log\left(1+\frac{p_1+2\alpha_{21}p_2}{n_1}\right). \tag{6}$$

From (4) and (6), $R_1' \geq R_1 \Leftrightarrow$

$$\log\left(\frac{p_1+2\alpha_{21}p_2+n_1}{n_1}\right) \geq \frac{p_1+2\alpha_{21}p_2}{p_1}\log\left(\frac{p_1+\alpha_{21}p_2+n_1}{\alpha_{21}p_2+n_1}\right). \tag{7}$$

Define $c \triangleq \frac{p_1+2\alpha_{21}p_2+2n_1}{p_1+2\alpha_{21}p_2}$, $x \triangleq \frac{p_1}{p_1+2\alpha_{21}p_2}$, and (7) can be rewritten as

$$\log\left(\frac{c+1}{c-1}\right) \geq \frac{1}{x}\log\left(\frac{c+x}{c-x}\right). \tag{8}$$

From Lemma 2, (8) always holds since $c>1$ and $x \in (0,1]$. Thus, $R_1' \geq R_1$. Similarly, we also have $R_2' \geq R_2$. Therefore, *for both users*, a proper flat FDMA power reallocation leads to rates higher than or equal to a flat frequency sharing. ■

Moreover, Theorem 1 can be generalized to the K-user case as follows.

Theorem 2: Consider a K-user frequency flat interference channel, $n_i(f) = n_i$, $\alpha_{ji}(f) = \alpha_{ji}$. Suppose the K users co-exist in a flat frequency sharing manner: $\mathbf{p}_i(f) = \mathbf{p}_i > 0$, $\forall f \in [0,1]$. If $\alpha_{ji}\alpha_{ij} \geq 1/4$, $\forall j \neq i$, then there exists a flat FDMA power reallocation such that *all* users' rates will be higher or unchanged.

To prove Theorem 2, we choose a proper set of reallocation bandwidths $W_1', W_2', \ldots, W_K'$ that generalizes (5). We note that while $W_1' + W_2' = 1$ is straightforward for the two-user case, showing that $W_1' + W_2' + \cdots + W_K' \leq 1$ for the K-user case is a much more involved task. The detailed proof of Theorem 2 is relegated to Appendix B. Theorem 2 can be immediately generalized to frequency selective channels as follows.

Corollary 1: Consider a K-user frequency selective interference channel. Suppose we have an arbitrary spectrum and power allocation scheme $\mathbf{p}(f)$ with some frequency sharing (overlapping) in the band. If $\alpha_{ji}(f)\alpha_{ij}(f) \geq 1/4, \forall j \neq i,, \forall f \in [0,1]$, then there exists an *FDMA* power reallocation scheme $\tilde{\mathbf{p}}(f)$, satisfying $\int_0^1 \tilde{p}_i(f)df = \int_0^1 p_i(f)df, , i = 1, \ldots, K$, such that *all* user's rates are higher or unchanged.

Proof: The proof is immediate as the strong coupling condition is for *all* frequencies. ∎

B. FDMA Within a Subset of Users Benefits All Other Users

We have seen that by properly separating a group of strongly coupled users to orthogonal channels, every user among them will have a rate higher than or equal to the rate of any frequency sharing (overlapping) scheme. In this section, we show that an FDMA allocation among a group of users *also benefits every other user outside this group*. This result completes the fundamental fact that to achieve *any K-user Pareto optimal rate tuple*, all the strongly coupled users (among all the users) must be separated into disjoint frequency bands.

We begin with the three-user (one user + two interferers) frequency flat channels, and extend the results to $K+1$-user (one user + K interferers) frequency selective channels.

Lemma 3: Consider a three-user frequency flat channel: $n_i(f) = n_i$, $\alpha_{ji}(f) = \alpha_{ji}$. Suppose the three users co-exist in a flat frequency sharing manner: $p_i(f) = p_i$, $\forall f \in [0,1]$, $i = 0,1,2$. From user 0's perspective, a flat FDMA power reallocation of its two interferers, namely users 1 and 2, always leads to a rate higher than or equal to the original rate *for user 0*.

Proof: At the receiver of user 0, the received PSDs before and after a flat FDMA power reallocation of its interferers are depicted in Fig. 4. User 0's rates before and after the reallocation are

$$\begin{aligned} R_0 &= \log\left(1 + \frac{p_0}{\alpha_{10}p_1 + \alpha_{20}p_2 + n_0}\right), \\ R_0' &= W_1' \log\left(1 + \frac{p_0}{\alpha_{10}p_1/W_1' + n_0}\right) \\ &\quad + W_2' \log\left(1 + \frac{p_0}{\alpha_{20}p_2/W_2' + n_0}\right) \\ &\quad + (1 - W_1' - W_2') \log\left(1 + \frac{p_0}{n_0}\right). \end{aligned}$$

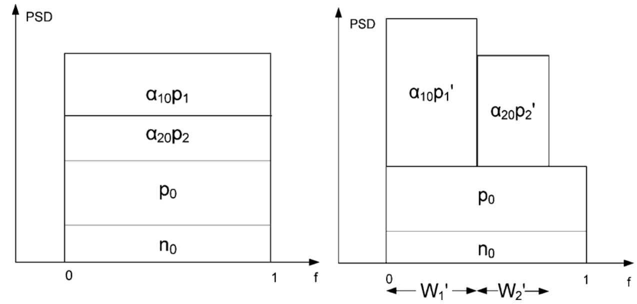

Fig. 4. PSD compositions at receiver 0 before and after a flat FDMA reallocation of users 1 and 2.

With straightforward calculations, one can verify that the function $\log(1 + \frac{P}{I+N})$ is convex in I. Therefore, by Jensen's Inequality, $R_0' \geq R_0, \forall p_1, p_2 \geq 0$. ∎

Lemma 3 can be generalized to an arbitrary number of users as in the following corollary.

Corollary 2.1: Consider a $K+1$-user (one user + K interferers) frequency flat channel: $n_i(f) = n_i$, $\alpha_{ji}(f) = \alpha_{ji}$. Suppose the $K+1$ users co-exist in a flat frequency sharing manner: $\mathbf{p}(f) = \mathbf{p}$, $\forall f \in [0,1]$. From user 0's perspective, a flat FDMA power reallocation of its K interferers, namely user 1, user 2, ..., user K, always leads to a rate higher than or equal to the original rate for user 0.

Proof: Similarly to the proof of Lemma 3, it follows from the convexity of $\log(1 + \frac{P}{I+N})$ in I. ∎

Finally, the benefits of an FDMA allocation within a subset of users to the other users can be generalized to frequency selective channels.

Corollary 2.2: Consider a $K+1$-user (one user + K interferers) frequency selective channel. Suppose we have an arbitrary spectrum management scheme $p_i(f)$, $i = 0,1,2,\ldots,K$, in which user 1, ..., user K are *not* completely using FDMA. Then, from user 0's perspective, there exists an FDMA power reallocation of its K interferers, namely user 1, ..., user K, that leads to a rate higher than or equal to the original rate for user 0.

Proof: $\forall \varepsilon > 0$, by Lemma 1, take a piecewise flat ε- approximation $\bar{\mathbf{p}}(f)$, $\{\bar{\alpha}_{ji}(f)\}$ and $\{\bar{n}_i(f)\}$, s.t.

$$\left|\bar{R}_0 - R_0\right| < \varepsilon$$

where $\bar{R}_0$ is user 0's rate computed with $\bar{\mathbf{p}}(f)$, $\{\bar{\alpha}_{ji}(f)\}$ and $\{\bar{n}_i(f)\}$. If $\bar{p}_1(f), \ldots, \bar{p}_K(f)$ is not completely FDMA yet, do a flat FDMA reallocation to $\bar{p}_1(f), \ldots, \bar{p}_K(f)$ in *every flat subchannel* that has a flat frequency sharing of any subset of the K interferers. By Corollary 2.1, the resulting rate of user 0 satisfies $\bar{R}_0' \geq \bar{R}_0 > R_0 - \varepsilon$. Finally, let $\varepsilon \to 0$. ∎

We summarize Theorem 1 and Lemma 3 as follows:

Theorem 3: For any two users i and j (among all the K users), for any frequency band $[f_1, f_2]$, if the normalized cross channel gains $\alpha_{ji}(f)\alpha_{ij}(f) \geq 1/4$, $\forall f \in [f_1, f_2]$, then no matter from which user's point of view, an FDMA of user i and user j within this band is always preferred.

Proof: Suppose the spectrum and power allocation for user i and j are not FDMA, take a piecewise flat ε-approximation $\bar{\mathbf{p}}(f)$, $\{\bar{\alpha}_{ji}(f)\}$, and $\{\bar{n}_i(f)\}$, s.t. $\left|\bar{R}_k - R_k\right| < \varepsilon, \forall k = 1, \ldots, K$. As in the proof of *Corollary*

2.2, with a flat FDMA reallocation of $\bar{p}_i(f)$ and $\bar{p}_j(f)$ in *every flat sub-channel* in $[f_1, f_2]$ that has a flat frequency sharing of user i and j, Theorem 1 implies that user i and j's rates are increased or unchanged, and Lemma 3 implies that every one of the other $K-2$ users' rate is increased or unchanged. Finally, let $\varepsilon \to 0$. ■

The pair-wise condition $\alpha_{ji}(f)\alpha_{ij}(f) \geq 1/4$ makes determining whether any two users should be orthogonally channelized depend only on the coupling conditions between the two of them. Furthermore, since this condition guarantees that an FDMA allocation between user i and user j benefits *every* one of the K users, under this condition, *all the Pareto optimal points* of the rate region can be achieved with these two users having an FDMA allocation. This pair-wise condition is thus an example of distributed decision making (on whether to orthogonalize any pair of users) with optimality guarantees.

We conclude this section by comparing our results with the previously developed conditions for the optimality of FDMA in [13] and [15].

1) In [13], it was shown that *all* Pareto optimal points of the rate region can be achieved with an FDMA allocation between user i and user j if $\alpha_{ji}(f)\alpha_{ij}(f) > 1, \forall f \in [0,1]$. In comparison, our result on the condition of $\alpha_{ji}(f)\alpha_{ij}(f) \geq 1/4$ improves this condition by a factor of four.
2) In [15], a *discrete* frequency model is considered. It was shown that the *sum-rate* optimal point of the rate region can be achieved with an FDMA allocation among *all* the users if

$$\alpha_{ji}^n > \frac{1}{2} \ and \ \alpha_{ji}^n\alpha_{ij}^n > \frac{1}{4}\left(1+\frac{1}{C-1}\right)^2, \forall 1 \leq i \neq j \leq K, \ \forall n$$

where n is the channel index in the discrete frequency model, and C is some constant. In comparison, we consider a continuous frequency model, and our result improves the above condition from the following three aspects: 1) Our result not only applies to the sum-rate optimal point, but also applies to all Pareto optimal points of the rate region. 2) Our result does not require all the users to be strongly coupled, but can be applied to any subset of the users who are strongly coupled. 3) Our result on the condition of $\alpha_{ji}(f)\alpha_{ij}(f) \geq 1/4$ strictly improves the above condition as the requirement of $\alpha_{ji}^n > \frac{1}{2}$ is dropped (cf., Theorem 2).

IV. Optimal Spectrum Management in Two-User Symmetric Channels

In this section, we continue to analyze the optimal spectrum management in the cases with $\alpha_{ji}(f)\alpha_{ij}(f) < 1/4$. We give a complete analysis of two-user (potentially frequency selective) symmetric Gaussian interference channels, defined as follows:

$$\begin{cases} \alpha_{12}(f) = \alpha_{21}(f) < 1/2, & \forall f \in [0,1] \\ n_1(f) = n_2(f), & \forall f \in [0,1]. \end{cases}$$

We choose the objective to be the *sum-rate* of the two users $R_1 + R_2$. Generalizations with $K \geq 2$ users and arbitrary weighted sum-rate objective functions in general (asymmetric) channels are discussed later in Section V.

Here, an *equal* power constraint

$$\int_0^1 p_i(f)df \leq p/2, \ i = 1,2$$

or equivalently, a sum-power constraint

$$\int_0^1 (p_1(f) + p_2(f))\, df \leq p$$

is assumed. (Equivalency is shown later in this section.) We begin with frequency flat channels, and solve the optimal spectrum management scheme. Based on this result, we show that finding the spectrum management scheme that maximizes the nonconcave sum-rate objective in symmetric frequency selective channels can be equivalently transformed into a convex optimization in the *primal* domain.

A. Optimal Solutions for Frequency Flat Channels With a Sum-Power Constraint, Or Equivalently, Equal Power Constraints

Consider a two-user symmetric frequency flat Gaussian interference channel model

$$\begin{cases} \alpha_{12}(f) = \alpha_{21}(f) = \alpha < 1/2, & \forall f \in [0,1] \\ n_1(f) = n_2(f) = n, & \forall f \in [0,1]. \end{cases}$$

WLOG, we can normalize the power and their constraints by the noise, $p_i(f) \leftarrow p_i(f)/n, p \leftarrow p/n$, and assume $n = 1$. First, we have the following lemma on the condition under which a flat FDMA scheme is better than a flat frequency sharing scheme. Denote p_i to be the PSD of user $i = 1,2$ in a flat frequency sharing scheme.

Lemma 4: For any flat frequency sharing power allocation, a flat FDMA power reallocation with $W_1' = p_1/(p_1+p_2)$ and $W_2' = p_2/(p_1+p_2)$ leads to a higher or unchanged sum-rate if and only if $p_1 + p_2 \geq 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$.

The proof is relegated to Appendix C.

Given the cross channel gains α, Lemma 4 provides us a power region P_{FDMA} within which flat FDMA has a higher sum-rate than flat frequency sharing, depicted as the shaded area in Fig. 5 (with the complement region P_{FDMA}^c also depicted). Clearly, if and only if $\alpha \geq 1/2$ (which implies $\alpha^2 \geq 1/4$), P_{FDMA} contains the entire nonnegative quadrant. This provides a "weak" converse argument on the necessity of the coupling condition $\alpha_{ji}(f)\alpha_{ij}(f) \geq 1/4$, as derived in Section III, for FDMA to be *always optimal regardless of the power budget*.

Next, we derive the optimal flat frequency sharing scheme and the optimal flat FDMA scheme. Denote the sum-rate of a flat frequency sharing by

$$f(p_1,p_2) = \log(1 + \frac{p_1}{1+\alpha p_2}) + \log(1 + \frac{p_2}{1+\alpha p_1}).$$

With a sum-power constraint $p_1 + p_2 \leq p$, the maximum achievable sum-rate with flat frequency sharing, denoted by $f^*(p)$, is defined as the optimal value of the following optimization problem.

Definition 5:

$$\begin{aligned} f^*(p) \triangleq & \max f(p_1,p_2) \\ & s.t.\ p_1 + p_2 \leq p, \\ & \quad p_1 \geq 0,\ p_2 \geq 0. \end{aligned} \tag{9}$$

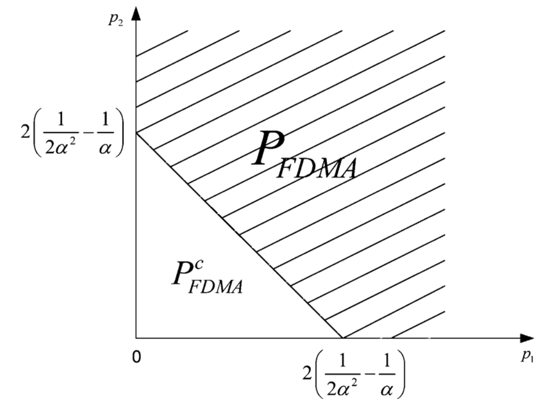

Fig. 5. The power region in which flat FDMA has a higher sum-rate than flat frequency sharing.

Next, we show the form and the concavity of $f^*(p)$ in the region of P^c_{FDMA}.

Lemma 5: When $0 < p \leq 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$

$$f^*(p) = 2\log\left(1 + \frac{p/2}{1+\alpha p/2}\right) \tag{10}$$

is a concave function of the constraint p. The optimal flat frequency sharing scheme is $p_1 = p_2 = p/2$.

The proof is relegated to Appendix C.

In comparison, we compute the maximum achievable sum-rate with a sum-power constraint for FDMA schemes, denoted by $h^*(p)$.

Definition 6:

$$h^*(p) = \max_{p_1(f), p_2(f)} \quad R_1 + R_2$$
$$s.t.\ R_1 = \int_0^1 \log(1+p_1(f))\,df, \quad R_2 = \int_0^1 \log(1+p_2(f))\,df$$
$$\int_0^1 (p_1(f) + p_2(f))\,df \leq p$$
$$p_1(f)p_2(f) = 0, \quad p_1(f) \geq 0, p_2(f) \geq 0, \quad \forall f \in [0,1].$$

From FDMA and the symmetry assumption of the channel, the sum-rate of both users is equivalent to the rate of a single user with a power constraint of p. With the water-filling principle, $h^*(p)$ is achieved when the PSD over the whole band is flat. In other words, both users' powers are allocated mutually nonoverlapped and collectively filling the whole band uniformly. Accordingly, we have the following lemma.

Lemma 6: The maximum achievable sum-rate with FDMA is

$$h^*(p) = \log(1+p). \tag{11}$$

Define the critical point $p_0 = 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$. Directly from Lemma 4, it can be verified that $f^*(p_0) = h^*(p_0)$. As $f^*(p)$ and $h^*(p)$ are both increasing and concave, the upper envelope of $f^*(p)$ and $h^*(p)$ is given by

$$r(p) \stackrel{\triangle}{=} \max\{f^*(p),\ h^*(p)\} = \begin{cases} f^*(p), & p \in [0, p_0] \\ h^*(p), & p \in [p_0, \infty). \end{cases}$$

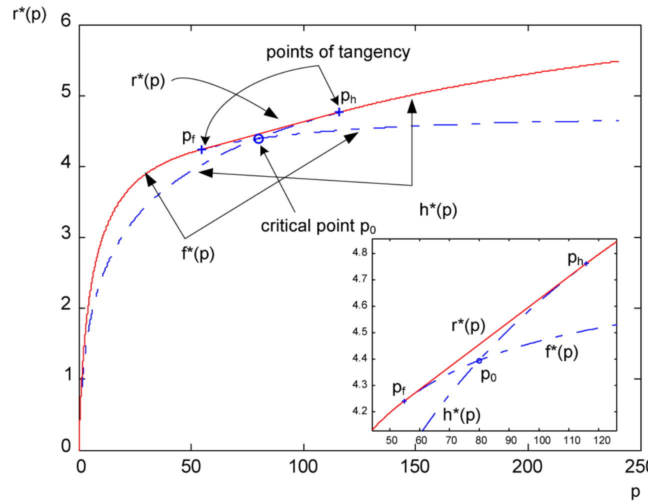

Fig. 6. The maximum achievable rate as the convex hull function of point-wise maximum of the achievable rate of flat FDMA and flat frequency sharing.

Furthermore, as $0 < \alpha < 1/2$

$$\left.\frac{d}{dp}f^*(p)\right|_{p=p_0} = \frac{4\alpha^3}{1-\alpha} < \frac{\alpha^2}{(1-\alpha)^2} = \left.\frac{d}{dp}h^*(p)\right|_{p=p_0} \tag{12}$$

and the upper envelope $r(p)$ is nonconcave in $[0, \infty)$.

Next, we define the convex hull function as follows.

Definition 7: The convex hull function of $f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n_+$, denoted by $conv_{\mathbf{x}}(f(\mathbf{x}))$, is a function of $\mathbf{x} \in \mathbb{R}^n_+$ that is the pointwise infimum of all the *concave* functions that upper bound $f(\mathbf{x})$

$$conv_{\mathbf{x}}(f(\mathbf{x})) \stackrel{\triangle}{=} \inf_{C_{f(\mathbf{x})}} \{g(\mathbf{x}), g(\mathbf{x}) \in C_{f(\mathbf{x})}\}$$

where $C_{f(\mathbf{x})} \stackrel{\triangle}{=} \{g(\mathbf{x}), \mathbf{x} \in R^n_+ | g(\mathbf{x}) \geq f(\mathbf{x}), g(\mathbf{x}) \text{ concave}\}$, provided that $C_{f(\mathbf{x})} \neq \emptyset$.

Since at any $\mathbf{x} \in R^n_+$, the infimum over $C_{f(\mathbf{x})} \neq \emptyset$ uniquely exists, $conv_{\mathbf{x}}(f(\mathbf{x}))$ is a well-defined function of $\mathbf{x} \in \mathbb{R}^n_+$. It is straightforward to check that $C_{r(p)} \neq \emptyset$, and we define

$$r^*(p) \stackrel{\triangle}{=} conv_p(r(p)). \tag{13}$$

A typical plot of $f^*(p)$, $h^*(p)$, and $r^*(p)$ is given in Fig. 6. Since $f^*(p)$ and $h^*(p)$ are concave, the convex hull function $r^*(p)$ is found by computing their *common tangent line*. For example, in Fig. 6, α is chosen to be 0.1. $f^*(p)$ and $h^*(p)$ intersect at $p_0 = 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right) = 80$. The two points of tangency are $p_f = 54.931$ and $p_h = 115.938$. In order to find the common tangent line of $f^*(p)$ and $h^*(p)$, the two points of tangency p_f and p_h are determined by

$$\left.\frac{d}{dp}f^*(p)\right|_{p=p_f} = \left.\frac{d}{dp}h^*(p)\right|_{p=p_h} = \frac{h^*(p_h) - f^*(p_f)}{p_h - p_f}$$

which simplifies to finding p_f by solving

$$\frac{p_f(\alpha(1+\alpha)p_f + 4\alpha - 2)}{(\alpha p_f + 2)((1+\alpha)p_f + 2)} = \log\left(\frac{(\alpha p_f + 2)^3}{4((1+\alpha)p_f + 2)}\right) \tag{14}$$

and computing p_h by

$$p_h = \frac{1}{4} p_f \left(\alpha(1+\alpha)p_f + 4\alpha + 2\right). \tag{15}$$

p_f and p_h can be obtained by solving the closed form (14) where various numerical methods can be applied. We note that (14) always has only one valid fixed point solution as shown in [2].

Next, we provide the main theorem of this section.

Definition 8: In a frequency flat symmetric Gaussian interference channel with $\alpha < 1/2$, define $r^o(p)$ to be the maximum achievable sum-rate with a sum-power constraint p

$$\begin{aligned} r^o(p) \triangleq \max_{\mathbf{p}(f)} \quad & \int_0^1 r_1(\mathbf{p}(f), f) + r_2(\mathbf{p}(f), f) df \qquad (16) \\ & r_1(\mathbf{p}(f), f) = \log\left(1 + \frac{p_1(f)}{1+\alpha p_2(f)}\right) \\ & r_2(\mathbf{p}(f), f) = \log\left(1 + \frac{p_2(f)}{1+\alpha p_1(f)}\right) \\ & \int_0^1 (p_1(f) + p_2(f))\, df \le p \\ & p_1(f) \ge 0, p_2(f) \ge 0, \quad \forall f \in (f_1, f_2). \end{aligned}$$

Theorem 4:

$$r^o(p) = r^*(p).$$

While the proof of the achievability of $r^*(p)$ is fairly straightforward, the proof of the converse follows from Jensen's inequality, as we recognize that *all* allocation schemes $\mathbf{p}(f)$ are *point-wise either flat frequency sharing or flat FDMA*.

Proof of Theorem 4:

i) $r^*(p) \le r^o(p)$ (Achievability of $r^*(p)$).

The achievability of $r^*(p)$ when $0 < p \le p_f$ or $p \ge p_h$ is immediate. When $p_f \le p \le p_h$

$$r^*(p) = f^*(p_f) + \lambda\left(h^*(p_h) - f^*(p_f)\right)$$

where $\lambda = \frac{p - p_f}{p_h - p_f}$, and $r^*(p)$ is achievable by the following scheme as depicted in Fig. 7: The band of the original channel is split into two disjoint channels: C_1 with bandwidth $1-\lambda$, and C_2 with bandwidth λ.

- In C_1, a flat frequency sharing with a PSD of $p_f/2$ for each user is applied, achieving a sum-rate of $f^*_{C_1} = (1-\lambda)f^*(p_f)$.
- In C_2, a flat FDMA with a PSD of p_h for each user is applied, achieving a sum-rate of $h^*_{C_2} = \lambda h^*(p_h)$.

Note that the sum-power constraint is satisfied by such a combination of flat frequency sharing and flat FDMA

$$(1-\lambda)p_f + \lambda p_h = p.$$

Therefore, the sum-rate

$$f^*_{C_1} + h^*_{C_2} = (1-\lambda)f^*(p_f) + \lambda h^*(p_h) = r^*(p)$$

can be achieved in the original problem (16).

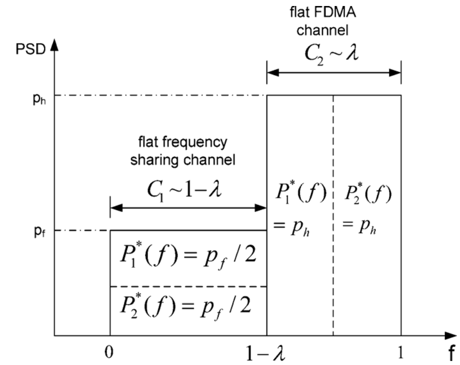

Fig. 7. The optimal spectrum management scheme as a mixture of flat FDMA and flat frequency sharing.

ii) $r^o(p) \le r^*(p)$ (Converse)

For any given p, let $\{p_1^o(f), p_2^o(f)\}$ be an optimal scheme that achieves $r^o(p)$. Define the sum-rate density function

$$r_p^o(f) \triangleq \log\left(1 + \frac{p_1^o(f)}{1+\alpha p_2^o(f)}\right) + \log\left(1 + \frac{p_2^o(f)}{1+\alpha p_1^o(f)}\right)$$

and $p^o(f) \triangleq p_1^o(f) + p_2^o(f)$. Clearly, $r^o(p) = \int_0^1 r_p^o(f) df$.

From Lemma 5, $\forall f \in [0,1]$, when $p^o(f) \le 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$

$$r_p^o(f) \le f^*(p^o(f)).$$

From Lemmas 4 and 6, $\forall f \in [0,1]$, when $p^o(f) > 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$

$$r_p^o(f) \le h^*(p^o(f)).$$

Thus, $r_p^o(f) \le \max\{f^*(p^o(f)), h^*(p^o(f))\} \le r^*(p^o(f))$, and

$$\begin{aligned} r^o(p) = \int_0^1 r_p^o(f) df &\le \int_0^1 r^*(p^o(f)) df \\ &\le r^*\left(\int_0^1 p^o(f) df\right) \le r^*(p). \end{aligned}$$

The second inequality arises from the concavity of $r^*(p)$ and Jensen's inequality, and the last inequality arises from the sum-power constraint and the fact that $r^*(p)$ is increasing. ■

The mixture of a flat frequency sharing and a flat FDMA shown in Fig. 7 represents the general form of the optimal spectrum management scheme that achieves $r^*(p)$.

The computation of the optimal spectrum management scheme is summarized in Procedure 1. Note that there always exists an optimal spectrum management scheme with two users each using the same total power of $p/2$. Therefore, the above optimal solution with a *sum*-power constraint directly leads to the optimal solution with *equal* individual power constraints:

Procedure 1

Computing the optimal spectrum management scheme for two-user symmetric frequency flat channels

1) Solve the two points of tangency p_f and p_h on the convex hull function of $\max\{f^*(x), h^*(x)\}$:

a. Solve equation (14) numerically to find p_f.

b. Compute p_h by (15).

2) Compute the maximum achievable sum-rate $r^*(p)$:

If $p \le p_f$, $r^*(p) = f^*(p)$.

Allocate $p_1(f) = p_2(f) = p/2, \forall f$.

If $p \ge p_h$, $r^*(p) = h^*(p)$.

Allocate $p_1(f)$ and $p_2(f)$ such that

$$\begin{cases} p_1(f)p_2(f) = 0, \forall f \\ p_1(f) + p_2(f) = p, \forall f \end{cases}.$$

If $p_f < p < p_h$,

$$r^*(p) = f^*(p_f) + \frac{h^*(p_h) - f^*(p_f)}{p_h - p_f}(p - p_f).$$

a. Compute $\lambda = \dfrac{p - p_f}{p_h - p_f}$

b. Separate [0,1] into two disjoint channels: C_1 with bandwidth $1-\lambda$, C_2 with bandwidth λ.

c. Allocate power as follows (Figure 7):

In C_1, $p_1(f) = p_2(f) = p_f/2$.

In C_2, $\begin{cases} p_1(f)p_2(f) = 0, \forall f \\ p_1(f) + p_2(f) = p_h, \forall f \end{cases}.$

Corollary 3: In a flat symmetric Gaussian interference channel with $\alpha < 1/2$, the maximum sum-rate defined as the optimal value of the following optimization problem

$$\begin{aligned} \max_{\mathbf{p}(f)} \quad & \int_0^1 r_1(\mathbf{p}(f), f) + r_2(\mathbf{p}(f), f) df \\ s.t. \quad & \int_0^1 p_i(f) df \le p/2, \;\; i = 1, 2, \\ & p_1(f) \ge 0, p_2(f) \ge 0, \quad \forall f \in [0, 1] \end{aligned}$$

is $r^*(p)$.

Proof: On the one hand, the equal power constraints imply the sum-power constraint. On the other hand, the optimal value with the sum-power constraint can be achieved with the equal power constraints. ∎

B. Generalizations to Frequency Selective Channels

In this section, we extend the sum-rate maximization problem to the symmetric frequency selective Gaussian interference channel

$$\begin{cases} \alpha_{12}(f) = \alpha_{21}(f) = \alpha(f), & \forall f \in [0, 1] \\ n_1(f) = n_2(f) = n(f), & \forall f \in [0, 1]. \end{cases}$$

With

$$r_1(\mathbf{p}(f), f) = \log\left(1 + \frac{p_1(f)}{n(f) + p_2(f)\alpha(f)}\right)$$
$$r_2(\mathbf{p}(f), f) = \log\left(1 + \frac{p_2(f)}{n(f) + p_1(f)\alpha(f)}\right)$$

define r^o to be the maximum achievable sum-rate with a sum-power constraint as follows:

Definition 9:

$$\begin{aligned} r^o \triangleq \max_{\mathbf{p}(f)} \quad & \int_0^1 r_1(\mathbf{p}(f), f) + r_2(\mathbf{p}(f), f) df \qquad (17) \\ s.t. \quad & \int_0^1 (p_1(f) + p_2(f)) \, df \le p \\ & p_1(f) \ge 0, p_2(f) \ge 0, \quad \forall f \in [0, 1]. \end{aligned}$$

Note that the objective function *is separable in f*. (The whole problem is, however, not immediately separable in f because of the total power constraint across the whole band.) Because *for every fixed* $f \in [0, 1]$, $r_1(\mathbf{p}(f), f) + r_2(\mathbf{p}(f), f)$ is *nonconcave* in $\{p_1(f), p_2(f)\}$, the above infinite-dimensional problem (17) is a *nonconvex optimization*. Next, we derive a *primal* domain convex relaxation of (17). We first normalize the PSD and the sum-PSD by $n(f)$.

Definition 10: At every frequency $f \in [0, 1]$

1. $\tilde{p}_1(f) \triangleq \frac{p_1(f)}{n(f)}, \tilde{p}_2(f) \triangleq \frac{p_2(f)}{n(f)}, \tilde{p}(f) \triangleq \tilde{p}_1(f) + \tilde{p}_2(f)$.
2. In the same form of (10) and (11) with $\alpha(f)$ instead of α: $f^*(p, f) \triangleq 2\log\left(1 + \frac{p/2}{1+\alpha(f)p/2}\right)$, $h^*(p, f) \triangleq \log(1 + p)$, and

$$r^*(p, f) \triangleq conv_p\left(\max\{f^*(p, f), h^*(p, f)\}\right).$$

Note that the convex hull operation is performed *along the power dimension for every fixed* f (*not along the frequency dimension.*) $\forall f \in [0, 1]$, $p_f(f)$, $p_h(f)$, and $r^*(p, f)$ are computed in the same way as in Procedure 1 with $\alpha(f)$ instead of α. In the (separable) objective function of (17), *at every frequency* f, we *replace the nonconcave* $r_1(\mathbf{p}(f), f) + r_2(\mathbf{p}(f), f)$ *with the concave* $r^*(\tilde{p}(f), f)$ (concave in the first variable $\tilde{p}(f)$), and define r^* to be the corresponding maximum achievable value as follows:

Definition 11:

$$\begin{aligned} r^* \triangleq \max_{\tilde{p}(f)} \quad & \int_0^1 r^*(\tilde{p}(f), f) \, df \qquad (18) \\ s.t. \quad & \int_0^1 \tilde{p}(f) n(f) df \le p, \;\; \tilde{p}(f) \ge 0, \; \forall f \in [0, 1]. \end{aligned}$$

Note that, *for every fixed* $f \in [0, 1]$, $r^*(\tilde{p}(f), f)$ is *concave in* $\tilde{p}(f)$. The constraint is *linear in* $\tilde{p}(f), \forall f$. Thus, the above infinite-dimensional problem (18) is a *convex optimization*. Now, we have the following theorem.

Theorem 5:

$$r^o = r^*.$$

The proof of the converse is similar to that in Theorem 4. For the proof of the achievability of r^*, as the channel is frequency selective, we need to introduce a piecewise flat ε-approximation, and the remaining proof follows that in Theorem 4.

Proof of Theorem 5:

i) $r^o \leq r^*$ (Converse).

It is sufficient to prove the inequality between the integrands in (17) and (18). From Lemmas 4, 5, and 6

$$\begin{aligned}
&r_1(\mathbf{p}(f), f) + r_2(\mathbf{p}(f), f)\\
&= \log\left(1 + \frac{\tilde{p}_1(f)}{1 + \tilde{p}_2(f)\alpha(f)}\right) + \log\left(1 + \frac{\tilde{p}_2(f)}{1 + \tilde{p}_1(f)\alpha(f)}\right)\\
&\leq \max\left\{f^*(\tilde{p}(f), f), h^*(\tilde{p}(f), f)\right\} \leq r^*(\tilde{p}(f), f)
\end{aligned}$$

ii) $r^* \leq r^o$ (Achievability).

Let sum-PSD $\tilde{p}^*(f)$ be an optimal solution of (18) such that $\int_0^1 r^*(\tilde{p}^*(f), f)\, df = r^*$. Then, $\forall \varepsilon > 0:$

By Lemma 1, based on $\tilde{p}^*(f)$ and $\{\alpha_{ji}(f)\}$, take a piecewise flat ε-approximation $\bar{p}^*(f)$ and $\{\bar{\alpha}_{ji}(f)\}$, s.t.

$$\left|\int_0^1 \bar{r}^*(\bar{p}^*(f), f)\, df - r^*\right| < \varepsilon$$

where $\bar{p}^*(f)$ is a piecewise flat *sum-PSD*, and $\bar{r}^*(\bar{p}^*(f), f)$ is computed with $\bar{p}^*(f)$ and $\{\bar{\alpha}_{ji}(f)\}$. (Note that, since the noise PSD is already normalized to 1 as in Definition 10, no further piecewise flat approximation of the noise is needed.)

Based on the piecewise flat ε-approximation, *in every flat sub-channel with a flat* $\bar{p}^*(f)$, as in the proof of Theorem 4, $\bar{r}^*(\bar{p}^*(f), f)$ can be achieved by further dividing this flat sub-channel into two sub-bands, applying a flat frequency sharing and a flat FDMA, respectively (cf., Fig. 7). Removing the normalization by multiplying by $n(f)$, we denote the resulting allocation scheme by $\mathbf{p}^o(f) = [p_1^o(f), p_2^o(f)]^T$, which achieves the same sum-rate

$$\int_0^1 \bar{r}_1(\mathbf{p}^o(f), f) + \bar{r}_2(\mathbf{p}^o(f), f)\, df = \int_0^1 \bar{r}^*(\bar{p}^*(f), f)\, df$$

where $\bar{r}_1(\mathbf{p}^o(f), f)$ and $\bar{r}_2(\mathbf{p}^o(f), f)$ are computed with the piecewise flat approximate channel responses $\{\bar{\alpha}_{ji}(f)\}$.

Then

$$\begin{aligned}
r^o \geq &\int_0^1 r_1(\mathbf{p}^o(f), f) + r_2(\mathbf{p}^o(f), f) df\\
\geq &\int_0^1 \bar{r}_1(\mathbf{p}^o(f), f) + \bar{r}_2(\mathbf{p}^o(f), f) df\\
= &\int_0^1 \bar{r}^*(\bar{p}^*(f), f)\, df > r^* - \varepsilon
\end{aligned}$$

where the first inequality occurs because $\mathbf{p}^o(f)$ is a *feasible* solution of (17); the second inequality arises because (by P2 from Lemma 1) $\{\bar{\alpha}_{ji}(f)\} \geq \{\alpha_{ji}(f)\}, \forall i, j, \forall f$, i.e., the ε-approximation leads to worse channel responses, resulting in lower rates. Finally, let $\varepsilon \to 0$. ■

Therefore, although the integrand in (18) is a convex relaxation of that in (17), the *optimal objective value of the problem does not change*, and the original nonconvex optimization (17) is equivalently transformed to the convex optimization (18). Finally, for the same reasons as in Section IV-A, the optimal solution with equal individual power constraints is the same as that with a corresponding sum-power constraint.

Remark 3: Throughout this section, we have worked with a sum-power constraint for brevity in derivations of the results for the fully symmetric cases. One may also derive the results directly with equal individual power constraints. In Section V, as we consider potentially *asymmetric* channels, we will directly work with individual power constraints.

V. Optimal Spectrum Management in the General Cases

In Section IV, we solved the sum-rate maximization problem in two-user symmetric frequency selective channels with equal power (or sum-power) constraints. In this section, we make the following generalizations:

1. two-user $\to$ K-user;
2. equal power constraints $\to$ arbitrary individual power constraints;
3. symmetric channels $\to$ arbitrary (including asymmetric) channels;
4. sum-rate $\to$ weighted sum-rate.

The general optimization problem is thus the following:

$$\max_{\mathbf{p}(f)} \quad \int_0^1 \sum_{i=1}^K w_i r_i\left(\mathbf{p}(f), f\right) df \tag{19}$$

$$s.t. \quad r_i\left(\mathbf{p}(f), f\right) = \log\left(1 + \frac{p_i(f)}{n_i(f) + \sum\limits_{j\neq i} p_j(f)\alpha_{ji}(f)}\right)$$

$$\int_0^1 \mathbf{p}(f) df \leq \mathbf{p}, \quad \mathbf{p}(f) \geq 0, \quad \forall f \in [0,1].$$

Next, we analyze (19) in parallel with the analysis in Section IV, and generalize the basic ideas in Section IV.

A. Optimal Solutions for Frequency Flat Channels

Consider a K-user (potentially asymmetric) frequency flat channel

$$\alpha_{ji}(f) = \alpha_{ji},\ n_i(f) = n_i, \forall f \in [0,1], \forall i, j.$$

First, consider the weighted sum-rate achieved with *flat* power allocations $\mathbf{p}(f) = \mathbf{P}, \forall f \in [0,1]$ defined as

$$R(\mathbf{P}) \triangleq \sum_{i=1}^K w_i \log\left(1 + \frac{P_i}{n_i + \sum\limits_{j\neq i} P_j \alpha_{ji}}\right). \tag{20}$$

Denote its K-dimensional convex hull function by

$$R^*(\mathbf{P}) \triangleq conv_{\mathbf{P}}\left(R(\mathbf{P})\right). \tag{21}$$

We have the following lemma on the monotonicity of $R^*(\mathbf{P})$ whose proof is relegated to Appendix C.

Lemma 7: $R^*(\mathbf{P})$ is strictly increasing in every component of $\mathbf{P}, \forall \mathbf{P} \leq \mathbf{0}$.

Next, the original problem (19) in frequency flat channels can be rewritten as

Definition 12:

$$R^o(\mathbf{p}) \stackrel{\triangle}{=} \max_{\mathbf{p}(f)} \int_0^1 R(\mathbf{p}(f))\, df$$
$$s.t. \int_0^1 \mathbf{p}(f) df \leq \mathbf{p}, \quad \mathbf{p}(f) \geq 0, \ \forall f \in [0,1].$$

Now, we have the following theorem.

Theorem 6:

$$R^o(\mathbf{p}) = R^*(\mathbf{p})$$

and the optimal spectrum and power allocation $\mathbf{p}^o(f)$ consists of $K+1$ sub-bands, with $\mathbf{p}^o(f)$ flat in each of the sub-bands.

Proof: The proof is in parallel with that of Theorem 4.

1. $R^*(\mathbf{p}) \leq R^o(\mathbf{p})$ (Achievability).
   As $R^*(\mathbf{p}) = conv_{\mathbf{p}}(R(\mathbf{p}))$, by Carathéodory's theorem
   $$\exists \sum_{k=1}^{K+1} c^{(k)} = 1, \quad \sum_{k=1}^{K+1} c^{(k)}\mathbf{p}^{(k)} = \mathbf{p}, \quad c^{(k)} \geq 0, \ s.t.$$
   $$R^*(\mathbf{p}) = \sum_{k=1}^{K+1} c^{(k)} R(\mathbf{p}^{(k)}).$$
   Accordingly, we can divide the band $[0,1]$ into $K+1$ sub-bands, each with a bandwidth of $c^{(k)}$ and uses the flat power levels of $\mathbf{p}^{(k)} = [p_1^{(k)}, \ldots, p_K^{(k)}]$ for the K users.
2. $R^o(\mathbf{p}) \leq R^*(\mathbf{p})$ (Converse).
   For any feasible allocation scheme $\mathbf{p}(f), f \in [0,1]$
   $$\int_0^1 R(\mathbf{p}(f))df \leq \int_0^1 R^*(\mathbf{p}(f))df \leq R^*\left(\int_0^1 \mathbf{p}(f)df\right) \leq R^*(\mathbf{p})$$
   where the first inequality is from definition (21), the second inequality arises from Jensen's inequality, and the third inequality arises from Lemma 7 that $R^*(\mathbf{P})$ is increasing in $\mathbf{P}$.

■

Remark 4: In the literature, it was first shown that allocation schemes consisting of $2K$ sub-bands of *frequency flat* power allocations are sufficient to achieve any Pareto optimal solution [13], and this sufficient number of sub-bands was later refined to $K+1$[19]. From Theorem 6, the sufficiency of $K+1$ sub-bands is also immediately implied by the fact that the optimal value and solution are obtained by computing the convex hull function (21) of a nonconcave function (20).

Now, for the special case of two-user symmetric frequency flat channels as discussed in Section IV-A, we compare $R^*(\mathbf{P})$ (21) and $r^*(p)$ (13) as follows.

1) $R^*(\mathbf{P})$ is defined over the 2-D nonnegative quadrant of *two individual powers*, and is the convex hull function of the achievable rate of *flat frequency sharing*.
2) $r^*(p)$ is defined over the 1-D nonnegative half-line of *sum-power*, and is the convex hull function of the point-wise maximum of the achievable rates of *flat frequency sharing and flat FDMA*.
3) By Theorems 4 and 6, $R^*(\frac{p}{2}, \frac{p}{2}) = r^*(p)$.

As previously mentioned in Remark 3, we see that the optimal solution in two-user symmetric frequency flat channels, solved in Section IV-A, can also be solved using the general approach derived in this section. In Section IV-A, by exploiting the *symmetry* of the channel and the power constraints, we showed that $R^*(\frac{p}{2}, \frac{p}{2}), p \geq 0$ can be characterized in a simpler form, namely, $r^*(p)$. Finally, as mentioned in Remark 2, the power allocation in each sub-band of a flat FDMA allocation can be viewed as a special case of flat frequency sharing with only one user's power strictly positive. This explains the intuition of why, to have Theorem 6, it is sufficient to define $R^*(\mathbf{P})$ as in (21) without explicitly considering flat FDMA as in $r^*(p)$(13).

B. Generalizations to Frequency Selective Channels

In frequency selective channels, define the *weighted sum-rate density function* as

$$R(\mathbf{P}, f) \stackrel{\triangle}{=} \sum_{i=1}^{K} w_i r_i(\mathbf{P}, f). \tag{22}$$

Problem (19) can then be rewritten as follows.

Definition 13:

$$R^o \stackrel{\triangle}{=} \max_{\mathbf{p}(f)} \int_0^1 R(\mathbf{p}(f), f)\, df \tag{23}$$
$$s.t. \int_0^1 \mathbf{p}(f) df \leq \mathbf{p}, \quad \mathbf{p}(f) \geq 0, \ \forall f \in [0,1].$$

Note that, for every fixed $f \in [0,1]$, $R(\mathbf{p}(f), f)$ is nonconcave in $\mathbf{p}(f)$, and (23) is an infinite-dimensional nonconvex optimization. At every frequency $f \in [0,1]$, define

$$R^*(\mathbf{P}, f) \stackrel{\triangle}{=} conv_{\mathbf{P}} R(\mathbf{P}, f)$$

i.e., *the convex hull function of* $R(\mathbf{P}, f)$ *along the* K *dimensions of power* $\mathbf{P}$. Note that the convex hull operation is *not* taken along the frequency dimension f. ($R^*(\mathbf{P}, f)$) is concave in $\mathbf{P}$ for every fixed f, but not necessarily jointly concave in $\mathbf{P}$ and f.)

Next, we derive the following primal domain convex relaxation of (23): *At every frequency* f, we replace the nonconcave $R(\mathbf{p}(f), f)$ with the concave $R^*(\mathbf{p}(f), f)$ (concave in the first variable $\mathbf{p}(f)$), and define R^* to be the corresponding maximum achievable value as follows.

Definition 14:

$$R^* \stackrel{\triangle}{=} \max_{\mathbf{p}(f)} \int_0^1 R^*(\mathbf{p}(f), f) df \tag{24}$$
$$s.t. \int_0^1 \mathbf{p}(f) df \leq \mathbf{p}, \quad \mathbf{p}(f) \geq 0, \ \forall f \in [0,1].$$

Clearly, (24) is an infinite-dimensional convex optimization, because $\forall f \in [0,1]$ the integrand is a concave function of the variables $\{\mathbf{p}(f)\}$, and the constraint is linear in $\{\mathbf{p}(f)\}$. Now, we have the following theorem whose proof is in parallel with that of Theorem 5 and is relegated to Appendix C.

Theorem 7:

$$R^o = R^*.$$

We see that the optimal value for the nonconvex optimization (23) *equals* that of its convex relaxation (24).

C. On the Complexity of Solving the General Problem

For general bounded piecewise continuous channel responses, problem (23) can have up to an uncountably infinite number of dimensions, for which describing the complexity of solving the continuous frequency optimal solution is pointless. However, one can first *approximate* the channel responses by *piecewise flat* functions of frequency, which is the approach with which spectrum management problems are addressed in practice. With piecewise frequency flat channel responses, denote the corresponding flat sub-channels by $I_1, I_2, \ldots, I_M$, each with a given bandwidth b_m.

One can consider two types of problems distinguished by the assumptions on power allocations.

Case a) $\mathbf{p}(f)$ consists of bounded piecewise continuous functions.

Case b) $\mathbf{p}(f)$ must be *flat in every flat sub-channel* I_m.

For example, consider a single frequency flat band. It makes a fundamental difference whether we allow a user to freely subdivide this flat band and use different PSD in different sub-bands. If so, it is Case a, and the problem model is still *continuous* frequency; otherwise, it is Case b, and it corresponds to the *discrete* frequency model.

For the *discrete* frequency model (Case b), it has been proven that finding the optimal solution of weighted sum-rate maximization is NP hard in *both the number of users K and the number of sub-channels M* [16]. Next, we discuss the complexity of solving the *continuous* frequency (Case a) optimal spectrum management problem (23) in *piecewise frequency flat channels*. From Theorem 7, it is sufficient to solve the convex optimization (24), which consists of two general steps.

Step 1: Compute the convex hull function $R^*(\mathbf{P}, f)$ for every frequency flat sub-channel I_m, $m = 1, \ldots, M$.

Step 2: Optimize $\mathbf{p}(f)$ with the objective $\int_0^1 R^*(\mathbf{p}(f), f) df$

In Step 1, given the channel parameters for each flat sub-channel I_m, $m = 1, \ldots, M$, a convex hull function $R_m^*(\mathbf{P})(\triangleq R^*(\mathbf{P}, f), f \in I_m,) \forall \mathbf{P} \geq 0$ is computed. In Step 2, given the M convex hull functions for all the flat sub-channels, as the number of sub-channels M is *finite*, problem (24) becomes *finite dimensional*, with an *increasing concave* utility function $R_m^*(\mathbf{p}(f))$ in each sub-channel I_m. Now, because each I_m has *frequency flat channel* parameters and $R_m^*(\mathbf{P})$ is increasing and concave, by Jensen's inequality, the optimal solution must satisfy that $\mathbf{p}(f)$ is *flat* in *each* sub-channel I_m, i.e., $\forall m = 1, \ldots, M, \exists \mathbf{p}(m) \geq 0, s,t. \mathbf{p}(f) = \mathbf{p}(m), \forall f \in I_m$

Problem (24) then becomes

$$\max_{\mathbf{p}(m)} \sum_{m=1}^{M} b_m \cdot R_m^* (\mathbf{p}(m))$$
$$s.t. \sum_{m=1}^{M} b_m \cdot \mathbf{p}(m) \leq \mathbf{p},\ \mathbf{p}(m) \geq 0,\ \forall m = 1, 2, \ldots, M. \tag{25}$$

(Note that b_m $(m = 1, \ldots, M)$ is the bandwidth of sub-channel I_m, and is *not* an optimization variable). Problem (25) is a *finite-dimensional convex optimization* problem that has efficient polynomial time algorithms to solve the global optimal solution. (For example, a dual decomposition algorithm works; see, e.g., [7] among many others.) In particular, the computational complexity of (25) grows linearly in the number of sub-channels M [7]. Finally, the optimal solution of (25) which is also the optimal solution of (24), denoted by $\mathbf{p}^*(m)$, $m = 1, \ldots, M$, is transformed back to the optimal solution of (23): In each sub-channel I_m, $m = 1, \ldots, M$, as $R_m^* (\mathbf{p}^*(m))$ is formed by *a convex combination of at most $K+1$ points of $R_m(\mathbf{p}(m))$* (cf., Remark 4), we further subdivide the sub-channel I_m into (at most) $K+1$ sub-bands in each of which a corresponding flat frequency sharing scheme is used.

We see that the critical complexity in solving the general problem (23) based on Theorem 7 lies in computing convex hull functions $(R_m^*(\mathbf{P}),\ m = 1, \ldots, M)$. Computing K-dimensional convex hull functions is known to be NP hard in the number of users K [12]. We note that similar complexities from computing convex hulls also appear in [5] where the objective is to find the optimal time shared power transmission modes in single carrier networks for network utility maximization.

Thus, the overall computational complexity of the above two-step approach is NP in K (although this does not directly follow from the results in [16] because the assumptions are different, i.e., Case a versus Case b). Nonetheless, this two-step method does provide the following advantage:

Remark 5: Once the channel parameters are given, the M convex hull functions $R_m^*(\mathbf{P})$ are computed for one time, consuming an NP complexity in the number of users K. Then, no matter how the power constraints may vary due to problem needs, the additional computational complexity of solving the optimal solution (Step 2) grows linearly in the number of sub-channels M.

We note that, for the discrete frequency model (Case b), the constraint that a user must use a flat PSD within every (flat) sub-channel leads to the well known NP hardness in both K and M. In comparison, for the continuous frequency model (Case a), the main complexity is from computing convex hull functions which is NP in K. Finally, better approximation of the continuous frequency channel can be obtained by increasing the number of sub-channels M in the piecewise flat channel approximation. This, however, does not lead to prohibitively greater computational cost since the overall complexity grows linearly in M for the continuous frequency model (cf., Remark 5).

D. On the Zero Duality Gap

It has been proven that the continuous frequency nonconvex optimization (23) has an *exact* zero duality gap [16], [20]. It is pointed out that the zero duality gap comes from a *time sharing condition* [20]. It is also proved using the nonatomic property of the Lebesgue measure [16].

We show that this is also immediately implied by Theorem 7.

Definition 15: For problem (23), its Lagrange dual is defined as

$$L(\mathbf{p}(f), \boldsymbol{\lambda}) \triangleq \int_0^1 R(\mathbf{p}(f), f) \ \ df - \boldsymbol{\lambda}^T (\ \ \int_0^1 \mathbf{p}(f) df - \mathbf{p}).$$

Its dual objective and dual optimal value are defined as

$$g(\boldsymbol{\lambda}) \triangleq \sup_{\mathbf{p}(f)\geq 0} L(\mathbf{p}(f),\boldsymbol{\lambda}), \text{ and } \quad D^o \triangleq \min_{\boldsymbol{\lambda}\geq 0} g(\boldsymbol{\lambda}).$$

Similarly, for problem (24), its Lagrange dual, dual objective, and dual optimal value are defined as

$$\hat{L}(\mathbf{p}(f),\boldsymbol{\lambda}) \triangleq \int_0^1 R^*(\mathbf{p}(f),f)\,df - \boldsymbol{\lambda}^T\left(\int_0^1 \mathbf{p}(f)df - \mathbf{p}\right)$$

$$\hat{g}(\boldsymbol{\lambda}) \triangleq \sup_{\mathbf{p}(f)\geq 0} \hat{L}(\mathbf{p}(f),\boldsymbol{\lambda}), \text{ and } \quad D^* \triangleq \min_{\boldsymbol{\lambda}\geq 0} \hat{g}(\boldsymbol{\lambda}).$$

Corollary 4: The nonconvex optimization (23) has a zero duality gap.

Proof: Since $R^*(\mathbf{P},f) \geq R(\mathbf{P},f), \forall \mathbf{P}, \forall f$, we have

$$\hat{L}(\mathbf{p}(f),\boldsymbol{\lambda}) \geq L(\mathbf{p}(f),\boldsymbol{\lambda}) \Rightarrow \hat{g}(\boldsymbol{\lambda}) \geq g(\boldsymbol{\lambda}), \quad \forall \boldsymbol{\lambda} \geq 0$$
$$\Rightarrow D^* \geq D^o.$$

Note that the primal optimal values for (23) and (24) are R^o and R^*. Therefore

$$R^* = D^* \geq D^o \geq R^o = R^* \Rightarrow D^o = R^o$$

where the first equality occurs because problem (24) is a convex optimization and has *strong* duality [3]; the second inequality is from the *weak* duality of the nonconvex optimization (23); the key step is the second equality $R^o = R^*$ from Theorem 7. ■

Furthermore, it has been shown that, under mild technical conditions, the nonconvex optimization for the *discrete* frequency model has an *asymptotically* zero duality gap as the number of sub-channels goes to infinity [20]. The result is rigorously generalized to include Lebesgue integrable PSDs in [16]. Indeed, for a bounded piecewise continuous frequency channel, as it is divided into more and finer/flatter sub-channels, the difference between the power allocation assumptions Cases a and b vanishes (discrete frequency model → continuous frequency model.) The intuition is that we can *bundle a large number of similar frequency flat* sub-channels, treat them as *one combined frequency flat channel*, compute the *continuous* frequency power allocation, and accordingly distribute the power within these roughly identical sub-channels (as a discrete approximation of the continuous allocation.)

VI. Conclusion

In this paper, we considered two general problems for continuous frequency optimal spectrum management in Gaussian interference channels: 1) the channel conditions under which FDMA schemes are Pareto optimal; and 2) equivalent convex formulations for the nonconvex weighted sum-rate maximization problem.

First, we have shown that for any two (among K) users, as long as the product of the two normalized cross channel gains between them is greater than or equal to 1/4, an FDMA allocation between these two users *benefits every one of the K users*. Therefore, under this pair-wise condition, any Pareto optimal point of the K-user rate region can be achieved with this pair of users using orthogonal channels. The pair-wise nature of the condition allows a completely distributed decision on whether any two users should use orthogonal channels, without loss of any Pareto optimality.

Next, we have shown that the classic nonconvex weighted sum-rate maximization in K-user asymmetric frequency selective channels can be equivalently transformed in the primal domain to a convex optimization. We first analyzed in detail the sum-rate maximization in two-user symmetric frequency flat channels, and showed that the optimal solution consists of one sub-band of flat frequency sharing, and one sub-band of flat FDMA. We generalized the results to weighted sum-rate maximization in K-user asymmetric frequency flat channels: we showed that the optimal value is computed as the convex hull function of the nonconcave objective function, and the piecewise frequency flat optimal solution is obtained based on the convex combination used in computing the point on the convex hull function. Finally, a primal domain convex formulation is established for frequency selective channels. For piecewise frequency flat channels, we showed that the overall computational complexity is NP in the number of users K from computing convex hull functions, and is linear in the number of sub-channels M.

This paper has focused on providing a unified and in-depth view on solving the optimal spectrum management problem for the *continuous* frequency model. The multicarrier *discrete* frequency model is different from (although related to) the continuous frequency model (even with piecewise frequency flat channel responses). As problems with the discrete frequency model are in general NP-complete in both K and M, finding practical algorithms to find approximately optimal solutions has attracted many research endeavors, and continues to be very interesting.

Appendix A

Proof of Lemma 1: First, we prove for the case that $\mathbf{p}(f), \{\alpha_{ji}(f)\}$ and $\{n_i(f)\}$ are *bounded continuous in* $[0,1]$, (*not piecewise*.) It is then immediate to generalize to bounded *piecewise* continuous functions with a *finite* number of discontinuities.

1) Since $U(\mathbf{p},\boldsymbol{\alpha},\mathbf{n})$ is a uniformly continuous function of $\{\{p_i\},\{\alpha_{ji}\},\{n_i\}\}$, $\forall \varepsilon > 0$, $\exists \varepsilon' > 0$,, s.t. $\forall \mathbf{p}^{(1)}, \boldsymbol{\alpha}^{(1)}, \mathbf{n}^{(1)}$ and $\mathbf{p}^{(2)}, \boldsymbol{\alpha}^{(2)}, \mathbf{n}^{(2)}$ satisfying

$$|p_i^{(1)} - p_i^{(2)}| < \varepsilon', |\alpha_{ji}^{(1)} - \alpha_{ji}^{(2)}| < \varepsilon', |n_i^{(1)} - n_i^{(2)}| < \varepsilon', \forall i,j$$

we have $|U(\mathbf{p}^{(1)}, \boldsymbol{\alpha}^{(1)}, \mathbf{n}^{(1)}) - U(\mathbf{p}^{(2)}, \boldsymbol{\alpha}^{(2)}, \mathbf{n}^{(2)})| < \varepsilon$.

2) For $\mathbf{p}(f), \{\alpha_{ji}(f)\}$, and $\{n_i(f)\}$, since bounded continuity implies uniform continuity, $\forall \varepsilon' > 0$, $\exists \delta > 0$, s.t. $\forall |f_1 - f_2| < \delta$, s. t. $f_1, f_2 \in [0,1]$, we have

$$|p_i(f_1) - p_i(f_2)| < \varepsilon', |n_i(f_1) - n_i(f_2)| < \varepsilon', \forall i$$
$$|\alpha_{ji}(f_1) - \alpha_{ji}(f_2)| < \varepsilon', \forall i,j.$$

Now, combining 1) and 2), $\forall \varepsilon > 0$, divide $[0,1]$ into consecutive intervals $I_1,\ldots,I_M$ with lengths all less than δ. $\forall m = 1,\ldots,M, \forall f \in I_m$, let

$$\bar{p}_i(f) = \min_{f\in I_m} p_i(f), \forall i$$
$$\bar{\alpha}_{ji}(f) = \max_{f\in I_m} \alpha_{ji}(f),\ \forall i \neq j,\ \bar{n}_i(f) = \max_{f\in I_m} n_i(f),\ \forall i.$$

Thus, Properties P1 and P2 are satisfied, and $\forall f \in [0,1], \forall i, j$

$$|p_i(f)-\bar{p}_i(f)|<\varepsilon', |n_i(f)-\bar{n}_i(f)|<\varepsilon', |\alpha_{ji}(f)-\bar{\alpha}_{ji}(f)|<\varepsilon'$$
$$\Rightarrow |U(\bar{\mathbf{p}}(f),\bar{\boldsymbol{\alpha}}(f),\bar{\mathbf{n}}(f))-U(\mathbf{p}(f),\boldsymbol{\alpha}(f),\mathbf{n}(f))|<\varepsilon.$$

Thus, we have proved the lemma for bounded continuous power allocations and channel responses. To generalize it to bounded piecewise continuous cases, simply use the fact that the number of discontinuities in $\mathbf{p}(f), \{\alpha_{ji}(f)\}$, and $\{n_i(f)\}$ are *finite*. Thus, we can construct piecewise flat functions $\bar{\mathbf{p}}(f)$, $\{\bar{\alpha}_{ji}(f)\}$, and $\{\bar{n}_i(f)\}$ in every sub-interval with bounded continuity, and the values on the discontinuities do not have any impact on the power and rates, as they form a set of measure zero. ■

APPENDIX B

Proof of Lemma 2: We want to show

$$\frac{df(x)}{dx} = \frac{2cx-(c^2-x^2)\log(\frac{c+x}{c-x})}{x^2(c^2-x^2)} \geq 0,\ \forall x \in (0,1].$$

Since $c > 1$, it is equivalent to show

$$\frac{2cx}{c^2-x^2} \geq \log(\frac{c+x}{c-x}),\ x \in (0,1].$$

Let $g(x) = \frac{2cx}{c^2-x^2}$ and $h(x) = \log(\frac{c+x}{c-x}),\quad x \in [0,\ 1]$. We have $\frac{dg(x)}{dx} = \frac{2c(c^2+x^2)}{(c^2-x^2)^2} = \frac{dh(x)}{dx}\frac{c^2+x^2}{c^2-x^2} \geq \frac{dh(x)}{dx}$.

Since $g(0) = h(0) = 0$, we have $g(x) \geq h(x),\ \forall x \in [0,\ 1]$. Thus, $\frac{df(x)}{dx} \geq 0,\ \forall x \in (0,1], \Rightarrow f(1) \geq f(x), \forall x \in (0,\ 1]$. ■

Proof of Theorem 2: It is sufficient to prove for the case of $\alpha_{ji}\alpha_{ij} = 1/4,\ \forall i \neq j$, since we are treating interference as noise. Generalizing (5), we apply a flat FDMA power reallocation with the following specific bandwidths:

$$W_i' = \frac{p_i}{p_i + \sum\limits_{j\neq i} 2\alpha_{ji}p_j},\quad \forall i = 1,\ldots,K. \tag{26}$$

Accordingly, $p_i' = p_i + \sum\limits_{j\neq i} 2\alpha_{ji}p_j,\quad \forall i = 1,\ldots,K$.

From exactly the same argument as in the proof of Theorem 1

$$R_i' \geq R_i,\ \forall i = 1,\ldots,K.$$

Now, the final and main part of the proof is to show that $\sum\limits_{i=1}^{K} W_i' \leq 1$, meaning that (26) is a feasible FDMA reallocation.

We use induction as follows.

1) For $K = 2$, as shown in the proof of Theorem 1, $\sum\limits_{i=1}^{K} W_i' = 1$.
2) Suppose for $K = N-1 \geq 2$, we always have $\sum\limits_{i=1}^{K} W_i' \leq 1$. Consider $K = N$.
   For users $2,\ldots,N$, we rewrite their reallocation bandwidths (26) as, $\forall i = 2,\ldots,N$

$$W_i' = \frac{p_i}{2\alpha_{1i}p_1 + p_i + \sum\limits_{j\neq i, j\neq 1} 2\alpha_{ji}p_j} = \frac{p_i}{\frac{p_1}{2\alpha_{i1}} + p_i + S_i}$$

where we define $S_i \triangleq \sum\limits_{j\neq i, j\neq 1} 2\alpha_{ji}p_j$, and we have used the fact that $\alpha_{1i}\alpha_{i1} = 1/4$. Note that, if $p_1 = 0$, then $W_i' = \frac{p_i}{p_i+S_i} (i \geq 2)$ is exactly the reallocation bandwidth corresponding to the case of $K = N-1$. From the induction assumption, we have

$$\Omega \triangleq \sum_{i=2}^{N} \frac{p_i}{(p_i + S_i)} \leq 1. \tag{27}$$

For notational simplicity, we use the following change of variables: $\forall i = 2,\ldots,N,\quad \beta_i \triangleq 2\alpha_{i1}$. Accordingly

$$\sum_{i=1}^{N} W_i' = W_1' + \sum_{i=2}^{N} W_i' = \frac{p_1}{p_1 + \sum\limits_{i=2}^{N}\beta_i p_i} + \sum_{i=2}^{N}\frac{p_i}{\frac{p_1}{\beta_i} + p_i + S_i}. \tag{28}$$

We now prove that $\sum\limits_{i=1}^{N} W_i' \leq 1$, *for any* $\{\beta_i > 0,\ i = 2,\ldots,N\}$. To do so, we *optimize* over $\{\beta_i > 0,\ i = 2,\ldots,N\}$ such that $\sum\limits_{i=1}^{N} W_i'$ is maximized. We first take the following partial derivative, $\forall j \geq 2$

$$\frac{\partial \sum\limits_{i=1}^{N} W_i'}{\partial \beta_j} = p_1 p_j \left(\frac{1}{(p_1+\beta_j p_j+\beta_j S_j)^2} - \frac{1}{\left(p_1+\beta_j p_j + \sum\limits_{i\neq j, i\geq 2}\beta_i p_i\right)^2} \right).$$

Clearly, for fixed $\{\beta_i > 0,\ i \neq j\}$, there is a unique $\beta_j > 0$ that maximizes $\sum\limits_{i=1}^{N} W_i'$

$$\beta_j = \frac{\sum\limits_{i\neq j, i\geq 2}\beta_i p_i}{S_j}. \tag{29}$$

Since $\sum\limits_{i=1}^{N} W_i'$ is upper bounded by N, we have the following:

1) There exists a globally optimal solution $\{\beta_2^*, \beta_3^*, \ldots, \beta_N^*\}$ that maximizes $\sum\limits_{i=1}^{N} W_i'$.
2) From (29), the globally optimal solution must satisfy the following set of linear equations:

$$\beta_j^* S_j - \sum_{i\neq j, i\geq 2}\beta_i^* p_i = 0,\quad \forall j = 2,3,\ldots,N. \tag{30}$$

With straightforward calculations, (30) implies the following:

$$\beta_i^*(p_i + S_i) = \beta_j^*(p_j + S_j),\quad \forall 2 \leq i \neq j \leq N. \tag{31}$$

Equation (31) implies that the globally optimal solution must take the following form:

$$\beta_i^* = \frac{C}{(p_i + S_i)},\quad \forall i = 2,3,\ldots,N \tag{32}$$

where $C > 0$ is some constant.

Substitute (32) for β_i in (28), we have

$$\sum_{i=1}^{N} W_i' = \frac{p_1}{p_1 + \sum_{i=2}^{N} \frac{Cp_i}{(p_i+S_i)}} + \sum_{i=2}^{N} \frac{p_i}{\frac{p_1(p_i+S_i)}{C} + p_i + S_i} = \frac{p_1}{p_1 + C\Omega} + \frac{C\Omega}{p_1 + C} \quad (33)$$

where Ω is the total bandwidth for the case of $K = N-1$, as defined in (27). Finally, from the induction assumption that $\Omega \le 1$, we have

$$(33) = \frac{p_1^2 + Cp_1 + p_1 C\Omega + C^2\Omega^2}{p_1^2 + Cp_1 + p_1 C\Omega + C^2\Omega} \le 1.$$

■

Appendix C

Proof of Lemma 4: With flat frequency sharing, the rates of users 1 and 2 are

$$R_1 = \log(1 + \frac{p_1}{n + p_2\alpha}), \ \ R_2 = \log(1 + \frac{p_2}{n + p_1\alpha}).$$

With a flat FDMA reallocation, $W_1' = p_1/(p_1+p_2)$ implies that $p_1' = p_1 + p_2$, and $W_2' = p_2/(p_1+p_2)$ implies that $p_2' = p_1 + p_2$. The rates of users 1 and 2 become

$$R_1' = \frac{p_1}{p_1+p_2}\log(1+\frac{p_1+p_2}{n}), R_2' = \frac{p_2}{p_1+p_2}\log(1+\frac{p_1+p_2}{n}).$$

Straightforward calculations lead to

$$R_1 + R_2 \le R_1' + R_2' \Leftrightarrow p_1 p_2\left(\alpha^2(p_1+p_2) - (2\alpha - 1)\right) \ge 0$$

which implies the conclusion of Lemma 4. ■

Proof of Lemma 5: Clearly, the condition of p implies $(p_1, p_2) \in P^c_{\text{FDMA}}$.

First, we find the solution to the optimization problem with an equality sum-power constraint instead of inequality, i.e.,

$$\max\ f(p_1, p_2), \ \ s.t.\ p_1 + p_2 = p,\ p_1 \ge 0, p_2 \ge 0.$$

With $p_1 + p_2 = p$

$$f(p_1,p_2) = \left(\log(1 + \frac{p_1}{1+\alpha(p-p_1)}) + \log(1 + \frac{p-p_1}{1+\alpha p_1})\right) \triangleq \tilde{f}(p_1).$$

Straightforward calculations lead to

$$\frac{d}{dp_1}\tilde{f}(p_1) = \frac{(1+\alpha p)(\alpha^2 p + 2\alpha - 1)(p - 2p_1)}{(1+\alpha(p-p_1))\,(1+\alpha(p-p_1)+p_1)\,(1+\alpha p_1)(1+\alpha p_1 + p - p_1)}.$$

Since $0 < p \le 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$, when $0 \le p_1 \le p/2$, $\tilde{f}(p_1)$ is nondecreasing. Furthermore, note that $\tilde{f}(p_1) = \tilde{f}(p - p_1)$, i.e., $\tilde{f}(p_1)$ is *symmetric* about $p_1 = p/2$. Therefore, $\tilde{f}(p_1)$ takes the maximum value $2\log\left(1 + \frac{p/2}{1+\alpha p/2}\right)$ when $p_1 = p/2$. With straightforward calculations, one can verify that $\log\left(1 + \frac{p/2}{1+\alpha p/2}\right)$ is an *increasing concave* function of p.

Consequently, the constraint $p_1 + p_2 \le p$ in the definition problem of $f^*(p)$ (9) can be equivalently replaced by $p_1 + p_2 = p$, and we have $f^*(p) = 2\log\left(1 + \frac{p/2}{1+\alpha p/2}\right)$. ■

Proof of Lemma 7: We use the proof by contradiction.

Suppose in some component of $\mathbf{P}$, $R^*(\mathbf{P})$ is not everywhere strictly increasing. WLOG, assume this component is P_1. Thus

$$\begin{aligned} &\exists \mathbf{P}^{(0)} \triangleq \left[P_1^{(0)}, P_2^{(0)}, \ldots, P_K^{(0)}\right]^T \\ &\mathbf{P}^{(1)} \triangleq \left[P_1^{(0)} + \Delta P_1, P_2^{(0)}, \ldots, P_K^{(0)}\right]^T \\ &\mathbf{P}^{(0)} \ge \mathbf{0}, \ \ \Delta P_1 > 0, \quad s.t.\ \ R^*(\mathbf{P}^{(0)}) \ge R^*(\mathbf{P}^{(1)}). \end{aligned} \quad (34)$$

Because $R^*(\mathbf{P})$ is a jointly concave function, *it is concave in P_1 for any fixed $P_2, P_3, \ldots, P_K$*. From this concavity and (34), for fixed $P_2 = P_2^{(0)}, \ldots, P_K = P_K^{(0)}$, $R^*(\mathbf{P})$ must be nonincreasing in P_1, $\forall P_1 > P_1^{(0)} + \Delta P_1$. In particular

$$\lim_{P_1\to\infty} R^*(P_1, P_2^{(0)}, \ldots, P_K^{(0)}) \le R^*(\mathbf{P}^{(0)}) < \infty. \quad (35)$$

However, from (20), $\lim_{P_1\to\infty} R(P_1, P_2^{(0)}, \ldots, P_K^{(0)}) \to \infty$. This contradicts with $R^*(\mathbf{P}) = conv_{\mathbf{P}}\left(R(\mathbf{P})\right) \ge R(\mathbf{P})$, $\forall \mathbf{P} \ge 0$ (cf., Definition 7). ■

Proof of Theorem 7:

1. $R^o \le R^*$ (Converse).
   It is immediately true, because the integrands in (23) and (24) by definition satisfy

$$R(\mathbf{p}(f), f) \le R^*(\mathbf{p}(f), f), \forall f \in [0,1].$$

2. $R^* \le R^o$ (Achievability).
   Let $\mathbf{p}^*(f)$ be an optimal solution of (24) such that $\int_0^1 R^*(\mathbf{p}^*(f), f)df = R^*$. Then, $\forall \varepsilon > 0$.

By Lemma 1, based on $\mathbf{p}^*(f), \{\alpha_{ji}(f)\}$ and $\{n_i(f)\}$, take a piecewise flat ε-approximation $\bar{\mathbf{p}}^*(f), \{\bar{\alpha}_{ji}(f)\}$ and $\{\bar{n}_i(f)\}$, s.t.

$$\left|\int_0^1 \bar{R}^*(\bar{\mathbf{p}}^*(f), f)\,df - R^*\right| < \varepsilon$$

where for every fixed $f \in [0,1]$, $\bar{R}^*(\mathbf{P}, f) \triangleq conv_{\mathbf{P}}\bar{R}(\mathbf{P}, f)$, with

$$\bar{R}(\mathbf{P}, f) = \sum_{i=1}^{K} w_i \log\left(1 + \frac{P_i}{\bar{n}_i(f) + \sum_{j\ne i} P_i \bar{\alpha}_{ji}(f)}\right).$$

Based on the piecewise flat ε-approximation, *in every flat sub-channel with a flat* $\bar{\mathbf{p}}^*(f)$, as in Theorem 6, $\bar{R}^*(\bar{\mathbf{p}}^*(f), f)$ can be achieved by further dividing this sub-channel into $K+1$ sub-bands, each applying a frequency flat power allocation. Denote the resulting allocation scheme by $\mathbf{p}^o(f)$, achieving the same sum-rate

$$\int_0^1 \bar{R}(\mathbf{p}^o(f), f)\,df = \int_0^1 \bar{R}^*(\bar{\mathbf{p}}^*(f), f)\,df.$$

Then

$$R^o \geq \int_0^1 R(\mathbf{p}^o(f), f)df$$
$$\geq \int_0^1 \bar{R}(\mathbf{p}^o(f), f)df = \int_0^1 \bar{R}^*(\bar{\mathbf{p}}^*(f), f) \ df > R^* - \varepsilon$$

where the first inequality occurs because $\mathbf{p}^o(f)$ is a feasible solution of (23); the second inequality occurs because (by P2 from Lemma 1) $\bar{\alpha}_{ji}(f) \geq \alpha_{ji}(f), \ \forall i \neq j, \bar{n}_i(f) \geq n_i(f), \forall i, \forall f$, i.e., the ε-approximation leads to worse channel responses, resulting in lower rates. Finally, let $\varepsilon \to 0$. ■

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