On the Maximum Achievable Sum-Rate With Successive Decoding in Interference Channels

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***Abstract*—In this paper, we investigate the maximum achievable sum-rate of the two-user Gaussian interference channel with Gaussian superposition coding and successive decoding. We first examine an approximate deterministic formulation of the problem, and introduce the complementarity conditions that capture the use of Gaussian coding and successive decoding. In the deterministic channel problem, we find the constrained sum-capacity and its achievable schemes with the minimum number of messages, first in symmetric channels, and then in general asymmetric channels. We show that the constrained sum-capacity *oscillates* as a function of the cross link gain parameters between the information theoretic sum-capacity and the sum-capacity with interference treated as noise. Furthermore, we show that if the number of messages of either of the two users is fewer than the minimum number required to achieve the constrained sum-capacity, the maximum achievable sum-rate drops to that with interference treated as noise. We provide two algorithms to translate the optimal schemes in the deterministic channel model to the Gaussian channel model. We also derive two upper bounds on the maximum achievable sum-rate of the Gaussian Han-Kobayashi schemes, which automatically upper bound the maximum achievable sum-rate using successive decoding of Gaussian codewords. Numerical evaluations show that, similar to the deterministic channel results, the maximum achievable sum-rate with successive decoding in the Gaussian channels oscillates between that with Han-Kobayashi schemes and that with single message schemes.**

***Index Terms*—Deterministic channel model, Gaussian interference channel, successive decoding, sum-rate maximization.**

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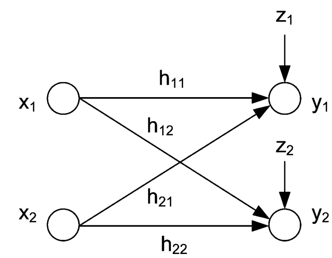

Fig. 1. Two-user Gaussian interference channel.

I. Introduction

We consider the sum-rate maximization problem in two-user Gaussian interference channels (cf. Fig. 1) under the constraints of successive decoding. While the information theoretic capacity region of the Gaussian interference channel is still not known, it has been shown that a Han-Kobayashi scheme with random Gaussian codewords can achieve within 1 bit/s/Hz of the capacity region [2], and hence within 2 bits/s/Hz of the sum-capacity. In this Gaussian Han-Kobayashi scheme, each user first decodes both users' common messages jointly, and then decodes its own private message. In comparison, the simplest commonly studied decoding constraint is that each user treats the interference from the other users as noise, i.e., without any decoding attempt. Using Gaussian codewords, the corresponding constrained sum-rate maximization problem can be formulated as a nonconvex optimization of power allocation, which has an analytical solution in the two-user case [3]. It has also been shown that within a certain range of channel parameters for *weak* interference channels, treating interference as noise achieves the information theoretic sum-capacity [4]–[6]. For general interference channels with more than two users, there is so far neither a near optimal solution information theoretically, nor a polynomial time algorithm that finds a near optimal solution with interference treated as noise [7], [8].

In this paper, we consider a decoding constraint—successive decoding of Gaussian superposition codewords—that bridges the complexity between joint decoding (e.g., in Han-Kobayashi schemes) and treating interference as noise. We investigate the maximum achievable sum-rate and its achievable schemes. Compared to treating interference as noise, allowing successive cancellation yields a much more complex problem structure. To clarify and capture the key aspects of the problem, we resort to the deterministic channel model [9]. In [10], the information theoretic capacity region for the two-user deterministic interference channel is derived as a special case of the El Gamal-Costa deterministic model [11], and is shown to be achievable using Han-Kobayashi schemes.

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We transmit messages using a superposition of Gaussian codebooks, and use successive decoding. To capture the use of successive decoding of Gaussian codewords, in the deterministic formulation, we introduce the *complementarity conditions* on the bit levels, which have also been characterized using a conflict graph model in [12]. We develop transmission schemes on the bit-levels, which in the Gaussian model corresponds to message splitting and power allocation of the messages. We then derive the constrained sum-capacity for the deterministic channel, and show that it *oscillates* (as a function of the cross link gain parameters) between the information theoretic sum-capacity and the sum-capacity with interference treated as noise. Furthermore, the minimum number of messages needed to achieve the constrained sum-capacity is obtained. Interestingly, we show that if the number of messages is limited to even *one less* than this minimum capacity achieving number, the maximum achievable sum-rate drops to that with interference treated as noise.

We then translate the optimal schemes in the deterministic channel to the Gaussian channel, using a rate constraint equalization technique. To evaluate the optimality of the translated achievable schemes, we derive and compute two upper bounds on the maximum achievable sum-rate of Gaussian Han-Kobayashi schemes[1]. Since a scheme using superposition coding with Gaussian codebooks and successive decoding is a special case of Han-Kobayashi schemes, these bounds automatically apply to the maximum achievable sum-rate with such successive decoding schemes as well. We select two mutually exclusive subsets of the inequality constraints that characterize the Gaussian Han-Kobayashi capacity region. Maximizing the sum-rate with each of the two subsets of inequalities leads to one of the two upper bounds. The two bounds are shown to be tight in different ranges of parameters. Numerical evaluations show that the maximum achievable sum-rate with Gaussian superposition coding and successive decoding oscillates between that with Han-Kobayashi schemes and that with single message schemes.

The remainder of the paper is organized as follows. Section II formulates the problem of sum-rate maximization with successive decoding of Gaussian superposition codewords in Gaussian interference channels, and compares it with Gaussian Han-Kobayashi schemes. Section III reformulates the problem with the deterministic channel model, and then solves for the constrained sum-capacity. Section IV translates the optimal schemes in the deterministic channel back to the Gaussian channel, and derives two upper bounds on the maximum achievable sum-rate. Numerical evaluations of the achievability against the upper bounds are provided. Section V concludes the paper with a short discussion on generalizations of the coding-decoding assumptions and their implications.

II. Problem Formulation in Gaussian Channels

We consider the two-user Gaussian interference channel shown in Fig. 1. The received signals of the two users are

$$y_1 = h_{11}x_1 + h_{21}x_2 + z_1$$
$$y_2 = h_{22}x_2 + h_{12}x_1 + z_2,$$

[1]Throughout this paper, when we refer to the Han-Kobayashi scheme, we mean the Gaussian Han-Kobayashi scheme, unless stated otherwise.

where $\{h_{ij}\}$ are constant complex channel gains, x_i $(i=1,2)$ is the transmitted signal of the encoded messages from the ith user, and $z_i \sim \mathcal{CN}(0, N_i)$. Define $g_{ij} \triangleq |h_{ij}|^2$, $(i,j=1,2)$.

There is an average power constraint equal to $\bar{p}_i$ for the ith user $(i=1,2)$. In the following, we first formulate the problem of finding the sum-rate optimal Gaussian superposition coding and successive decoding scheme, and then provide an illustrative example to show that successive decoding schemes do not necessarily achieve the same maximum achievable sum-rate as Han-Kobayashi schemes.

A. Gaussian Superposition Coding and Successive Decoding: A Power and Decoding Order Optimization

Suppose the ith user uses a superposition of L_i messages $M_i^{(\ell)} (1 \le \ell \le L_i)$. Denote by $r_i^{(\ell)}$ the information rate of message $M_i^{(\ell)}$. For the ith user, the transmit signal x_i is a superposition of L_i codewords $x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(L_i)}$, where each $x_i^{(\ell)}$ has a block length n, and is chosen from a codebook of size $2^{nr_i^{(\ell)}}$ that encodes message $M_i^{(\ell)}$, generated using independent and identically distributed (i.i.d.) random variables of $\mathcal{CN}(0,1)$. With the power constraints $\bar{p}_i$ $(i=1,2)$, we have

$$x_i = \sum_{\ell=1}^{L_i} \sqrt{p_i^{(\ell)}} x_i^{(\ell)}$$
$$\sum_{\ell=1}^{L_i} p_i^{(\ell)} \le \bar{p}_i, \quad i=1,2. \tag{1}$$

where $p_i^{(\ell)}$ is the power allocated to message $M_i^{(\ell)}$.

The ith receiver attempts to decode all $M_i^{(\ell)}, \ell = 1, \ldots, L_i$, using successive decoding as follows. It chooses a decoding order $\mathcal{O}_i$ of all the $L_1 + L_2$ messages from both users. It starts decoding from the first message in this order (by treating all other messages that are not yet decoded as noise,) then peeling it off and moving to the next one, until it decodes all the messages intended for itself—$M_i^{(\ell)}, \ell = 1, \ldots, L_i$.

Denote the message that has order q in $\mathcal{O}_i$ by $M_{t_{q,i}}^{(\ell_{q,i})}$, i.e., it is the $\ell_{q,i}$th message of the $t_{q,i}$th user. Then, for the successive decoding procedure to have a vanishing error probability as the block length $n \to \infty$, we have the following constraints on the rates of the messages:

$$r_{t_{q,i}}^{(\ell_{q,i})} \le \log\left(1 + \frac{p_{t_{q,i}}^{(\ell_{q,i})} g_{t_{q,i}i}}{\sum_{s=q+1}^{L_1+L_2} p_{t_{s,i}}^{(\ell_{s,i})} g_{t_{s,i}i} + N_i}\right)$$
$$\forall 1 \le q \le \max_{1\le\ell\le L_i} \{\text{order of } \mathrm{M_i^{(\ell)}} \text{ in } \mathcal{O}_\mathrm{i}\},\ i=1,2. \tag{2}$$

Now, we can formulate the sum-rate maximization problem as

$$\max_{\substack{\{p_i^{(\ell)}\},\mathcal{O}_i,\\ i=1,2}} \sum_{i=1}^{2}\sum_{\ell=1}^{L_i} r_i^{(\ell)} \tag{3}$$
$$\text{subject to: } (1), (2).$$

Note that (3) involves both a *combinatorial optimization* of the decoding orders $\{\mathcal{O}_i\}$ and a *nonconvex optimization* of the transmit power $\{p_i^{(\ell)}\}$. As a result, it is a hard problem from an

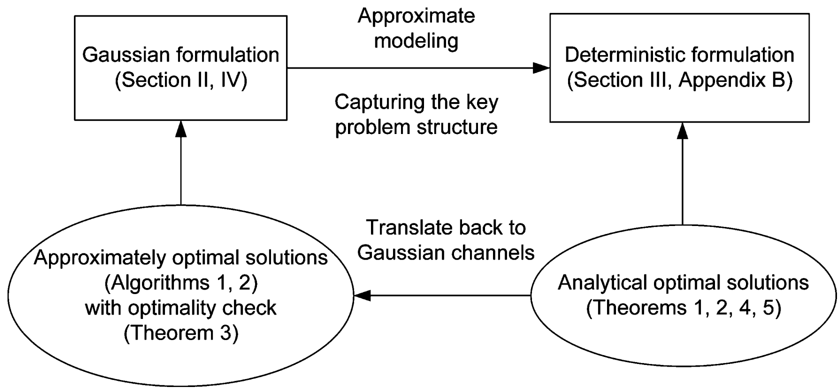


Fig. 2. Our approach to solving problem (3).

optimization point of view which has not been addressed in the literature.

Interestingly, we show that an "indirect" approach can effectively and fruitfully provide approximately optimal solutions to the above problem (3). Instead of directly working with the Gaussian model, we approximate the problem using the recently developed deterministic channel model [9]. The approximate formulation successfully captures the key structure and intuition of the original problem, for which we give a complete analytical solution that achieves the constrained sum-capacity in all channel parameters. Next, we translate this optimal solution in the deterministic formulation back to the Gaussian formulation, and show that the resulting solution is indeed close to the optimum. This indirect approach of solving (3) is outlined in Fig. 2.

Next, we provide an illustration of the following point: Although the constraints for the achievable rate region with Han-Kobayashi schemes share some similarities with those for the capacity region of multiple access channels, successive decoding in interference channels does *not* always have the same achievability as Han-Kobayashi schemes, (whereas time-sharing of successive decoding schemes does achieve the capacity region of multiple access channels.)

B. Successive Decoding of Gaussian Codewords versus Gaussian Han-Kobayashi Schemes With Joint Decoding

We first note that Gaussian superposition coding—successive decoding is a special case of the Han-Kobayashi scheme, using the following observations. For the first user, if its message $M_1^{(\ell)} (1 \le \ell \le L_1)$ is *decoded* at the second receiver according to the decoding order $\mathcal{O}_2$, we categorize it into the *common* information of the first user. Otherwise, $M_1^{(\ell)}$ is treated as noise at the second receiver, i.e., it appears *after* all the messages of the second user in $\mathcal{O}_2$, and we categorize it into the *private* information of the first user. The same categorization is performed for the L_2 messages of the second user. Note that every message of the two users is either categorized as private information or common information. Thus, every successive decoding scheme is a special case of the Han-Kobayashi scheme, and hence the capacity region with successive decoding of Gaussian codewords is included in that with Han-Kobayashi schemes.

However, the inclusion in the other direction is untrue, since Han-Kobayashi schemes allow joint decoding. In Sections III–V, we will give a characterization of the difference between the maximum achievable sum-rate using Gaussian successive decoding schemes and that using Gaussian Han-Kobayashi schemes. This difference appears *despite* the fact that the sum-capacity of a Gaussian multiple access channel is achievable using successive decoding of Gaussian codewords. In the remainder of this section, we show an illustrative example that provides some intuition into this difference.

Suppose the ith user $(i = 1, 2)$ uses *two* messages: a common message M_i^c and a private message M_i^p. We consider a power allocation to the encoded messages, and denote the power allocated to M_i^c and M_i^p by q_i^c and q_i^p, $(i = 1, 2.)$ Denote the achievable rates of M_i^c and M_i^p by r_i^c and r_i^p. In a Han-Kobayashi scheme, at each receiver, the common messages and the intended private message are *jointly* decoded, treating the unintended private message as noise. This gives rise to the achievable rate region with any given power allocation as follows:

$$r_1^c + r_1^p + r_2^c \le \log(1 + \frac{q_1^c + q_1^p + g_{21} q_2^c}{g_{21} q_2^p + N_1}) \tag{4}$$

$$r_2^c + r_2^p + r_1^c \le \log(1 + \frac{q_2^c + q_2^p + g_{12} q_1^c}{g_{12} q_1^p + N_2}) \tag{5}$$

$$r_1^c + r_2^c \le \log(1 + \frac{q_1^c + g_{21} q_2^c}{g_{21} q_2^p + N_1}) \tag{6}$$

$$r_2^c + r_1^c \le \log(1 + \frac{q_2^c + g_{12} q_1^c}{g_{12} q_1^p + N_2}) \tag{7}$$

$$r_1^c + r_1^p \le \log(1 + \frac{q_1^c + q_1^p}{g_{21} q_2^p + N_1}) \tag{8}$$

$$r_2^c + r_2^p \le \log(1 + \frac{q_2^c + q_2^p}{g_{12} q_1^p + N_2}) \tag{9}$$

$$r_1^p + r_2^c \le \log(1 + \frac{q_1^p + g_{21} q_2^c}{g_{21} q_2^p + N_1}) \tag{10}$$

$$r_2^p + r_1^c \le \log(1 + \frac{q_2^p + g_{12} q_1^c}{g_{12} q_1^p + N_2}) \tag{11}$$

$$r_1^c \le \log(1 + \frac{q_1^c}{g_{21} q_2^p + N_1})$$

$$r_2^c \le \log(1 + \frac{q_2^c}{g_{12} q_1^p + N_2}) \tag{12}$$

$$r_2^c \le \log(1 + \frac{g_{21} q_2^c}{g_{21} q_2^p + N_1})$$

$$r_1^c \le \log(1 + \frac{g_{12} q_1^c}{g_{12} q_1^p + N_2}) \tag{13}$$

$$r_1^p \le \log(1 + \frac{q_1^p}{g_{21}q_2^p + N_1})$$
$$r_2^p \le \log(1 + \frac{q_2^p}{g_{12}q_1^p + N_2}). \tag{14}$$

In a successive decoding scheme, depending on the different decoding orders applied, the achievable rate regions have different expressions. In the following, we provide and analyze the achievable rate region with the decoding orders at receiver 1 and 2 being $(M_1^c \to M_2^c \to M_1^p)$ and $(M_2^c \to M_1^c \to M_2^p)$, respectively. The intuition obtained with these decoding orders holds similarly for other decoding orders. With any given power allocation, we have

$$r_1^c \le \min\left\{ \log(1 + \frac{q_1^c}{q_1^p + g_{21}(q_2^c + q_2^p) + N_1}) \quad \log(1 + \frac{g_{12}q_1^c}{q_2^p + g_{12}q_1^p + N_2}) \right\} \tag{15}$$
$$r_2^c \le \min\left\{ \log(1 + \frac{q_2^c}{q_2^p + g_{12}(q_1^c + q_1^p) + N_2}) \quad \log(1 + \frac{g_{21}q_2^c}{q_1^p + g_{21}q_2^p + N_1}) \right\} \tag{16}$$
$$r_1^p \le \log(1 + \frac{q_1^p}{g_{21}q_2^p + N_1})$$
$$r_2^p \le \log(1 + \frac{q_2^p}{g_{12}q_1^p + N_2}). \tag{17}$$

It is immediate to check that (15)–(17)⇒(4)–(14), but not vice versa.

To observe the difference between the maximum achievable sum-rate with (4)–(14) and that with(15)–(17), we examine the following symmetric channel,

$$g_{11} = g_{22} = 1, g_{12} = g_{21} = 0.17, N_1 = N_2 = 1, \tag{18}$$

in which we apply symmetric power allocation schemes with $q_1^c = q_2^c$ and $q_1^p = q_2^p$, and a power constraint of $\bar{p} = \bar{p}_i = q_i^c + q_i^p = 1000$, $i = 1, 2$.

Remark 1: Note that $\mathrm{SNR} = \frac{g_{11}\bar{p}}{N_i} = 1000 \sim 30\,\mathrm{dB}$, $\mathrm{INR} = \frac{g_{21}\bar{p}}{N_j} = 170 \sim 22.5\,\mathrm{dB} \Rightarrow \frac{\log \mathrm{INR}}{\log \mathrm{SNR}} \approx \frac{3}{4}$. As indicated in Fig. 19 of [10], under this parameter setting, simply using successive decoding of Gaussian codewords can have an arbitrarily large maximum achievable sum-rate loss compared to joint decoding schemes, as $\mathrm{SNR} \to \infty$.

We plot the sum-rates with the private message power q_i^p sweeping from nearly zero (-30 dB) to the maximum (30 dB) as in Fig. 3. As observed, the difference between the two schemes is evident when the private message power q_i^p is sufficiently smaller than the common message power q_i^c (with $q_i^p + q_i^c = 1000$.) The intuition of why successive decoding of Gaussian codewords is not equivalent to the Han-Kobayashi schemes is best reflected in the case of $q_i^p = 0$. In the above parameter setting, with $q_i^p = 0$, (4)–(14) translate to

$$r_1^c + r_2^c \le \log(1 + \frac{q_1^c + g_{21}q_2^c}{N_1}) = 10.19 \text{ bits}$$
$$r_1^c \le \log(1 + \frac{g_{12}q_1^c}{N_2}) = 7.42 \text{ bits}$$
$$r_2^c \le \log(1 + \frac{g_{21}q_2^c}{N_1}) = 7.42 \text{ bits}$$

whereas (15)–(17) translate to

$$r_1^c \le \min\left\{ \log(1 + \frac{q_1^c}{g_{21}q_2^c + N_1}), \log(1 + \frac{g_{12}q_1^c}{N_2}) \right\}$$
$$= \min\{2.78, 7.42\} = 2.78 \text{ bits}$$
$$r_2^c \le \min\left\{ \log(1 + \frac{q_2^c}{g_{12}q_1^c + N_2}), \log(1 + \frac{g_{21}q_2^c}{N_1}) \right\}$$
$$= \min\{2.78, 7.42\} = 2.78 \text{ bits}.$$

As a result, the maximum achievable sum-rates with the Han-Kobayashi scheme and that with the successive decoding scheme are 10.19 and 5.56 bits, respectively. Here, the key intuition is as follows: for a common message, its individual rate constraints at the two receivers in a successive decoding scheme (15) and (16) are tighter than those in a joint decoding scheme (12) and (13). In Sections III–V, we will see that (15) and (16) lead to a nonsmooth behavior of the maximum achievable sum-rate using successive decoding of Gaussian codewords. Finally, we connect the results shown in Fig. 3 to the results shown later in Fig. 13 of Section IV-C:

Remark 2: In Fig. 3, the optimal symmetric power allocation for a Han-Kobayashi scheme and that for a successive decoding scheme are $q_1^p/N_1 = 6.2$ and 14.5 dB, respectively, leading to sum-rates of 11.2 and 10.2 bits. This result corresponds to the performance evaluation at $\alpha = \frac{\log(\mathrm{INR})}{\log(\mathrm{SNR})} = 0.75$ in Fig. 13.

III. Sum-Capacity in Deterministic Interference Channels

A. *Channel Model and Problem Formulation*

In this section, we apply the deterministic channel model [9] as an approximation of the Gaussian model on the two-user interference channel. We define

$$n_{11} \triangleq \log(\mathrm{SNR}_1) = \log(\frac{g_{11}\bar{p}_1}{N_1}) = \log(\tilde{g}_{11}\bar{p}_1) \tag{19}$$
$$n_{22} \triangleq \log(\mathrm{SNR}_2) = \log(\frac{g_{22}\bar{p}_2}{N_2}) = \log(\tilde{g}_{22}\bar{p}_2) \tag{20}$$
$$n_{12} \triangleq \log(\mathrm{INR}_1) = \log(\frac{g_{21}\bar{p}_2}{N_1}) = \log(\tilde{g}_{21}\bar{p}_2) \tag{21}$$
$$n_{21} \triangleq \log(\mathrm{INR}_2) = \log(\frac{g_{12}\bar{p}_1}{N_2}) = \log(\tilde{g}_{12}\bar{p}_1) \tag{22}$$

where $\tilde{g}_{ij} \triangleq g_{ij}/N_j$ are the channel gains normalized by the noise power. Without loss of generality (WLOG), we assume that $n_{11} \ge n_{22}$. We note that the logarithms used in this paper are taken to base 2. Now, n_{ji} counts the bit levels of the signal sent from the ith transmitter that are above the noise level at the jth receiver. Further, we define

$$\delta_1 \triangleq n_{11} - n_{21} = -\log(\frac{\tilde{g}_{12}}{\tilde{g}_{11}}),\ \delta_2 \triangleq n_{22} - n_{12} = -\log(\frac{\tilde{g}_{21}}{\tilde{g}_{22}}) \tag{23}$$

which represent the cross channel gains relative to the direct channel gains, in terms of the number of bit-level shifts. To formulate the optimization problem, we consider $\{n_{ji}\}$ to be *real* numbers. (As will be shown later in Remark 5, with *integer* bit-level channel parameters, our derivations automatically give integer bit-level optimal solutions.)

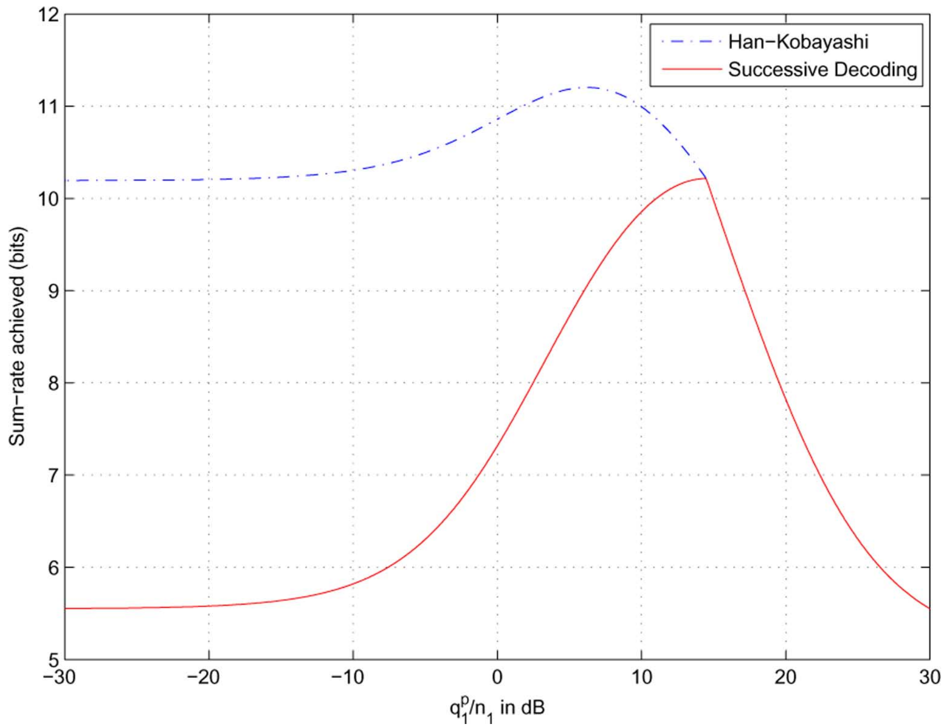


Fig. 3. Illustrations of the difference between the achievable sum-rate with Han-Kobayashi schemes and that with successive decoding of Gaussian codewords.

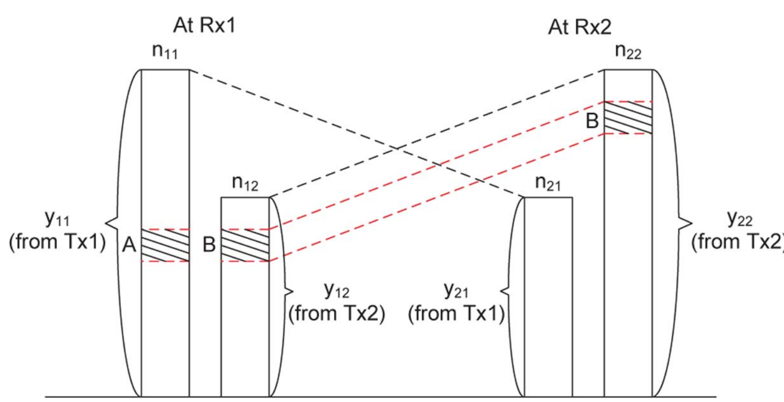


Fig. 4. Two-user deterministic interference channel. Levels A and B interfere at the first receiver, and cannot be fully active simultaneously.

In Fig. 4, the desired signal and the interference signal at both receivers are depicted. y_{11} and y_{12} are the sets of received *information levels* at receiver 1 that are above the noise level, from users 1 and 2, respectively. y_{21} and y_{22} are the sets of received information levels at receiver 2. A more concise representation is provided in Fig. 5.

- The sets of information levels of the *desired* signals at receivers 1 and 2 are represented by the continuous intervals $I_1 = [0, n_{11}]$ and $I_2 = [n_{11} - n_{22}, n_{11}]$ on two parallel lines, where the leftmost points correspond to the most significant (i.e., highest) information levels, and the points at n_{11} correspond to the positions of the noise levels at both receivers.
- The positions of the information levels of the *interfering* signals are indicated by the dashed lines crossing between the two parallel lines.

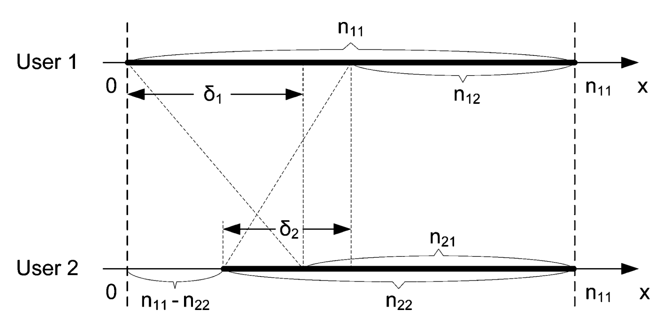


Fig. 5. Interval representation of the two-user deterministic interference channel.

Note that an information level (or simply termed "*level*") is a real *point* on a line, and the measure of a set of levels (e.g., the length of an interval) equals the amount of information that this set can carry. The design variables are whether or not each level of a user's received desired signal carries information for this user, characterized by the following definition.

Definition 1: $f_i(x)$ is the indicator function on whether the levels inside I_i carry information for the ith user.

$$f_i(x) = \begin{cases} 1, & \text{if } x \in I_i, \text{ and level } x \text{ carries} \\ & \text{information for the } i\text{th user} \qquad (i = 1, 2.) \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

As a result, the rates of the two users are

$$R_1 = \int_0^{n_{11}} f_1(x)\mathrm{d}x, R_2 = \int_0^{n_{11}} f_2(x)\mathrm{d}x.$$

For an information level x s.t. $f_i(x) = 1$, we call it an *active* level for the ith user, and otherwise an *inactive* level.

The constraints from superposition of Gaussian codewords with successive decoding (15)–(17) translate to the following *Complementarity Conditions* in the deterministic formulation.

$$f_1(x)f_2(x+\delta_1) = 0, \forall -\infty < x < \infty \tag{25}$$
$$f_2(x)f_1(x+\delta_2) = 0, \forall -\infty < x < \infty \tag{26}$$

where δ_1 and δ_2 are defined in (23). The interpretation of (25) and (26) are as follows: for any two levels each from one of the two users, if they interfere with each other at any of the two receivers, they cannot be simultaneously active. For example, in Fig. 4, information levels A from the first user and B from the second user interfere at the first receiver, and hence cannot be fully active simultaneously. These complementarity conditions have also been characterized using a conflict graph model in [12].

Remark 3: For any given function $f_i(x)$, $x \in I_i$, every disjoint segment within I_i with $f_i(x) = 1$ on it corresponds to a distinct message. Adjacent segments that can be so combined as a super-segment having $f_i(x) = 1$ on it, are viewed as *one* segment, i.e., the combined super-segment. Thus, for two segments $s_1 = [a, b] \in I_i$ and $s_2 = [c, d] \in I_i$, $(b < c,)$ satisfying $f_i(x) = 1, \forall x \in s_1 \cup s_2$, if $\exists_{x0} \in (b, c)$, $f(x_0) = 0$, then s_1, s_2 separated by the point x_0 have to correspond to two distinct messages.

Finally, we note that

$$(25) \Leftrightarrow f_2(x)f_1(x-\delta_1) = 0, \forall -\infty < x < \infty$$
$$\text{and } (26) \Leftrightarrow f_1(x)f_2(x-\delta_2) = 0, \forall -\infty < x < \infty.$$

Thus, we have the following result:

Lemma 1: The parameter settings $\begin{cases} \delta_1 = a \\ \delta_2 = b \end{cases}$ and $\begin{cases} \delta_1 = -b \\ \delta_2 = -a \end{cases}$ correspond to the same set of complementarity conditions.

We consider the problem of maximizing the sum-rate $R^{\text{sum}} \triangleq R_1 + R_2$ of the two users employing successive decoding, formulated as the following continuous support (infinite dimensional) optimization problem:

$$\max_{f_1(x), f_2(x)} (R^{\text{sum}} =) \int_0^{n_{11}} f_1(x) + f_2(x)\mathrm{d}x \tag{27}$$
$$\text{subject to } (24), (25), (26).$$

Problem (27) does not include upper bounds on the number of messages L_1, L_2. Such upper bounds can be added based on Remark 3. We will analyze the cases without and with upper bounds on the number of messages. We first derive the constrained sum-capacity in *symmetric* interference channels in the remainder of this section. Results are then generalized using similar approaches to *general (asymmetric)* interference channels in Appendix B.

B. Symmetric Interference Channels

In this section, we consider the case where $n_{11} = n_{22}$, $n_{12} = n_{21}$. Define $\alpha \triangleq \frac{n_{12}}{n_{11}}$, $\beta \triangleq 1 - \alpha$. WLOG, we normalize the amount of information levels by n_{11}, and consider $n_{11} = n_{22} = 1$, and $n_{12} = n_{21} = \alpha$. Note that in symmetric channels, $\beta = \delta_1 = \delta_2$.

Now, (25) and (26) becomes

$$f_1(x)f_2(x+\beta) = 0, \forall -\infty < x < \infty \tag{28}$$
$$f_2(x)f_1(x+\beta) = 0, \forall -\infty < x < \infty. \tag{29}$$

Problem (27) becomes

$$\max_{f_1(x), f_2(x)} (R^{\text{sum}} =) \int_0^{1} f_1(x) + f_2(x)\mathrm{d}x \tag{30}$$
$$\text{subject to } (24), (28), (29).$$

From Lemma 1, it is sufficient to only consider the case with $\beta \geq 0$, i.e., $\alpha \leq 1$, and the case with $1 \leq \alpha \leq 2$ can be obtained by symmetry as in Corollary 3 later.

We next derive the constrained sum-capacity using successive decoding for $\alpha \in [0, 1]$, first without upper bounds on the number of messages, then with upper bounds. We will see that in symmetric channels, the constrained sum-capacity $R^{\text{sum}*}$ is achievable with $R_1 = R_2$. Thus, we also use the maximum achievable *symmetric* rate, denoted by $R(\alpha)$ as a function of α, as an equivalent performance measure. $R(\alpha)$ is thus one half of the optimal value of (30).

1) Symmetric Capacity Without Constraint on the Number of Messages:

Theorem 1: In symmetric weak interference channels ($\alpha \in [0, 1]$), the constrained symmetric capacity, i.e., the maximum achievable symmetric rate using successive decoding [with (28) and (29)], $R(\alpha)$, is characterized by

- $R(\alpha) = 1 - \frac{\alpha}{2}$, when $\alpha = \frac{2n}{2n+1}, n = 0, 1, 2, \ldots$.
- $R(\alpha) = \frac{1}{2}$, when $\alpha = \frac{2n-1}{2n}, n = 1, 2, 3, \ldots$.
- In every interval $[\frac{2n}{2n+1}, \frac{2n+1}{2n+2}], n = 0, 1, 2, \ldots, R(\alpha)$ is a decreasing linear function.
- In every interval $[\frac{2n-1}{2n}, \frac{2n}{2n+1}], n = 1, 2, 3, \ldots, R(\alpha)$ is an increasing linear function.
- $R(1) = \frac{1}{2}$.

Remark 4: We plot $R(\alpha)$ in Fig. 6, compared with the information theoretic capacity [10].

The key idea in deriving the constrained sum-capacity is to *decouple* the effects of the complementarity conditions. Before we present the complete proof of Theorem 1, we first analyze the following two examples that illustrate this decoupling idea.

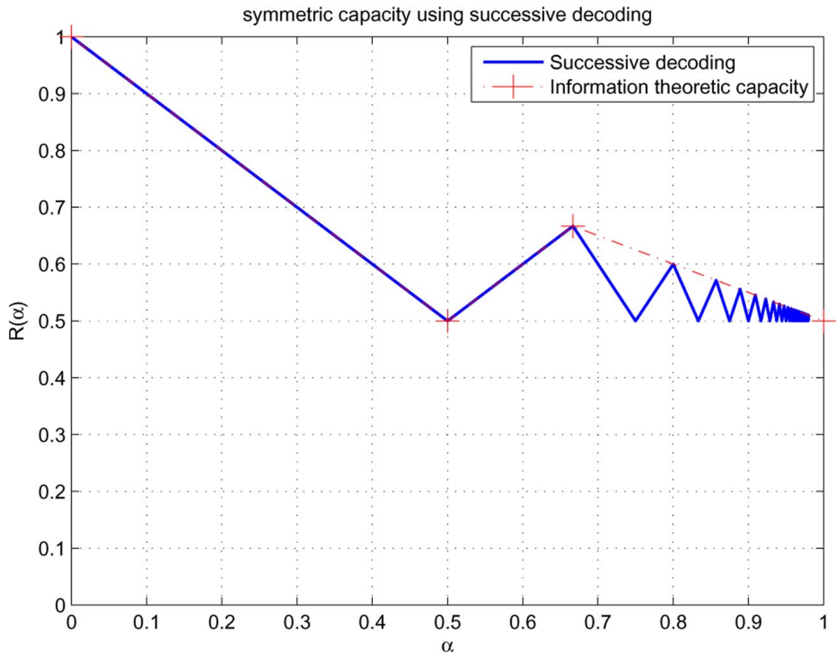


Fig. 6. The symmetric capacity with successive decoding in symmetric deterministic weak interference channels.

Example 1, $\alpha = \frac{3}{4}$, $\beta = \frac{1}{4}$: As in Fig. 7(a), we divide the interval $[0,1]$ into 4 segments s_1, s_2, s_3, s_4 with equal lengths. From the complementarity conditions (28) and (29),

$$\begin{cases} \forall x \in s_1 \left(\Leftrightarrow x + \frac{1}{4} \in s_2\right), \; f_1(x) f_2\left(x + \frac{1}{4}\right) = 0 \\ \forall x \in s_3 \left(\Leftrightarrow x + \frac{1}{4} \in s_4\right), \; f_1(x) f_2\left(x + \frac{1}{4}\right) = 0. \end{cases} \tag{31}$$

As a result,

$$\begin{cases} \int_{s_1} f_1(x)\mathrm{d}x + \int_{s_2} f_2(x)\mathrm{d}x \le \frac{1}{4} \\ \int_{s_3} f_1(x)\mathrm{d}x + \int_{s_4} f_2(x)\mathrm{d}x \le \frac{1}{4}, \end{cases}$$

$$\Rightarrow \int_{s_1 \cup s_3} f_1(x)\mathrm{d}x + \int_{s_2 \cup s_4} f_2(x)\mathrm{d}x \le \frac{1}{2}.$$

Similarly, $\int_{s_2 \cup s_4} f_1(x)\mathrm{d}x + \int_{s_1 \cup s_3} f_2(x)\mathrm{d}x \le \frac{1}{2}$, and we have

$$\int_0^1 f_1(x) + f_2(x)\mathrm{d}x \le 1 \Rightarrow R(\frac{3}{4}) \le \frac{1}{2}. \tag{32}$$

Clearly, $R(\frac{3}{4}) = \frac{1}{2}$ can be achieved by letting

$$\begin{cases} f_1(x) = 1, & x \in s_1 \cup s_3 \\ f_2(x) = 1, & x \in s_1 \cup s_3 \end{cases} \text{and} \begin{cases} f_1(x) = 0, & x \in s_2 \cup s_4 \\ f_2(x) = 0, & x \in s_2 \cup s_4 \end{cases}.$$

Example 2, $\alpha = \frac{4}{5}$, $\beta = \frac{1}{5}$: As in Fig. 7(b), we divide the interval $[0,1]$ into 5 segments s_1, s_2, s_3, s_4, s_5 with equal lengths. For the same reasons as in the last example, $\int_0^{\frac{4}{5}} f_1(x) + f_2(x)\mathrm{d}x \le \frac{4}{5}$. Therefore

$$\int_0^1 f_1(x) + f_2(x)\mathrm{d}x \le \frac{6}{5} \Rightarrow R(\frac{4}{5}) \le \frac{3}{5}. \tag{33}$$

Clearly, $R(\frac{4}{5}) = \frac{3}{5}$ can be achieved by letting

$$\begin{cases} f_1(x) = 1, & x \in s_1 \cup s_3 \cup s_5 \\ f_2(x) = 1, & x \in s_1 \cup s_3 \cup s_5 \end{cases} \text{and} \begin{cases} f_1(x) = 0, & x \in s_2 \cup s_4 \\ f_2(x) = 0, & x \in s_2 \cup s_4 \end{cases}.$$

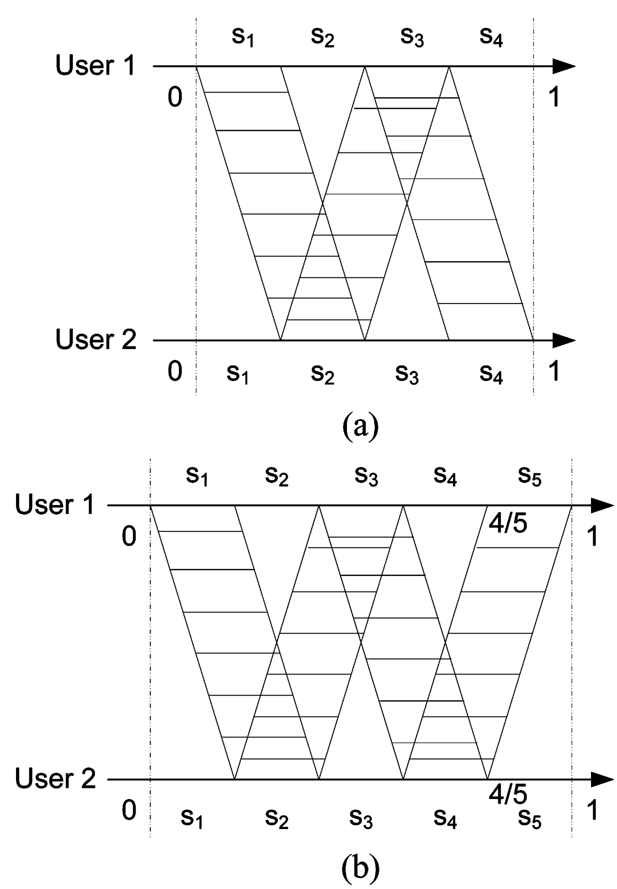


Fig. 7. Two examples that illustrate the proof ideas of Theorem 1. (a) The example of $\alpha = \frac{3}{4}$. (b) The example of $\alpha = \frac{4}{5}$.

Proof of Theorem 1:

i) When $\frac{2n-1}{2n} < \alpha \le \frac{2n}{2n+1}$, $n = 1, 2, 3, \ldots$, $\frac{1}{2n+1} \le \beta < \frac{1}{2n}$. We divide the interval $[0,1]$ into $2n+1$ segments

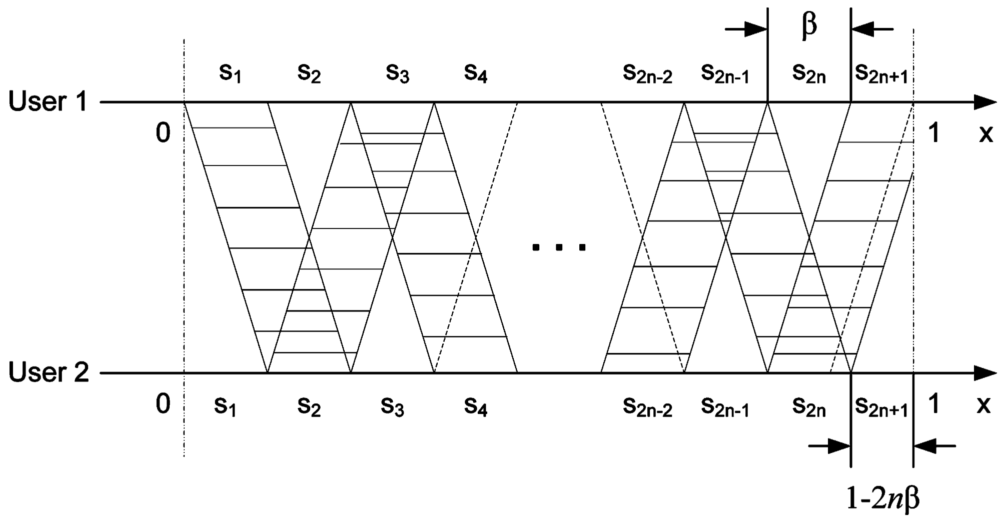


Fig. 8. Segmentation of the information levels, $\frac{2n-1}{2n} < \alpha \le \frac{2n}{2n+1}$.

$\{s_1, \ldots, s_{2n+1}\}$, where the first $2n$ segments have length β, and the last segment has length $1-2n\beta \in (0, \frac{1}{2n+1}]$ (cf. Fig. 8.) With these, the complementarity conditions (28) and (29) are equivalent to the following:

$$\begin{cases} \forall x \in s_1 (\Leftrightarrow x+\beta \in s_2), f_1(x)f_2(x+\beta) = 0 \\ \forall x \in s_2 (\Leftrightarrow x+\beta \in s_3), f_2(x)f_1(x+\beta) = 0 \\ \cdots \\ \forall x \in s_{2n-1} (\Leftrightarrow x+\beta \in s_{2n}), f_1(x)f_2(x+\beta) = 0, \end{cases} \tag{34}$$

$$\text{and } \forall x+\beta \in s_{2n+1},\ f_2(x)f_1(x+\beta) = 0 \tag{35}$$

[Equations (34) and (35) correspond to the shaded strips in Fig. 8.]

Similarly

$$\begin{cases} \forall x \in s_1 (\Leftrightarrow x+\beta \in s_2), f_2(x)f_1(x+\beta) = 0 \\ \forall x \in s_2 (\Leftrightarrow x+\beta \in s_3), f_1(x)f_2(x+\beta) = 0 \\ \cdots \\ \forall x \in s_{2n-1} (\Leftrightarrow x+\beta \in s_{2n}), f_2(x)f_1(x+\beta) = 0 \end{cases} \tag{36}$$

$$\text{and } \forall x+\beta \in s_{2n+1},\ f_1(x)f_2(x+\beta) = 0. \tag{37}$$

We partition the set of all segments into two groups: $\mathcal{G}_1 = s_1 \cup s_3 \cup \ldots \cup s_{2n+1}$ and $\mathcal{G}_2 = s_2 \cup s_4 \cup \ldots \cup s_{2n}$. Note that

- Equation (34) and (35) are constraints on $f_1(x)$ with support in $\mathcal{G}_1$, and on $f_2(x)$ with support in $\mathcal{G}_2$.
- Equation (36) and (37) are constraints on $f_1(x)$ with support in $\mathcal{G}_2$, and on $f_2(x)$ with support in $\mathcal{G}_1$.

Consequently, instead of viewing the (infinite number of) optimization variables as $f_1(x)|_{[0,1]}$ and $f_2(x)|_{[0,1]}$, it is more convenient to view them as

$$C_1 \triangleq \{f_1(x)|_{\mathcal{G}_1}, f_2(x)|_{\mathcal{G}_2}\} \text{ and } C_2 \triangleq \{f_1(x)|_{\mathcal{G}_2}, f_2(x)|_{\mathcal{G}_1}\}, \tag{38}$$

because there is no constraint between C_1 and C_2 from the complementarity conditions. In other words, C_1 and C_2 can be optimized *independently* of each other. Define

$$R_{C_1}^{\text{sum}} \triangleq \int_{\mathcal{G}_1} f_1(x)\mathrm{d}x + \int_{\mathcal{G}_2} f_2(x)\mathrm{d}x$$
$$R_{C_2}^{\text{sum}} \triangleq \int_{\mathcal{G}_2} f_1(x)\mathrm{d}x + \int_{\mathcal{G}_1} f_2(x)\mathrm{d}x.$$

Clearly, $R^{\text{sum}} = R_{C_1}^{\text{sum}} + R_{C_2}^{\text{sum}}$. Hence (30) can be solved by separately solving the following two subproblems:

$$\max_{f_1(x)|_{\mathcal{G}_1}, f_2(x)|_{\mathcal{G}_2}} (R_{C_1}^{\text{sum}} =) \int_{\mathcal{G}_1} f_1(x)\mathrm{d}x + \int_{\mathcal{G}_2} f_2(x)\mathrm{d}x \quad \text{subject to (24), (34), (35),} \tag{39}$$

and

$$\max_{f_1(x)|_{\mathcal{G}_2}, f_2(x)|_{\mathcal{G}_1}} (R_{C_2}^{\text{sum}} =) \int_{\mathcal{G}_2} f_1(x)\mathrm{d}x + \int_{\mathcal{G}_1} f_2(x)\mathrm{d}x \quad \text{subject to (24), (36), (37).} \tag{40}$$

We now prove that the optimal value of (39) is $R_{C_1}^{\text{sum}*} = 1 - n\beta$:

- (Achievability:) $1 - n\beta$ is achievable with $f_1(x) = 1$, $\forall x \in \mathcal{G}_1$, and $f_2(x) = 0$, $\forall x \in \mathcal{G}_2$.
- (Converse:) (34) $\Rightarrow$ $\forall i \in \{1, 2, \ldots, n\}$, $\int_{s_{2i-1}} f_1(x)\mathrm{d}x + \int_{s_{2i}} f_2(x)\mathrm{d}x \le \beta$,

$$\begin{aligned} &\Rightarrow \int_{\mathcal{G}_1} f_1(x)\mathrm{d}x + \int_{\mathcal{G}_2} f_2(x)\mathrm{d}x \\ &\quad = \sum_{i=1}^{n} \left(\int_{s_{2i-1}} f_1(x)\mathrm{d}x + \int_{s_{2i}} f_2(x)\mathrm{d}x \right) \\ &\qquad + \int_{s_{2i+1}} f_1(x)\mathrm{d}x \\ &\quad \le \beta \cdot n + (1 - 2n\beta) = 1 - n\beta. \end{aligned} \tag{41}$$

By symmetry, the solution of (40) can be obtained similarly, and the optimal value is $R_{C_2}^{\text{sum}*} = 1 - n\beta$ as well. Therefore, the optimal value of (30) is $R^{\text{sum}*} = 2(1 - n\beta)$.

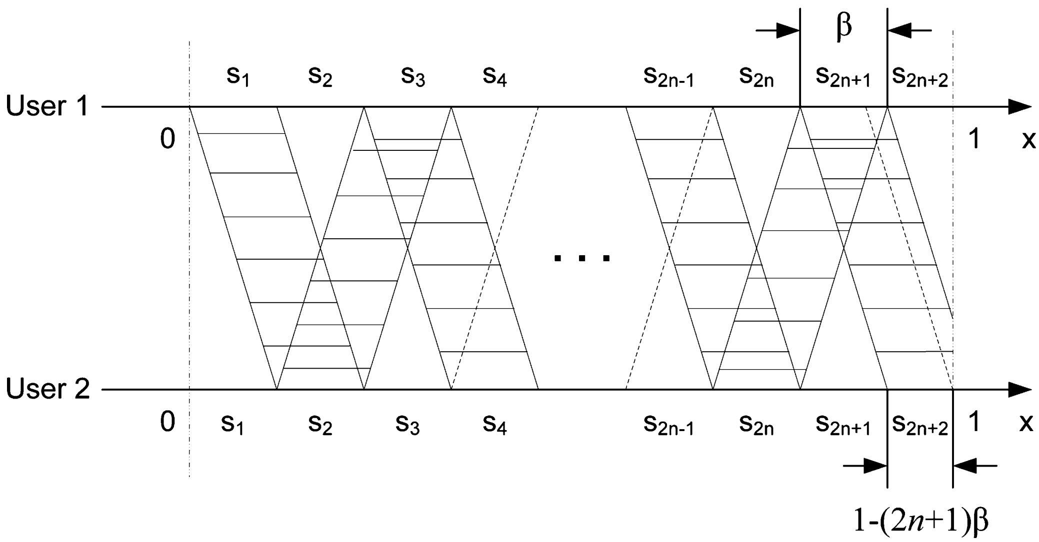


Fig. 9. Segmentation of the information levels, $\frac{2n}{2n+1} < \alpha \leq \frac{2n+1}{2n+2}$.

As the above maximum achievable scheme is symmetric, i.e.

$$f_1(x) = f_2(x) = \begin{cases} 1, & \forall x \in \mathcal{G}_1 \\ 0 & \forall x \in \mathcal{G}_2 \end{cases} \tag{42}$$

the symmetric capacity is

$$R(\alpha) = 1 - n\beta = n\alpha + 1 - n. \tag{43}$$

Clearly, $R(\alpha)$ is an *increasing linear* function of α in every interval $(\frac{2n-1}{2n}, \frac{2n}{2n+1}]$, $n = 1, 2, 3, \ldots$. It can be verified that $R(\alpha)|_{\frac{2n-1}{2n}} = \frac{1}{2}$, and $R(\alpha)|_{\frac{2n}{2n+1}} = 1 - \frac{\alpha}{2}$.

ii) When $\frac{2n}{2n+1} < \alpha \leq \frac{2n+1}{2n+2}$, $n = 0, 1, 2, \ldots$, $\frac{1}{2n+2} \leq \beta < \frac{1}{2n+1}$. Similarly to i), we divide the interval $[0, 1]$ into $2n + 2$ segments $\{s_1, \ldots, s_{2n+2}\}$, where the first $2n + 1$ segments have length β, and the last segment has length $1 - (2n + 1)\beta \in (0, \frac{1}{2n+2}]$ (cf. Fig. 9). Then, the complementarity conditions (28) and (29) are equivalent to the following:

$$(34), (35) \text{ and } f_1(x)f_2(x + \beta) = 0, \forall x + \beta \in s_{2n+2}, \tag{44}$$

$$\text{and } (36), (37) \text{ and } f_2(x)f_1(x + \beta) = 0, \forall x + \beta \in s_{2n+2}. \tag{45}$$

Similarly to i), with $\mathcal{G}_1 = s_1 \cup s_3 \cup \ldots \cup s_{2n+1}$ and $\mathcal{G}_2 = s_2 \cup s_4 \cup \ldots \cup s_{2n+2}$, (30) can be solved by separately solving the following two subproblems:

$$\max_{f_1(x)|_{\mathcal{G}_1}, f_2(x)|_{\mathcal{G}_2}} (R^{\text{sum}}_{C_1} =) \int_{\mathcal{G}_1} f_1(x)\mathrm{d}x + \int_{\mathcal{G}_2} f_2(x)\mathrm{d}x \tag{46}$$
$$\text{subject to } (24), (34), (35), (44),$$

and

$$\max_{f_1(x)|_{\mathcal{G}_2}, f_2(x)|_{\mathcal{G}_1}} (R^{\text{sum}}_{C_2} =) \int_{\mathcal{G}_2} f_1(x)\mathrm{d}x + \int_{\mathcal{G}_1} f_2(x)\mathrm{d}x \tag{47}$$
$$\text{subject to } (24), (36), (37), (45).$$

We now prove that the optimal value of (46) is $(n + 1)\beta$:

- (Achievability:) $(n+1)\beta$ is achievable with $f_1(x) = 1$, $\forall x \in \mathcal{G}_1$, and $f_2(x) = 0$, $\forall x \in \mathcal{G}_2$.
- (Converse:) $(34), (35), (44) \Rightarrow \forall i \in \{1, 2, \ldots, n+1\}$, $\int_{s_{2i-1}} f_1(x)\mathrm{d}x + \int_{s_{2i}} f_2(x)\mathrm{d}x \leq \beta$,

$$\begin{aligned} &\Rightarrow \int_{\mathcal{G}_1} f_1(x)\mathrm{d}x + \int_{\mathcal{G}_2} f_2(x)\mathrm{d}x \\ &\quad = \sum_{i=1}^{n+1} \Big(\int_{s_{2i-1}} f_1(x)\mathrm{d}x + \int_{s_{2i}} f_2(x)\mathrm{d}x \Big) \\ &\quad \leq (n + 1)\beta. \end{aligned} \tag{48}$$

By symmetry, the solution of (47) can be obtained similarly. Thus, the optimal value of (30) is $2(n + 1)\beta$. The maximum achievable scheme is also characterized by (42), and the symmetric rate is

$$R(\alpha) = (n + 1)\beta = -(n + 1)\alpha + n + 1. \tag{49}$$

Clearly, $R(\alpha)$ is a *decreasing linear* function of α in every interval $(\frac{2n}{2n+1}, \frac{2n+1}{2n+2}]$, $n = 0, 1, 2, \ldots$. It can be verified that $R(\alpha)|_{\frac{2n}{2n+1}} = 1 - \frac{\alpha}{2}$, and $R(\alpha)|_{\frac{2n+1}{2n+2}} = \frac{1}{2}$.

iii) It is clear that $R(0) = 1$, which is achievable with $f_1(x) = f_2(x) = 1$, $\forall x \in (0, 1)$, and $R(1) = \frac{1}{2}$, which is achievable by $\begin{cases} f_1(x) = 1, & x \in [\frac{1}{2}, 1] \\ f_2(x) = 0, & x \in [\frac{1}{2}, 1] \end{cases}$ and $\begin{cases} f_1(x) = 0, & x \in [0, \frac{1}{2}] \\ f_2(x) = 1, & x \in [0, \frac{1}{2}] \end{cases}$. ■

We summarize the optimal scheme that achieves the constrained symmetric capacity as follows:

Corollary 1: When $\alpha \in (0, 1)$, the constrained symmetric capacity is achievable with

$$f_1(x) = f_2(x) = \begin{cases} 1, & \forall x \in \mathcal{G}_1 \\ 0, & \forall x \in \mathcal{G}_2 \end{cases} \tag{50}$$

where $\mathcal{G}_1 = \bigcup_{i=1,2,\ldots} s_{2i-1}$ and $\mathcal{G}_2 = \bigcup_{i=1,2,\ldots} s_{2i}$.

In the special cases when $\alpha = \frac{2n-1}{2n}$, $(n = 1, 2, 3, \ldots,)$ and $\alpha = 1$, the constrained symmetric capacity drops to $\frac{1}{2}$ which is

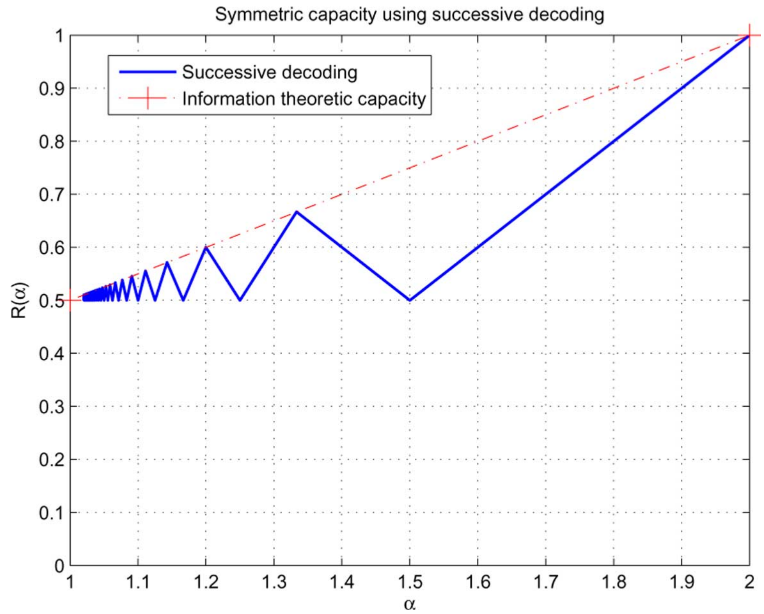


Fig. 10. The symmetric capacity with successive decoding in symmetric deterministic strong interference channels.

also achievable by time sharing $\begin{cases} f_1(x) = 1, & x \in [0,1] \\ f_2(x) = 0, & x \in [0,1] \end{cases}$ and $\begin{cases} f_1(x) = 0, & x \in [0,1] \\ f_2(x) = 1, & x \in [0,1] \end{cases}$.

We observe that the numbers of messages used by the two users—L_1, L_2—in the above optimal schemes are as follows.

Corollary 2:

- when $\alpha \in (\frac{2n-1}{2n}, \frac{2n+1}{2n+2})$, $(n = 1, 2, 3, \ldots)$, $L_1 = L_2 = n + 1$;
- when $\alpha \in [0, \frac{1}{2}]$, $\alpha = \frac{2n-1}{2n}$, $(n = 1, 2, 3, \ldots)$, or $\alpha = 1$, $L_1 = L_2 = 1$.

Remark 5: In the original formulation of the deterministic channel model [9], $\{n_{ij}\}$ are considered to be *integers*, and the achievable scheme must also have integer bit-levels. In this case, $\alpha = \frac{n_{12}}{n_{11}}$ is a *rational* number. As a result, the optimal scheme (50) will consist of active segments $\mathcal{G}_1$ that have rational boundaries with the same denominator n_{11}. This indeed corresponds to an integer bit-level solution.

From Theorem 1 (cf. Fig. 6), it is interesting to see that the constrained symmetric capacity oscillates as a function of α between the information theoretic capacity and the baseline of $\frac{1}{2}$. This phenomenon is a consequence of the complementarity conditions. In Section V, we further discuss the connections of this result to other coding-decoding constraints.

Finally, from Lemma 1, we have the following corollary on the maximum achievable symmetric rate with successive decoding in *strong* interference channels.

Corollary 3: In symmetric strong interference channels ($\alpha \geq 1$), $R(\alpha) = \begin{cases} R(2-\alpha), & \forall \alpha \in [1,2] \\ 1, & \forall \alpha \geq 2 \end{cases}$.

We plot $R(\alpha)$, $1 \leq \alpha \leq 2$ in Fig. 10, compared with the information theoretic capacity [10].

2) The Case With a Limited Number of Messages: In this subsection, we find the maximum achievable sum/symmetric rate using successive decoding when there are constraints on the maximum number of messages for the two users, respectively. Clearly, the maximum achievable symmetric rate achieved will be lower than $R(\alpha)$. We start with the following two lemmas, whose proofs are relegated to Appendix A.

Lemma 2: If there exists a segment with an even index s_{2i} $(i \geq 1)$ and s_{2i} does not end at 1, such that

$$f_1(x) = 1, \forall x \in s_{2i}, \text{ or } f_2(x) = 1, \forall x \in s_{2i}$$

[with $f_i(x)$ defined as in (24)] then $R^{\text{sum}} \leq 1$.

Lemma 3: If there exists a segment with an odd index s_{2i-1} $(i \geq 1)$, such that

$$f_1(x) = 0, \forall x \in s_{2i-1}, \text{ or } f_2(x) = 0, \forall x \in s_{2i-1}$$

then $R^{\text{sum}} \leq 1$.

Recall that the optimal scheme (50) requires that, for both users, *all* segments in $\mathcal{G}_2$ are fully inactive, and *all* segments in $\mathcal{G}_1$ are fully active. The above two lemmas show the cost of violating (50): if one of the segments in $\mathcal{G}_2$ becomes fully active for either user (cf. Lemma 2), or one of the segments in $\mathcal{G}_1$ becomes fully inactive for either user (cf. Lemma 3), the resulting sum-rate cannot be greater than 1. We now establish the following theorem.

Theorem 2: Denote by $L_i (i = 1, 2)$ the number of messages used by the ith user. When $\alpha \in (\frac{2n-1}{2n}, \frac{2n+1}{2n+2})$, $(n = 1, 2, \ldots,)$ if $L_1 \leq n$ or $L_2 \leq n$, the maximum achievable sum-rate is 1.

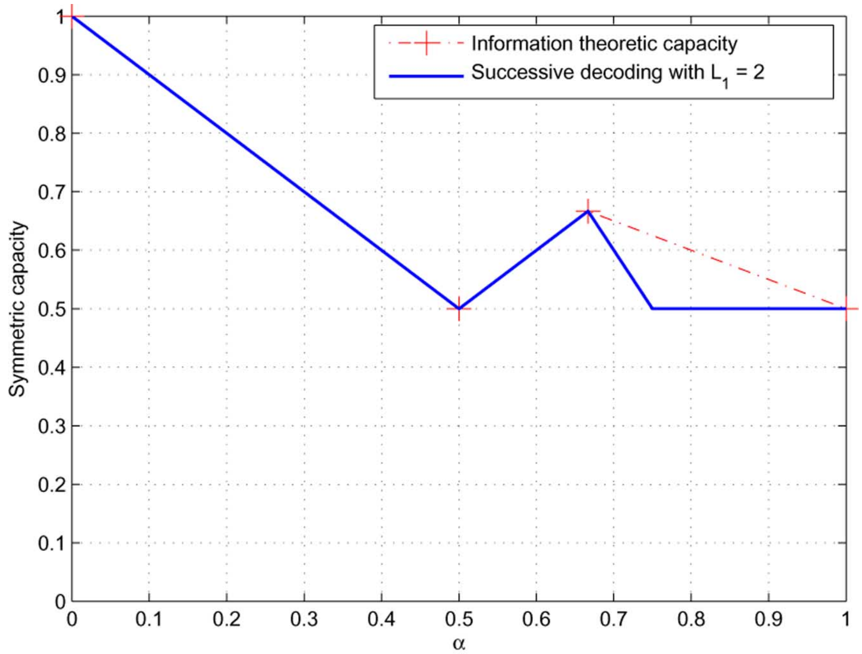
Proof: WLOG, assume that there is a constraint of $L_1 \leq n$.

i) First, the sum-rate of 1 is always achievable with

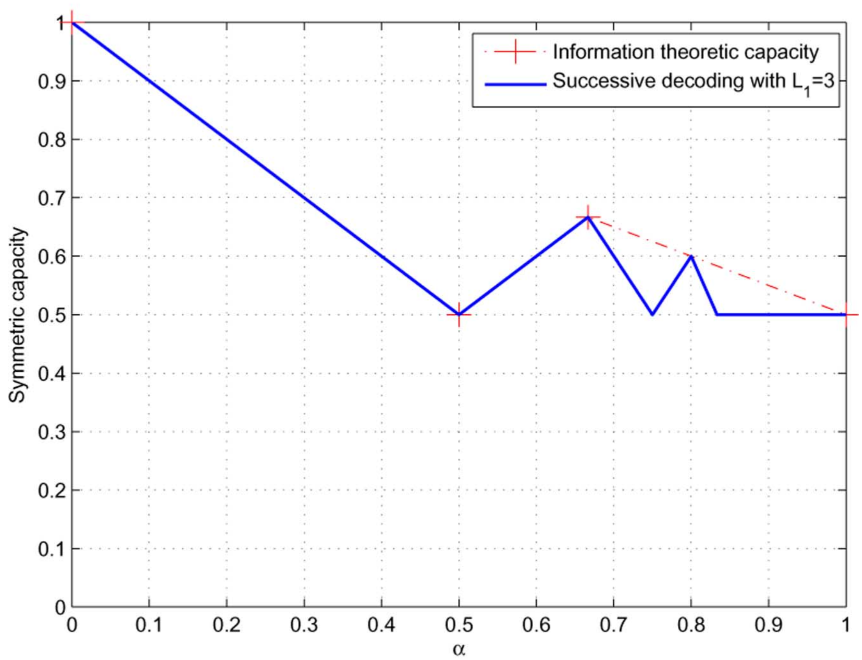
$$f_1(x) = 1, f_2(x) = 0, \forall x \in [0, 1].$$

ii) If there exists s_{2i}, $1 \leq i \leq n$, such that *either* $f_1(x) = 1$, $\forall x \in s_{2i}$, *or* $f_2(x) = 1$, $\forall x \in s_{2i}$, then from Lemma 2, the achieved sum-rate is no greater than 1.

iii) If for all s_{2i}, $1 \leq i \leq n$, there exists x_i in the interior of s_{2i} such that $f_1(x_i) = 0$.



(a)



(b)

Fig. 11. The maximum achievable symmetric rate with a limited number of messages. (a) Maximum achievable symmetric rate with $L_1 \le 2$. (b) Maximum achievable symmetric rate with $L_1 \le 3$.

Note that x_i *separates* the two segments s_{2i-1}, s_{2i+1} for the first user. From Remark 3, s_{2i-1} and s_{2i+1} have to be two distinct messages provided that both of them are (at least partly) active for the first user. On the other hand, there are $n+1$ such segments $\mathcal{G}_1 = \{s_1, s_3, \ldots, s_{2n+1}\}$ (cf. Figs. 8 and 9), whereas the number of messages of the first user is upper bounded by $L_1 \le n$. Consequently, $\exists 1 \le i_1 \le n+1$, such that $f_1(x) = 0$, $\forall x \in s_{2i_1-1}$. In other words, there must be a segment in $\mathcal{G}_1$ that is fully inactive for the first user. By Lemma 3, in this case, the achieved sum-rate is no greater than 1. ∎

Comparing Theorem 2 to Corollary 2, we conclude that if the number of messages used for *either* of the two users is fewer than the number used in the optimal scheme (50) (as in Corollary 2), the maximum achievable symmetric rate drops to $\frac{1}{2}$. This is illustrated in Fig. 11(a) with $L_1 \le 2$ (or $L_2 \le 2$), and in Fig. 11(b) with $L_1 \le 3$ (or $L_2 \le 3$).

Complete solutions (without and with constraints on the number of messages) in *asymmetric* channels follow similar ideas, albeit more tediously. Detailed discussions are relegated to Appendix B.

IV. Approximate Maximum Achievable Sum-Rate With Successive Decoding in Gaussian Interference Channels

In this section, we turn our focus back to the two-user Gaussian interference channel, and consider the sum-rate maximization problem (3). Based on the relation between the deterministic channel model and the Gaussian channel model, we *translate* the optimal solution of the deterministic channel into the Gaussian channel. We then derive upper bounds on the optimal value of (3), and evaluate the achievability of our translation against these upper bounds.

A. Achievable Sum-Rate Motivated by the Optimal Scheme in the Deterministic Channel

As the deterministic channel model can be viewed as an approximation to the Gaussian channel model, optimal schemes of the former suggest approximately optimal schemes of the latter. In this subsection, we show the translation of the optimal scheme of the deterministic channel to that of the Gaussian channel. We show in detail *two forms* (simple and fine) of the translation for symmetric interference channels

$$g_{11}=g_{22}, g_{12}=g_{21}, N_1=N_2, \bar{p}_1=\bar{p}_2=\bar{p}.$$

The translation for asymmetric channels can be derived similarly, albeit more tediously.

1) A Simple Translation of Power Allocation for the Messages: Recall the optimal scheme for symmetric deterministic interference channels (Corollary 1,) as plotted in Fig. 12. $x_i^{(\ell)}$, $\ell=1,\ldots,L$ represent the segments (or *messages* as translated to the Gaussian channel) that are active for the ith user. Recall that

$$-\beta=-(1-\alpha)=n_{21}-n_{11}=\log(\frac{g_{12}}{g_{11}}). \tag{51}$$

Thus, a shift of $|\beta|$ to the right (i.e., lower information levels) in the deterministic channel approximately corresponds to a power scaling factor of $\frac{g_{12}}{g_{11}}$ in the Gaussian channel. Accordingly, a simple translation of the symmetric optimal schemes (cf. Fig. 12) into the Gaussian channel is given as follows.

Algorithm 1: A simple translation by direct power scaling

Step 1: Determine the number of messages $L_1=L_2=L$ for each user as the same number used in the optimal deterministic channel scheme.

Step 2: If $g_{12}\leq g_{11}$,
let $\frac{p^{(2)}}{p^{(1)}}=\frac{p^{(3)}}{p^{(2)}}=\ldots=\frac{p^{(L)}}{p^{(L-1)}}=(\frac{g_{12}}{g_{11}})^2$, and normalize the power by $\sum_{\ell=1}^{L}p^{(\ell)}=\bar{p}$.

If $g_{12}>g_{11}$,
let $\frac{p^{(2)}}{p^{(1)}}=\frac{p^{(3)}}{p^{(2)}}=\ldots=\frac{p^{(L)}}{p^{(L-1)}}=(\frac{g_{11}}{g_{12}})^2$, and normalize the power by $\sum_{\ell=1}^{L}p^{(\ell)}=\bar{p}$.

2) A Finer Translation of Power Allocation for the Messages: In this part, for notational simplicity, we assume WLOG that the noise power $N_1=N_2=1$ and $g_{11}=1$. In the optimal deterministic scheme, the key property that ensures optimality is the following:

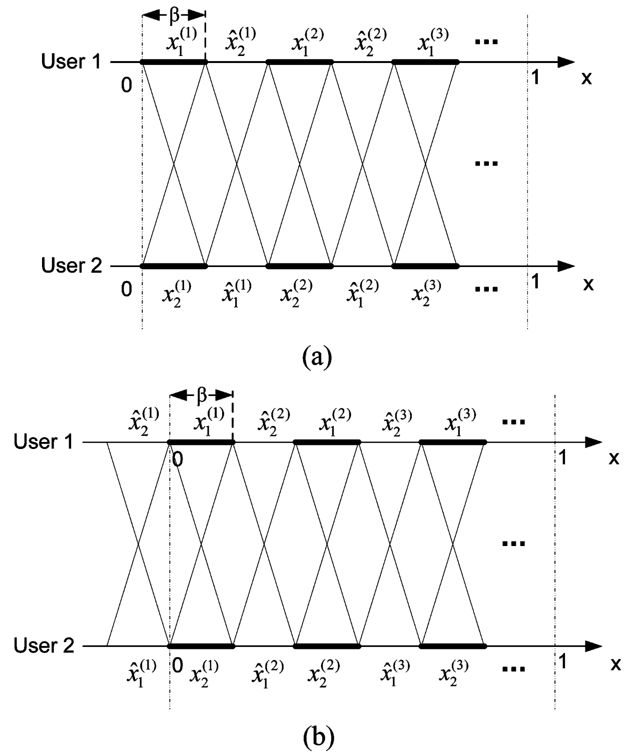


Fig. 12. The optimal schemes in the symmetric deterministic interference channel. (a) Weak interference channel. (b) Strong interference channel.

Corollary 4: A message $M_i^{(\ell)}$ that is decoded at *both* receivers is subject to the *same* achievable rate constraint at both receivers.

For example, in the optimal deterministic schemes (cf. Fig. 12), message $M_1^{(1)}$ is subject to an achievable rate constraint of $|x_1^{(1)}|$ at the first receiver, and that of $|\hat{x}_1^{(1)}|$ at the second receiver, with $|x_1^{(1)}|=|\hat{x}_1^{(1)}|=\beta$. In weak interference channels, $M_1^{(1)},\ldots,M_2^{(L-1)}$ and $M_2^{(1)},\ldots,M_2^{(L-1)}$ are the messages that are decoded at both receivers, whereas $M_1^{(L)}$, $M_2^{(L)}$ are decoded only at their intended receiver (and treated as noise at the other receiver.) In strong interference channels, all messages are decoded at both receivers.

According to Corollary 4, we show that a finer translation of the power allocation for the messages is achieved by equalizing the two rate constraints for every common message. (However, rates of different common messages are not necessarily the same.) In what follows, we present this translation for weak interference channel and strong interference channel, respectively.

Weak Interference Channel, $0\leq g_{12}\leq g_{11}=1$*:* As the first step of determining the power allocations, we give the following lemma on the power allocation of message $M_1^{(1)}$ (with the proof found in Appendix C).

Lemma 4:
1) If $\bar{p}\leq\frac{1-g_{12}}{g_{12}^2}$, then $L=1$, and $x_1^{(1)}(x_2^{(1)})$ is treated as noise at the second (first) receiver, with $p^{(1)}=\bar{p}$. In this

case, there is only one message for each user (as its private message.) Rate constraint equalization is *not* needed.

2) If $\bar{p} > \frac{1-g_{12}}{g_{12}^2}$, then $L \geq 2$, and $M_1^{(1)}(M_2^{(1)})$ are decoded at both receivers. To *equalize* their rate constraints at both receivers, we must have the power allocation as follows:

$$p^{(1)} = 1 - g_{12} + (1 - g_{12}^2)\bar{p} \quad (< \bar{p}). \tag{52}$$

Next, we observe that after decoding $M_1^{(1)}$, $M_2^{(1)}$ at both receivers, determining $p^{(2)}$ for $M_1^{(2)}$, $M_2^{(2)}$ can be transformed to an equivalent first step problem with $\bar{p} \leftarrow \bar{p} - p^{(1)}$: solving the new $p^{(1)}$ of the transformed problem gives the correct equalizing solution for $p^{(2)}$ of the original problem. In general, we have the following recursive algorithm in determining L and $p^{(1)}, \ldots, p^{(L)}$.

Algorithm 2.1, A finer translation by adapting L and the powers using rate constraint equalization; weak interference channel

---

Initialize $L = 1$.

Step 1: If $\bar{p} \leq \frac{1-g_{12}}{g_{12}^2}$, then $p^{(L)} \leftarrow \bar{p}$ and terminate.

Step 2: $p^{(L)} \leftarrow 1 - g_{12} + (1 - g_{12}^2)\bar{p}$.

$L \leftarrow L + 1$. $\bar{p} \leftarrow \bar{p} - p^{(1)}$. Go to Step 1.

Strong Interference Channel, $1 = g_{11} < g_{12}$*:* As the first step of determining the power allocations, we give the following lemma on the power allocation of $M_1^{(1)}$ (with the proof found in Appendix C).

Lemma 5: $M_1^{(1)}$ and $M_2^{(1)}$ are always decoded at both receivers. Moreover,

1) If $\bar{p} \leq g_{12} - 1$, then $L = 1$, and the power allocation of $M_1^{(1)}$ and $M_2^{(1)}$ is $p^{(1)} = \bar{p}$. In this case, there is only one message for each user. Rate constraint equalization is *not* needed.

2) If $\bar{p} > g_{12} - 1$, then $L \geq 2$. To *equalize* the rate constraints of $M_1^{(1)}$ (and $M_2^{(1)}$) at both receivers, we must have the power allocation as follows:

$$p^{(1)} = \frac{g_{12} - 1 + (g_{12}^2 - 1)\bar{p}}{g_{12}^2} \quad (< \bar{p}). \tag{53}$$

Next, we observe that after decoding $M_1^{(1)}$, $M_2^{(1)}$ at both receivers, determining $p^{(2)}$ for $M_1^{(2)}$, $M_2^{(2)}$ can be transformed to an equivalent first step problem with $\bar{p} \leftarrow \bar{p} - p^{(1)}$: solving the new $p^{(1)}$ of the transformed problem gives the correct equalizing solution for $p^{(2)}$ of the original problem. In general, we have the following recursive algorithm in determining L and $p^{(1)}, \ldots, p^{(L)}$.

Algorithm 2.2, A finer translation by adapting L and the powers using rate constraint equalization; strong interference channel

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Initialize $L = 1$.

Step 1: If $\bar{p} \leq g_{12} - 1$, then $p^{(L)} \leftarrow \bar{p}$ and terminate.

Step 2: $p^{(L)} \leftarrow \frac{g_{12}-1+(g_{12}^2-1)\bar{p}}{g_{12}^2}$.

$L \leftarrow L + 1$. $\bar{p} \leftarrow \bar{p} - p^{(1)}$. Go to Step 1.

Numerical evaluations of the above simple and finer translations of the optimal schemes for the deterministic channel into that for the Gaussian channel are provided later in Figs. 13 and 15.

B. Upper Bounds on the Maximum Achievable Sum-Rate With Successive Decoding of Gaussian Codewords

In this subsection, we provide two upper bounds on the optimal solution of (3) for general (asymmetric) weak interference channels. More specifically, the bounds are derived for the maximum achievable sum-rate with Han-Kobayashi schemes, which automatically upper bound that with successive decoding of Gaussian codewords (as shown in Section II-B.) We will observe that, for weak interference channels, the two bounds have complementary efficiencies, i.e., each being tight in a different regime of parameters. For strong interference channels, the information theoretic capacity is known [13], which is achievable by jointly decoding of all the messages from both users.

Similarly to Section II-B, we denote by M_i^p the private message of the ith user, and M_i^c the common message $(i = 1, 2.)$ We denote q_i to be the power allocated to each *private* message M_i^p, $i = 1, 2$. Then, the power of the common message M_i^c equals $\bar{p}_i - q_i$. WLOG, we normalize the channel parameters such that $g_{11} = g_{22} = 1$. Denote the rates of M_i^p and M_i^c by r_i^p and r_i^c. The maximum achievable sum-rate of Gaussian Han-Kobayashi schemes is thus the following:

$$\begin{aligned} \max_{q_1, q_2} \quad & r_1^c + r_1^p + r_2^c + r_2^p \\ s.t. \quad & (4) \sim (14). \end{aligned} \tag{54}$$

To bound (54), we select two mutually exclusive *subsets* of $\{(4), (5), (14)\}$ and $\{(10), (11)\}$. Then, with each subset of the constraints, a relaxed sum-rate maximization problem can be solved, leading to an *upper bound* on the original maximum achievable sum-rate (54).

The first upper bound on the maximum achievable sum-rate is as follows [whose proof is immediate from (4), (5) and (14)].

Lemma 6: The maximum achievable sum-rate using Han-Kobayashi schemes is upper bounded by

$$\begin{aligned} \mathsf{UB}_1 \triangleq \max_{q_1, q_2} \min \Bigg\{ & \log(1 + \frac{\bar{p}_1 + g_{21}(\bar{p}_2 - q_2)}{g_{21}q_2 + N_1}) \\ & + \log(1 + \frac{q_2}{g_{12}q_1 + N_2}) \\ & \log(1 + \frac{\bar{p}_2 + g_{12}(\bar{p}_1 - q_1)}{g_{12}q_1 + N_2}) \\ & + \log(1 + \frac{q_1}{g_{21}q_2 + N_1}) \Bigg\}. \end{aligned} \tag{55}$$

Computation of the Upper Bound (55): Note that

$$\begin{aligned} & \log(1 + \frac{\bar{p}_1 + g_{21}(\bar{p}_2 - q_2)}{g_{21}q_2 + N_1}) + \log(1 + \frac{q_2}{g_{12}q_1 + N_2}) \\ & = \log(c_1) - \log(g_{21}q_2 + N_1) - \log(g_{12}q_1 + N_2) \\ & \quad + \log(g_{12}q_1 + q_2 + N_2) \end{aligned} \tag{56}$$

$$\begin{aligned}&\text{and}\quad \log(1+\frac{\bar{p}_2+g_{12}(\bar{p}_1-q_1)}{g_{12}q_1+N_2})+\log(1+\frac{q_1}{g_{21}q_2+N_1})\\&\quad=\log(c_2)-\log(g_{12}q_1+N_2)-\log(g_{21}q_2+N_1)\\&\qquad+\log(g_{21}q_2+q_1+N_1)\end{aligned}\tag{57}$$

where $c_1 \triangleq N_1+\bar{p}_1+g_{21}\bar{p}_2$, $c_2 \triangleq N_2+\bar{p}_2+g_{12}\bar{p}_1$. Clearly, the minimum of (56) and (57)) is

$$\begin{aligned}&-\log(g_{21}q_2+N_1)-\log(g_{12}q_1+N_2) \qquad (58)\\&+\log\left(\min\{c_1(g_{12}q_1+q_2+N_2),c_2(g_{21}q_2+q_1+N_1)\}\right).\end{aligned}$$

Now, consider the halfspace $(q_1,q_2)\in\mathcal{H}$ defined by the linear constraint

$$\begin{aligned}&c_1(g_{12}q_1+q_2+N_2)\le c_2(g_{21}q_2+q_1+N_1)\\&\Leftrightarrow (c_1g_{12}-c_2)q_1\le(c_2g_{21}-c_1)q_2+c_2N_1-c_1N_2.\end{aligned}\tag{59}$$

In $\mathcal{H}$,

$$\begin{aligned}(58)=&\log(c_1)-\log(g_{21}q_2+N_1)-\log(g_{12}q_1+N_2)\\&+\log(g_{12}q_1+q_2+N_2)\triangleq f(q_1,q_2).\end{aligned}\tag{60}$$

Note that $\frac{\partial f(q_1,q_2)}{\partial q_1}<0, \forall q_1\ge 0$. Thus, depending on the sign of $c_1g_{12}-c_2$, we have the following two cases.

Case 1: $c_1g_{12}-c_2\ge 0$. Then, (59) gives an *upper* bound on q_1. Consequently, to maximize (60), the optimal solution is achieved with $q_1=0$. Thus, maximizing (60) is equivalent to

$$\max_{q_2}\quad -\log(g_{21}q_2+N_1)+\log(q_2+N_2) \tag{61}$$
$$\text{s.t.}\quad 0\le q_2\le\bar{p}_2 \tag{62}$$

in which the objective (61) is *monotonic*, and the solution is either $q_2=0$ or $q_2=\bar{p}_2$.

Case 2: $c_1g_{12}-c_2<0$. Then, (59) gives a *lower* bound on q_1

$$q_1\ge\frac{(c_1-c_2g_{21})q_2+c_1N_2-c_2N_1}{c_2-c_1g_{12}}. \tag{63}$$

Consequently, to maximize (60), the optimal solution is achieved with $q_1=\frac{(c_1-c_2g_{21})q_2+c_1N_2-c_2N_1}{c_2-c_1g_{12}}$, which is a linear function of q_2. Substituting this into (60), we need to solve the following problem:

$$\begin{aligned}&\max_{q_2}\ -\log(a_1q_2+b_1)-\log(a_2q_2+b_2)+\log(a_3q_2+b_3)\\&\text{s.t.}\ \ 0\le q_2\le\bar{p}_2\end{aligned}\tag{64}$$

where a_i, b_i, $(i=1,2,3)$ are constants determined by c_1, c_2, g_{12}, g_{21}, N_1, N_2. Now, (64) can be solved by taking the first derivative w.r.t. q_2, and checking the two stationary points and the two boundary points.

In the other halfspace $\mathcal{H}^c$, the same procedure as above can be applied, and the maximizer of (58) within $\mathcal{H}^c$ can be found. Comparing the two maximizers within $\mathcal{H}$ and $\mathcal{H}^c$, respectively, we get the global maximizer of (55). ∎

The second upper bound on the maximum achievable sum-rate is as follows [whose proof is immediate from (10) and (11)].

Lemma 7: The maximum achievable sum-rate using Han-Kobayashi schemes is upper bounded by

$$\begin{aligned}\mathsf{UB}_2\triangleq\max_{q_1,q_2}&\log\left(1+\frac{q_1+g_{21}(\bar{p}_2-q_2)}{g_{21}q_2+N_1}\right)\\&+\log\left(1+\frac{q_2+g_{12}(\bar{p}_1-q_1)}{g_{12}q_1+N_2}\right).\end{aligned}\tag{65}$$

Computation of the Upper Bound (65): Note that

$$\log\left(1+\frac{q_1+g_{21}(\bar{p}_2-q_2)}{g_{21}q_2+N_1}\right)+\log\left(1+\frac{q_2+g_{12}(\bar{p}_1-q_1)}{g_{12}q_1+N_2}\right)$$
$$=\log(q_1+g_{21}\bar{p}_2+N_1)-\log(g_{12}q_1+N_2) \tag{66}$$
$$+\log(q_2+g_{12}\bar{p}_1+N_2)-\log(g_{21}q_2+N_1) \tag{67}$$

where (66) is a function only of q_1, and (67) is a function only of q_2. Clearly, max (66), $s.t.\ 0\le q_1\le\bar{p}_1$ and max (67), $s.t.\ 0\le q_2\le\bar{p}_2$ can each be solved by taking the first order derivatives, and checking the stationary points and the boundary points. ∎

We combine the two upper bounds (55) and (65) as the following theorem.

Theorem 3: The maximum achievable sum-rate using Gaussian superposition coding-successive decoding is upper bounded by $\min(\mathsf{UB}_1,\mathsf{UB}_2)$.

C. Performance Evaluation

We numerically evaluate our results in symmetric Gaussian interference channels. The SNR is set to be 30 dB. We first evaluate the performance of successive decoding in weak interference channels and then in strong interference channels.

1) Weak Interference Channel: We sweep the parameter range of $\alpha=\frac{\log(\mathrm{INR})}{\log(\mathrm{SNR})}\in[0.5,1]$, as when $\alpha\in[0,0.5]$, the approximate optimal transmission scheme is simply treating interference as noise without successive decoding.

In Fig. 13, the simple translation by Algorithm 1 and the finer translation by Algorithm 2.1 are evaluated, and the two upper bounds derived above (55), (65) are computed. The maximum achievable sum-rate with a single message for each user $(L_1=L_2=1)$ is also computed, and is used as a baseline scheme for comparison.

We make the following observations:

- The finer translation of the optimal deterministic scheme by Algorithm 2.1 is strictly better than the simple translation by Algorithm 1, and is also strictly better than the optimal single message scheme.
- The first upper bound (55) is tighter for higher INR ($\alpha\ge 0.608$ in this example), while the second upper bound (65) is tighter for lower INR ($\alpha<0.608$ in this example).
- A phenomenon similar to that in the deterministic channels appears: the maximum achievable sum-rate with successive decoding of Gaussian codewords oscillates between that with Han-Kobayashi schemes and that with single message schemes.
- The largest difference between the maximum achievable sum-rate of successive decoding and that of single message schemes appears at around $\frac{\log(\mathrm{INR})}{\log(\mathrm{SNR})}=0.64$, which is about 1.8 bits.

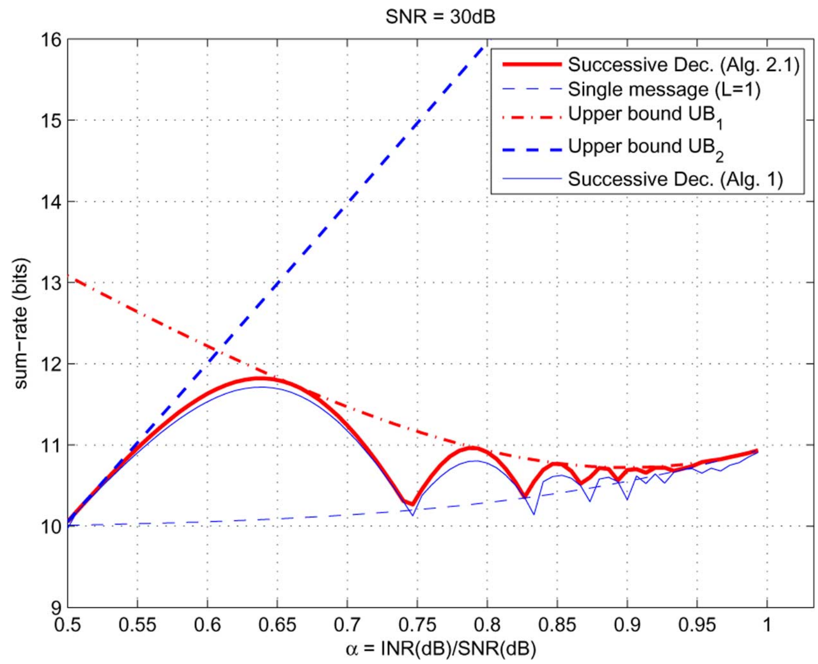

Fig. 13. Performance evaluation in symmetric weak interference channel: achievability versus upper bounds.

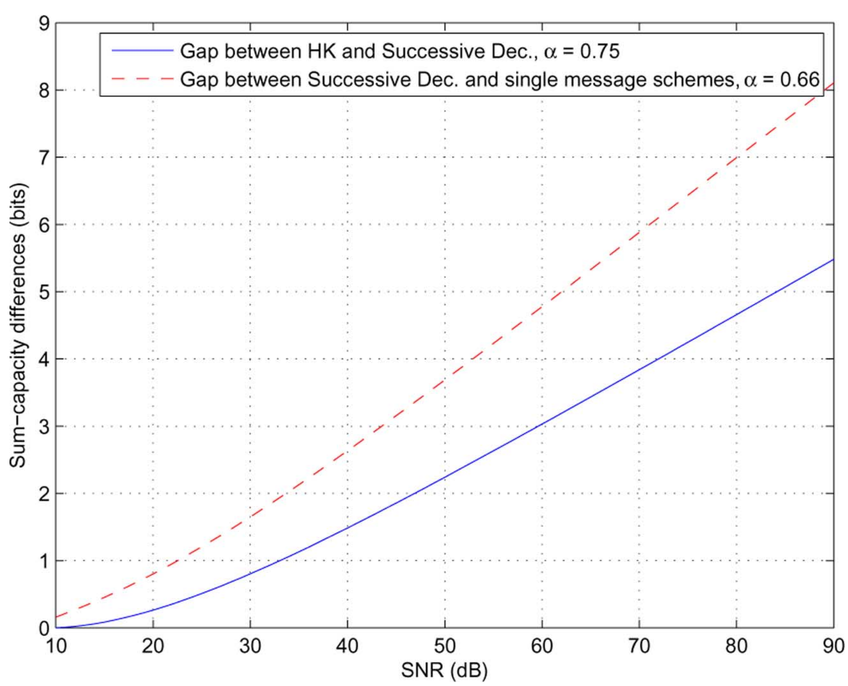

Fig. 14. Maximum achievable sum-rate differences: Han-Kobayashi versus successive decoding at $\alpha = 0.75$, and successive decoding versus the optimal single message scheme at $\alpha = 0.66$.

- The largest difference between the maximum achievable sum-rate of successive decoding and that of joint decoding (Han-Kobayashi schemes) appears at around $\frac{\log(\mathrm{INR})}{\log(\mathrm{SNR})} = 0.74$. This corresponds to the same parameter setting as discussed in Section II-B (cf. Fig. 3). We see that with 30 dB SNR, this largest maximum achievable sum-rate difference is about 1.0 bits.

For this particular case with $\mathrm{SNR} = 30\ \mathrm{dB}$, the observed maximum achievable sum-rate differences (1.8 bits and 1.0 bits) may not seem very large. However, the capacity curves shown with the deterministic channel model (cf. Fig. 6) indicate that these differences can go to infinity as $\mathrm{SNR} \to \infty$. This is because a rate point $d_{\mathrm{sym}}(\alpha)$ on the symmetric capacity curve in the deterministic channel has the following interpretation of generalized degrees of freedom in the Gaussian channel [2], [10]:

$$d_{\mathrm{sym}}(\alpha) = \lim_{\mathrm{SNR},\mathrm{INR}\to\infty, \frac{\log \mathrm{INR}}{\log \mathrm{SNR}}=\alpha} \frac{C_{\mathrm{sym}}(\mathrm{INR},\mathrm{SNR})}{C_{\mathrm{awgn}}(\mathrm{SNR})} \quad (68)$$

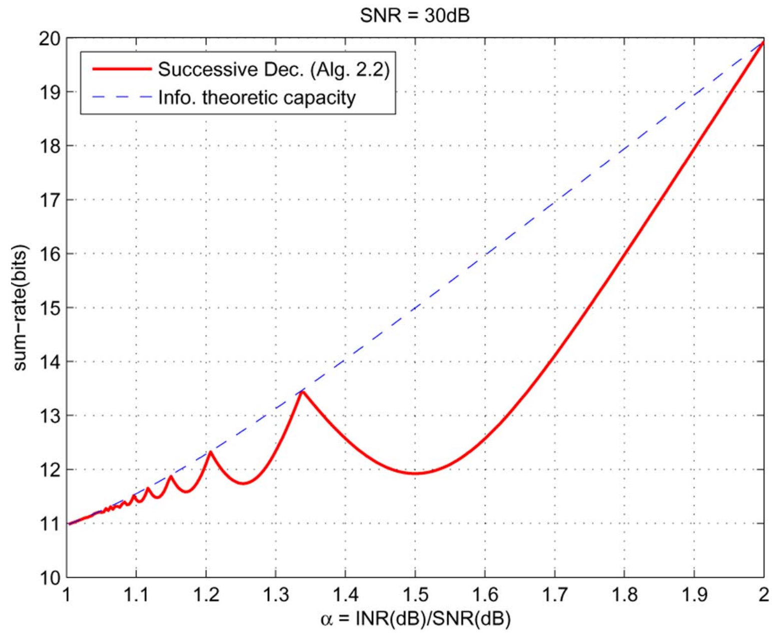

Fig. 15. Performance evaluation in symmetric strong interference channel: successive decoding versus information theoretic capacity.

where $C_{\mathrm{awgn}}(\mathrm{SNR}) = \log(1+\mathrm{SNR})$, and $C_{\mathrm{sym}}(\mathrm{INR},\mathrm{SNR})$ is the symmetric capacity in the two-user symmetric Gaussian channel as a function of INR and SNR.

Since $C_{\mathrm{awgn}}(\mathrm{SNR}) \to \infty$ as $\mathrm{SNR} \to \infty$, for a fixed α, any finite gap of the achievable rates in the deterministic channel indicates a rate gap that goes to infinity as $\mathrm{SNR} \to \infty$ in the Gaussian channel. To illustrate this, we plot the following maximum achievable sum-rate differences in the Gaussian channel, with SNR growing from 10 to 90 dB:

- The maximum achievable sum-rate gap between Gaussian superposition coding-successive decoding schemes and single message schemes, with $\alpha = \frac{\log(\mathrm{INR})}{\log(\mathrm{SNR})} = 0.66$.
- The maximum achievable sum-rate gap between Han-Kobayashi schemes and Gaussian superposition coding—successive decoding schemes, with $\alpha = \frac{\log(\mathrm{INR})}{\log(\mathrm{SNR})} = 0.75$.

As observed, the maximum achievable sum-rate gaps increase asymptotically linearly with $\log \mathrm{SNR}$, and will go to infinity as $\mathrm{SNR} \to \infty$.

2) Strong Interference Channel: We sweep the parameter range of $\alpha = \frac{\log(\mathrm{INR})}{\log(\mathrm{SNR})} \in [1,2]$. As the information theoretic sum-capacity in strong interference channel can be achieved by having each receiver jointly decode all the messages from both users [13], we directly compare the achievable sum-rate using successive decoding with this joint decoding sum-capacity (instead of upper bounds on it). This joint decoding sum-capacity can be computed as follows:

$$\max_{r_1,r_2} r_1 + r_2 \tag{69}$$

$$s.t.\ r_1 \le \log(1+\frac{g_{11}\bar{p}_1}{N_1}),\ r_2 \le \log(1+\frac{g_{22}\bar{p}_2}{N_2})$$

$$r_1 + r_2 \le \log(1+\frac{\bar{p}_1 + g_{21}\bar{p}_2}{N_1})$$

$$r_1 + r_2 \le \log(1+\frac{\bar{p}_2 + g_{12}\bar{p}_1}{N_2}).$$

In Fig. 15, the finer translation by Algorithm 2.2 is evaluated and compared with the information theoretic sum-capacity (69). Interestingly, an oscillation phenomenon similar to that in the deterministic channel case (cf. Fig. 10) is observed.

V. Concluding Remarks and Discussion

In this paper, we studied the problem of sum-rate maximization with Gaussian superposition coding and successive decoding in two-user interference channels. This is a hard problem that involves both a combinatorial optimization of decoding orders and a nonconvex optimization of power allocation. To approach this problem, we used the deterministic channel model as an educated approximation of the Gaussian channel model, and introduced the complementarity conditions that capture the use of successive decoding of Gaussian codewords. We solved the constrained sum-capacity of the deterministic interference channel under the complementarity conditions, and obtained the constrained capacity achieving schemes with the minimum number of messages. We showed that the constrained sum-capacity oscillates as a function of the cross link gain parameters between the information theoretic sum-capacity and the sum-capacity with interference treated as noise. Furthermore, we showed that if the number of messages used by either of the two users is fewer than its minimum capacity achieving number, the maximum achievable sum-rate drops to that with interference treated as noise. Next, we translated the optimal schemes in the deterministic channel to the Gaussian channel using a rate constraint equalization technique, and provided two upper bounds on the maximum achievable sum-rate with Gaussian superposition coding and successive decoding. Numerical evaluations of the translation and the upper bounds showed that the maximum achievable sum-rate with successive decoding of Gaussian codewords oscillates between that with Han-Kobayashi schemes and that with single message schemes.

Next, we discuss some intuitions and generalizations of the coding-decoding assumptions.

A. Complementarity Conditions and Gaussian Codewords

The complementarity conditions (25) and (26) in the deterministic channel model has played a central role that leads to the discovered oscillating constrained sum-capacity (cf. Theorem 1). The intuition behind the complementarity conditions is as follows: At any receiver, if two active levels from different users interfere with each other, then *no* information can be recovered at this level. In other words, the sum of interfering codewords provides nothing helpful.

This is exactly the case when random Gaussian codewords are used in Gaussian channels with successive decoding, because the sum of two codewords from random Gaussian codebooks cannot be decoded as a valid codeword. This is the reason why the usage of Gaussian codewords with successive decoding is translated to complementarity conditions in the deterministic channels. (Note that the preceding discussions do not apply to *joint* decoding of Gaussian codewords as in Han-Kobayashi schemes.)

B. Modulo-2 Additions, Lattice Codes and Feedback

In the deterministic channel, a relaxation on the complementarity conditions is that the *sum* of two interfering active levels can be decoded as their *modulo-2 sum*. As a result, the aggregate of two interfering codewords still provides something valuable that can be exploited to achieve higher capacity. This assumption is part of the original formulation of the deterministic channel model [9], with which the information theoretic capacity of the two-user interference channel (cf. Fig. 6 for the symmetric case) can be achieved with Han-Kobayashi schemes [10].

In Gaussian channels, to achieve an effect similar to decoding the modulo-2 sum with successive decoding, *Lattice* codes are natural candidates of the coding schemes. This is because Lattice codebooks have the group property such that the *sum* of two lattice codewords can still be decoded as a valid codeword. Such intermediate information can be decoded first and exploited later during a successive decoding procedure, in order to increase the achievable rate. For this to succeed in interference channels, alignment of the signal scales becomes essential [14]. However, our preliminary results have shown that the ability to decode the sum of the Lattice codewords does *not* increase the maximum achievable sum-rate for low and medium SNRs. In the above setting of $\mathrm{SNR} = 30$ dB (which is typically considered as a high SNR in practice) numerical computations show that the maximum achievable sum-rate using successive decoding of lattice codewords with alignment of signal scales is *lower* than the previously shown achievable sum-rate using successive decoding of Gaussian codewords (cf. Fig. 13), for the entire range of $\alpha = \frac{\log \mathrm{INR}}{\log \mathrm{SNR}} \in [0.5, 1]$. The reason is that the cost of alignment of the signal scales turns out to be higher than the benefit from it, if SNR is not sufficiently high. In summary, no matter using Gaussian codewords or Lattice codewords, the gap between the achievable rate using successive decoding and that using joint decoding can be significant for typical SNRs in practice.

Recently, the role of feedback in further increasing the information theoretic capacity region has been studied [15], [16]. In these work, the deterministic channel model was also employed as an approximation of the Gaussian channel model, leading to useful insights in the design of near-optimal transmission schemes with feedback. We note that, in deterministic channels, allowing feedback implicitly assumes that modulo-2 sums can be decoded. In Gaussian channels, it remains an interesting open question to find the maximum achievable sum-rate using successive decoding of Lattice codewords with feedback.

C. Symbol Extensions and Asymmetric Complex Signaling

We have focused on two-user complex Gaussian interference channels with constant channel coefficients, and have assumed that symbol extensions are not used, and *circularly symmetric* complex Gaussian distribution is employed in codebook generation. With symbol extensions and *asymmetric* complex signaling [17], the maximum achievable sum-rate using successive decoding can be potentially higher. It has been shown that, in three or more user interference channels, higher sum-degrees of freedom can be achieved by interference alignment if symbol extensions and asymmetric complex signaling are used [17]. In *two-user* interference channels, however, interference alignment is not applicable, and it remains an interesting open question to find the maximum achievable sum-rate with successive decoding considering symbol extensions and asymmetric complex signaling.

Appendix A
Proofs of Lemma 2 and 3

Proof of Lemma 2: By symmetry, it is sufficient to prove for the case $f_2(x) = 1, \forall x \in s_{2i}$, for some s_{2i} that does not end at 1.

Now, consider the sum-rate achieved within C_1 (38). As shown in Fig. 16, C_1 can be partitioned into three parts: $C_{11} = \{f_1(x)|_{s_1,s_3,\ldots,s_{2i-3}}, f_2(x)|_{s_2,s_4,\ldots,s_{2i-2}}\}$, $C_{12} = \{f_1(x)|_{s_{2i-1},s_{2i+1}}, f_2(x)|_{s_{2i}}\}$, and $C_{13} = \{f_1(x)|_{s_{2i+3},\ldots}, f_2(x)|_{s_{2i+2},\ldots}\}$, ($C_{11}$, C_{12}, C_{13} can be degenerate.) Note that

- From the achievable schemes in the proof of Theorem 1, the maximum achievable sum-rate within $C_{11} \cup C_{13}$ can be achieved with $f_2(x) = 1, \forall x \in s_2 \cup s_4 \cup \ldots \cup s_{2i-2} \cup s_{2i+2} \cup \ldots$, and $f_1(x) = 0, \forall x \in s_1 \cup s_3 \cup \ldots \cup s_{2i-3} \cup s_{2i+3} \cup \ldots$.
- By the assumed condition, $f_2(x) = 1, \forall x \in s_{2i} \Rightarrow f_1(x) = 0, \forall x \in s_{2i-1} \cup s_{2i+1}$.

Therefore, under the assumed condition, the maximum achievable sum-rate within C_1 is achievable with $\{f_2(x) = 1, \forall x \in \mathcal{G}_2$, and $f_1(x) = 0, \forall x \in \mathcal{G}_1\}$.

Furthermore, from the proof of Theorem 1, we know that the maximum achievable sum-rate within C_2 is achievable with $\{f_2(x) = 1, \forall x \in \mathcal{G}_1$, and $f_1(x) = 0, \forall x \in \mathcal{G}_2\}$. Combining the maximum achievable schemes within C_1 and C_2, by letting $\{f_2(x) = 1, \forall x \in [0,1]$, and $f_1(x) = 0, \forall x \in [0,1]\}$,

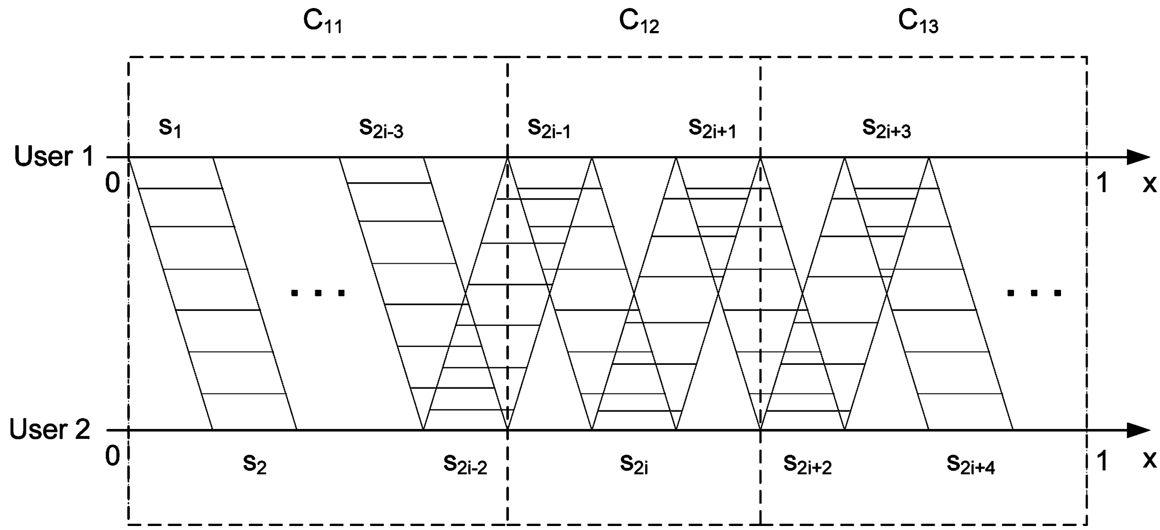

Fig. 16. C_1 partitioned into three parts for Lemma 2.

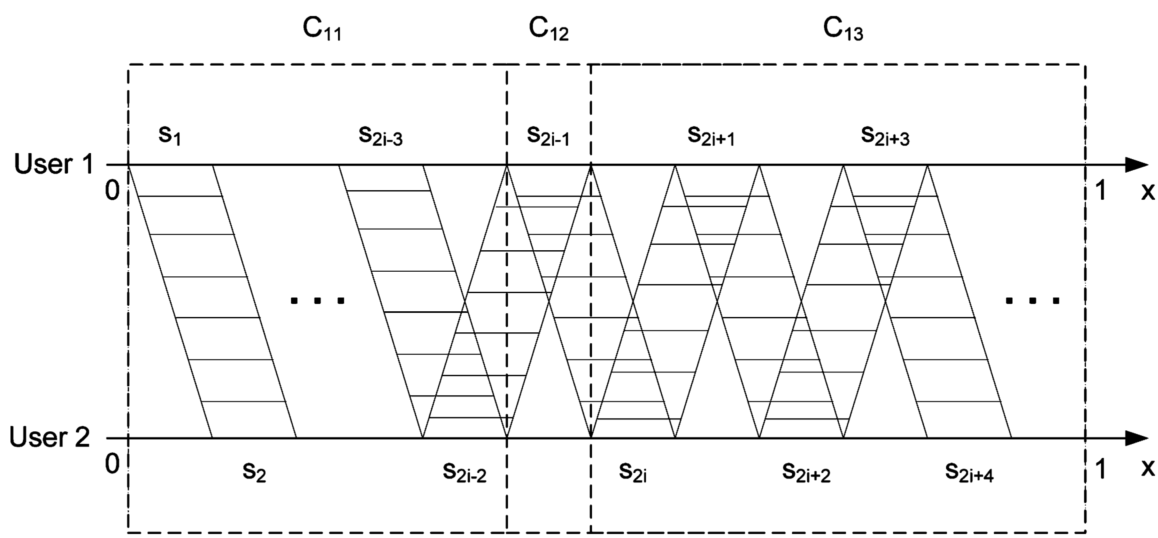

Fig. 17. C_1 partitioned into three parts for Lemma 3.

a sum-rate of 1 is achieved, and this is the maximum achievable sum-rate given the assumed condition. ■

Proof of Lemma 3: By symmetry, it is sufficient to prove for the case $f_1(x) = 0, \forall x \in s_{2i-1}$, for some s_{2i-1}.

Now, consider the sum-rate achieved within C_1. As shown in Fig. 17, C_1 can be partitioned into three parts: $C_{11} = \{f_1(x)|_{s_1,s_3,\ldots,s_{2i-3}}, f_2(x)|_{s_2,s_4,\ldots,s_{2i-2}}\}$, $C_{12} = f_1(x)|_{s_{2i-1}}$, and $C_{13} = \{f_1(x)|_{s_{2i+1},s_{2i+3},\ldots}, f_2(x)|_{s_{2i},s_{2i+2},\ldots}\}$, ($C_{11}$, C_{12}, C_{13} can be degenerate.) Note that:

- From the achievable schemes in the proof of Theorem 1, the maximum achievable sum-rate within $C_{11} \cup C_{13}$ can be achieved with $f_2(x) = 1, \forall x \in s_2 \cup s_4 \cup \ldots \cup s_{2i-2} \cup s_{2i} \cup \ldots$, and $f_1(x) = 0, \forall x \in s_1 \cup s_3 \cup \ldots \cup s_{2i-3} \cup s_{2i+1} \cup \ldots$.
- By the assumed condition, $f_1(x) = 0, \forall x \in s_{2i-1}$.

Therefore, under the assumed condition, the maximum achievable sum-rate within C_1 is achievable with $\{f_2(x) = 1, \forall x \in \mathcal{G}_2,$ and $f_1(x) = 0, \forall x \in \mathcal{G}_1\}$.

Furthermore, from the proof of Theorem 1, we know that the maximum achievable sum-rate within C_2 is achievable with $\{f_2(x) = 1, \forall x \in \mathcal{G}_1,$ and $f_1(x) = 0, \forall x \in \mathcal{G}_2\}$. Combining the maximum achievable schemes within C_1 and C_2, by letting $\{f_2(x) = 1, \forall x \in [0,1],$ and $f_1(x) = 0, \forall x \in [0,1]\}$, a sum-rate 1 is achieved, and this is the maximum achievable sum-rate given the assumed condition. ■

APPENDIX B
SUM-CAPACITY OF DETERMINISTIC ASYMMETRIC INTERFERENCE CHANNELS

In this section, we consider the general two-user interference channel where the parameters n_{11}, n_{22}, n_{12}, n_{21} can be arbitrary. Still, WLOG, we make the assumptions that $n_{11} \geq n_{22}$ and $n_{11} = 1$. We will see that our approaches in the symmetric channel can be similarly extended to solving the constrained

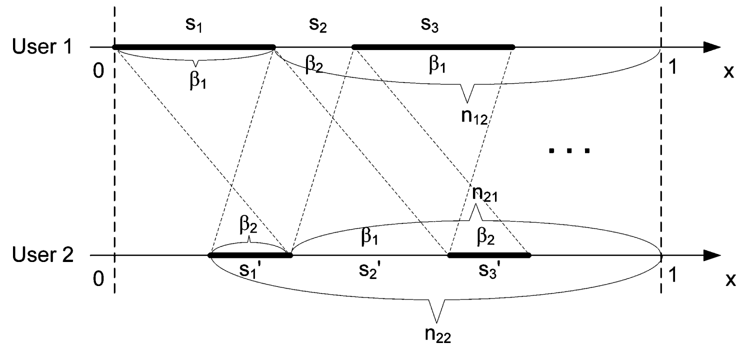

Fig. 18. $n_{11} \geq n_{21}$, $n_{22} \geq n_{12}$, and $n_{22} \geq n_{21}$.

sum-capacity in asymmetric channels, without and with constraints on the number of messages.

From Lemma 1, it is sufficient to consider the following three cases:

$$\begin{aligned} &\text{i) } \delta_1 \geq 0 \text{ and } \delta_2 \geq 0; \\ &\text{ii) } \delta_1 \geq 0 \text{ and } \delta_2 < 0; \\ &\text{iii) } \delta_1 < 0 \text{and} \delta_2 \geq 0. \end{aligned} \tag{70}$$

A. Sum-Capacity Without Constraint on the Number of Messages

We provide the optimal scheme that achieves the constrained sum-capacity in each of the three cases in (70), respectively.

$\delta_1 \geq 0$ and $\delta_2 \geq 0$: This is by definition (23) equivalent to $n_{21} \leq 1$ and $n_{22} \geq n_{12}$.

Case 1, $n_{22} \geq n_{21}$: Define $\beta_1 \triangleq 1 - n_{12}, \beta_2 \triangleq n_{22} - n_{21}$. As depicted in Fig. 18, interval $I_1 (= [0,1])$ is partitioned into segments $\{s_1, s_2, s_3, \ldots\}$, with $|s_1| = |s_3| = \ldots = \beta_1$ and $|s_2| = |s_4| = \ldots = \beta_2$; the last segment ending at 1 has the length of the proper residual. Interval $I_2 (= [1 - n_{22}, 1])$ is partitioned into segments $\{s_1', s_2', s_3' \ldots\}$, with $|s_1'| = |s_3'| = \ldots = \beta_2$ and $|s_2'| = |s_4'| = \ldots = \beta_1$; the last segment ending at 1 has the length of the proper residual.

Similarly to (38) as in the previous analysis for the symmetric channels, we partition the optimization variables $f_1(x)|_{[0,1]}$ and $f_2(x)|_{[1-n_{22},1]}$ into

$$\begin{aligned} &C_1 \triangleq \{f_1(x)|_{s_1,s_3,\ldots}, f_2(x)|_{s_2',s_4',\ldots}\} \\ \text{and } &C_2 \triangleq \{f_1(x)|_{s_2,s_4,\ldots}, f_2(x)|_{s_1',s_3',\ldots}\}. \end{aligned} \tag{71}$$

As there is no constraint between C_1 and C_2 from the complementarity conditions (25) and (26), similarly to (39) and (40), the sum-rate maximization (27) is decoupled into two separate problems

$$\max_{\substack{f_1(x)|_{s_1,s_3,\ldots}, \\ f_2(x)|_{s_2',s_4',\ldots}}} (R_{C_1}^{\text{sum}} =) \int_{s_1,s_3,\ldots} f_1(x)\mathrm{d}x + \int_{s_2',s_4',\ldots} f_2(x)\mathrm{d}x \tag{72}$$

$$\text{subject to } (24), (25), (26)$$

and

$$\max_{\substack{f_1(x)|_{s_2,s_4,\ldots}, \\ f_2(x)|_{s_1',s_3',\ldots}}} (R_{C_2}^{\text{sum}} =) \int_{s_2,s_4,\ldots} f_1(x)\mathrm{d}x + \int_{s_1',s_3',\ldots} f_2(x)\mathrm{d}x \tag{73}$$

$$\text{subject to } (24), (25), (26).$$

By the same argument as in the proof of Theorem 1, the optimal solution of (72) is given by

$$f_1(x) = 1, \forall x \in s_1 \cup s_3 \cup \ldots, \text{ and } f_2(x) = 0, \forall x \in s_2' \cup s_4' \cup \ldots. \tag{74}$$

Also, the optimal solution of (73) is given by

$$\begin{aligned} &f_1(x) = 0, \forall x \in s_2 \cup s_4 \cup \ldots \\ \text{and } &f_2(x) = 1, \forall x \in s_1' \cup s_3' \cup \ldots. \end{aligned} \tag{75}$$

Consequently, we have the following theorem.

Theorem 4: A constrained sum-capacity achieving scheme is given by

$$f_1(x) = \begin{cases} 1, & \forall x \in s_1 \cup s_3 \cup \ldots \\ 0, & otherwise \end{cases}$$
$$\text{and } f_2(x) = \begin{cases} 1, & \forall x \in s_1' \cup s_3' \cup \ldots \\ 0, & otherwise \end{cases} \tag{76}$$

and the maximum achievable sum-rate is readily computable based on (76).

Case 2, $n_{21} > n_{22}$: Define $\beta_1 \triangleq 1 - n_{12} - (n_{21} - n_{22})$. As depicted in Fig. 19, interval $I_1 (= [0,1])$ is partitioned into segments $\{s_0, s_1, s_3, s_5, \ldots\}$, with $|s_0| = n_{21} - n_{22}$, and $|s_1| = |s_3| = \ldots = \beta_1$; the last segment ending at 1 has the length of the proper residual. Interval $I_2 (= [1 - n_{22}, 1])$ is partitioned into segments $\{s_2', s_4' \ldots\}$, with $|s_2'| = |s_4'| = \ldots = \beta_1$; the last segment ending at 1 has the length of the proper residual. (The indexing is not consecutive as we consider $\{s_{2i}\}$ and $\{s_{2i-1}'\} (i \geq 1)$ as degenerating to empty sets.)

Clearly, s_0 of I_1 does not conflict with any levels of I_2, and thus we let $f_1(x) = 1, \forall x \in s_0$. On all the other segments, the sum-rate maximization problem is

$$\max_{\substack{f_1(x)|_{s_1,s_3,\ldots}, \\ f_2(x)|_{s_2',s_4',\ldots}}} \int_{s_1,s_3,\ldots} f_1(x)\mathrm{d}x + \int_{s_2',s_4',\ldots} f_2(x)\mathrm{d}x \tag{77}$$

$$\text{subject to } (24), (25), (26).$$

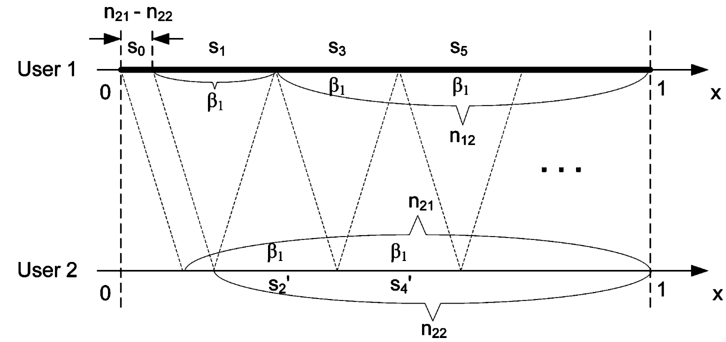

Fig. 19. $n_{11} \geq n_{21}, n_{22} \geq n_{12}$, and $n_{21} > n_{22}$.

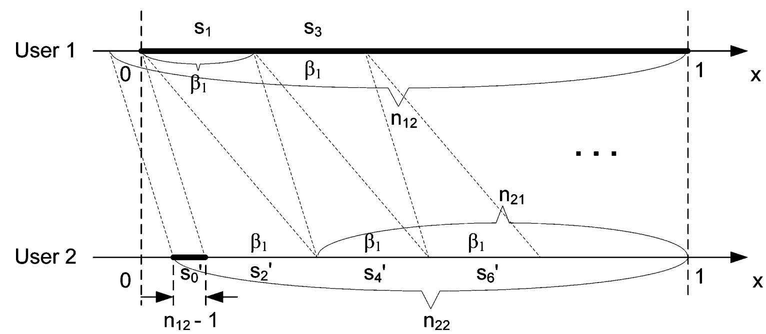

Fig. 20. $n_{11} \geq n_{21}, n_{22} < n_{12}, n_{22} \geq n_{21}$, and $n_{12} > n_{11}$.

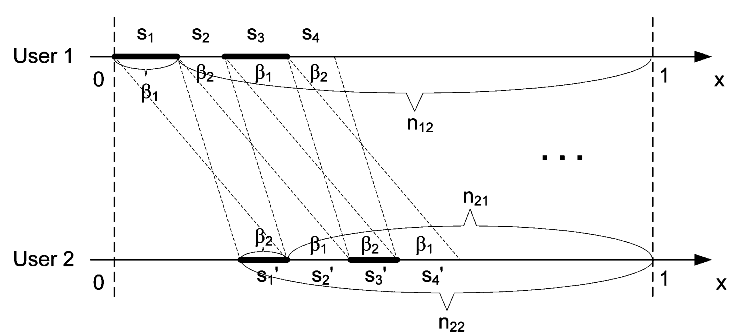

Fig. 21. $n_{11} \geq n_{21}, n_{22} < n_{12}, n_{22} \geq n_{21}$, and $n_{12} \leq n_{11}$, scheme I (nonoptimal).

By the same argument as in the proof of Theorem 1, the optimal solution of (77) is given by

$$f_1(x) = 1, \forall x \in s_1 \cup s_3 \cup \ldots, \text{ and } f_2(x) = 0, \forall x \in s_2' \cup s_4' \cup \ldots.$$

Thus, a sum-capacity achieving scheme is simply $f_1(x) = 1$, $\forall x \in I_1$, and $f_2(x) = 0, \forall x \in I_2$.

$\delta_1 \geq 0$ *and* $\delta_2 < 0$*:*

This is by definition (23) equivalent to $n_{21} \leq 1$ and $n_{22} < n_{12}$. Note that by Lemma 1, it is sufficient to only consider the case where $|\delta_1| \geq |\delta_2|$, (because in case $|\delta_1| < |\delta_2|$, we have $|-\delta_2| > |-\delta_1|$.)

Case 1, $n_{22} \geq n_{21}$*, and* $n_{12} > 1$*:* Define $\beta_1 \triangleq n_{22} - n_{21} - (n_{12} - 1)$. As depicted in Fig. 20, interval $I_1 (= [0,1])$ is partitioned into segments $\{s_1, s_3, \ldots\}$, with $|s_1| = |s_3| = \ldots = \beta_1$; the last segment ending at 1 has the length of the proper residual. Interval $I_2 (= [1 - n_{22}, 1])$ is partitioned into segments $\{s_0', s_2', s_4' \ldots\}$, with $|s_0'| = n_{12} - 1$ and $|s_2'| = |s_4'| = \ldots = \beta_1$; the last segment ending at 1 has the length of the proper residual.

Clearly, s_0' of I_2 does not conflict with any levels of I_1, and thus we let $f_2(x) = 1, \forall x \in s_0'$. On all the other segments, the sum-rate maximization problem is again (77), and the optimal solution is given by

$$f_1(x) = 1, \forall x \in s_1 \cup s_3 \cup \ldots, \text{ and } f_2(x) = 0, \forall x \in s_2' \cup s_4' \cup \ldots.$$

Thus, a sum-capacity achieving scheme is $f_1(x) = 1, \forall x \in I_1$, and $f_2(x) = \begin{cases} 1, & \forall x \in s_0' \\ 0, & otherwise \end{cases}$.

Case 2, $n_{22} \geq n_{21}$*, and* $n_{12} \leq 1$*:*

Define $\beta_1 \triangleq 1 - n_{12}, \beta_2 \triangleq n_{22} - n_{21}$. As depicted in Fig. 21, interval $I_1 (= [0,1])$ is partitioned into segments $\{s_1, s_2, s_3, \ldots\}$, with $|s_1| = |s_3| = \ldots = \beta_1$ and $|s_2| = |s_4| = \ldots = \beta_2$; the last segment ending at 1 has the length of the proper residual. Interval $I_2 (= [1 - n_{22}, 1])$ is partitioned into segments $\{s_1', s_2', s_3' \ldots\}$, with $|s_1'| = |s_3'| = \ldots = \beta_2$ and $|s_2'| = |s_4'| = \ldots = \beta_1$; the last segment ending at 1 has the length of the proper residual.

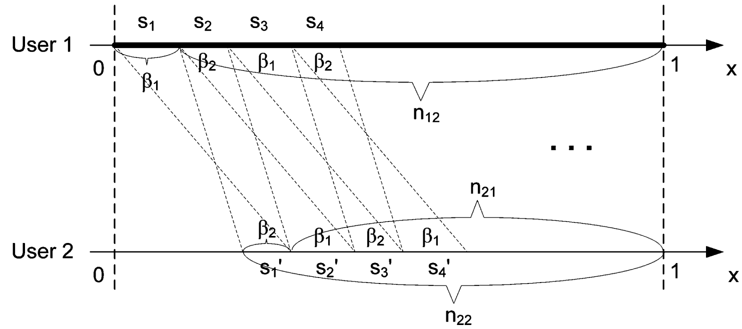

Fig. 22. $n_{11} \geq n_{21}, n_{22} < n_{12}, n_{22} \geq n_{21}$, and $n_{12} \leq n_{11}$, scheme II (optimal).

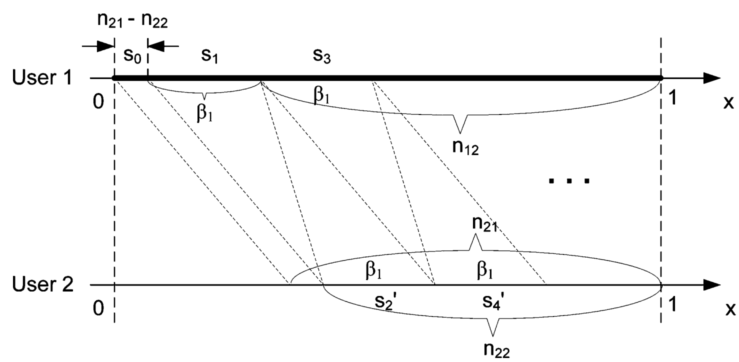

Fig. 23. $n_{11} \geq n_{21}, n_{22} < n_{12}$, and $n_{22} < n_{21}$.

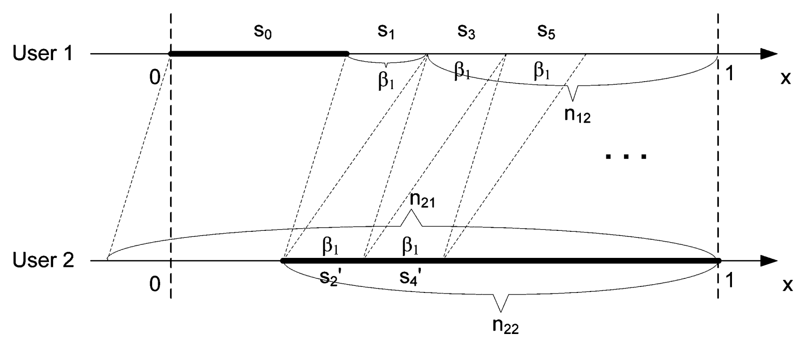

Fig. 24. $n_{11} < n_{21}$ and $n_{22} \geq n_{12}$.

Compare with Case 1 of Appendix B-A0a and note the similarities between Figs. 21 and 18: we apply the same partition of the optimization variables (71), and the sum-rate maximization (27) is decoupled in the same way into two separate problems (72) and (73). However, while the optimal solution of (72) is still given by (74), the optimal solution of (73) is no longer given by (75). Instead, as $\delta_2 < 0$, the optimal solution of (73) is given by

$$f_1(x) = 1, \forall x \in s_2 \cup s_4 \cup \ldots, \text{ and } f_2(x) = 0, \forall x \in s'_1 \cup s'_3 \cup \ldots.$$

Thus, a sum-capacity achieving scheme is given by $f_1(x) = 1$, $\forall x \in I_1$, and $f_2(x) = 0, \forall x \in I_2$, depicted as in Fig. 22.

Case 3, $n_{22} < n_{21}$*:* Comparing with Case 2 of Appendix B-A0a (cf. Fig. 19), with the same definition of β_1 and the same partition of I_1 and I_2, the segmentation is depicted in Fig. 23.

Noting the similarities between Figs. 19 and 23, we see that the optimal solution of the two cases are the same: $f_1(x) = 1$, $\forall x \in I_1$, and $f_2(x) = 0, \forall x \in I_2$.

$\delta_1 < 0$ *and* $\delta_2 \geq 0$*:* This is by (23) equivalent to $n_{21} > 1$ and $n_{22} \geq n_{12}$. Note that by Lemma 1, it is sufficient to only consider the case where $|\delta_1| \leq |\delta_2|$, (because in case $|\delta_1| > |\delta_2|$, we have $|-\delta_2| \leq |-\delta_1|$.)

Define $\beta_1 \triangleq 1 - n_{12} - (n_{21} - n_{22})$. As depicted in Fig. 24, interval $I_1 (= [0, 1])$ is partitioned into segments $\{s_0, s_1, s_3, s_5, \ldots\}$, with $|s_0| = n_{21} - n_{22}$ and $|s_1| = |s_3| = \ldots = \beta_1$; the last segment ending at 1 has the length of the proper residual. Interval $I_2 (= [1 - n_{22}, 1])$ is partitioned into segments $\{s'_2, s'_4 \ldots\}$, with $|s'_2| = |s'_4| = \ldots = \beta_1$; the last segment ending at 1 has the length of the proper residual.

Clearly, s_0 of I_1 does not conflict with any levels of I_2, and thus we let $f_1(x) = 1, \forall x \in s_0$. On all the other segments, the sum-rate maximization problem is again (77). As $\delta_1 < 0$, the optimal solution is given by

$$f_1(x) = 0, \forall x \in s_1 \cup s_3 \cup \ldots, \text{ and } f_2(x) = 1, \forall x \in s_2' \cup s_4' \cup \ldots.$$

Thus, a sum-capacity achieving scheme is $f_1(x) = \begin{cases} 1, & \forall x \in s_0 \\ 0, & otherwise \end{cases}$, and $f_2(x) = 1, \forall x \in I_2$.

Summarizing the discussions of the six parameter settings (cf. Figs. 18–20 and 22–24) in this subsection, we observe:

Remark 6: Except for Case 1 of Appendix B-A0a, the optimal schemes for the other cases all have the property that only one message is used for each user.

The Case With a Limited Number of Messages: In this subsection, we extend the sum-capacity results in Section III-B-II to the asymmetric channels when there are upper bounds on the number of messages L_1, L_2 for the two users, respectively. From Remark 6, we only need to discuss Case 1 of Appendix B-A0a (cf. Fig. 18,) with its corresponding notations.

Similarly to the symmetric channels, we generalize Lemma 2 and 3 to the following two lemmas for the general (asymmetric) channels, whose proofs are exact parallels to those of Lemma 2 and 3.

Lemma 8:

1. If $\exists s_{2i,s_{2i}}$ does not end at 1, such that $f_1(x) = 1, \forall x \in s_{2i}$, then $R^{\text{sum}} \leq 1$.
2. If $\exists s_{2i',s'_{2i}}$ does not end at 1, such that $f_2(x) = 1, \forall x \in s'_{2i}$, then $R^{\text{sum}} \leq n_{22}$.

Lemma 9:

1. If $\exists s_{2i-1}$, such that $f_1(x) = 0, \forall x \in s_{2i-1}$, then $R^{\text{sum}} \leq n_{22}$.
2. If $\exists s'_{2i-1}$, such that $f_2(x) = 0$, $\forall x \in s'_{2i-1}$, then $R^{\text{sum}} \leq 1$.

We then have the following generalization of Theorem 2 to the general (asymmetric) channels.

Theorem 5: Denote by L_i the number of messages used by the ith user in any scheme, and denote by n_i the dictated number of messages used by the ith user in the constrained sum-capacity achieving scheme (76). Then, if $L_1 \leq n_1 - 1$ *or* $L_2 \leq n_2 - 1$, we have $R^{\text{sum}} \leq 1$.

Proof: Consider $L_2 \leq n_2 - 1$. (The case of $L_1 \leq n_1 - 1$ can be proved similarly.)

i) The sum-rate of 1 is always achievable with

$$f_1(x) = 1, \forall x \in I_1, f_2(x) = 0, \forall x \in I_2.$$

ii) If there exists s'_{2i}, $(i \geq 1)$ and s'_{2i} does not end at 1, such that $f_2(x) = 1$, $\forall x \in s'_{2i}$, then from Lemma 8, $R^{\text{sum}} \leq n_{22} \leq 1$.

iii) If for every s'_{2i}, $i \geq 1$ and s'_{2i} does not end at 1, there exists x_i in the interior of s'_{2i} such that $f_2(x_i) = 0$.

For every x_i, since s'_{2i} does not end at 1, s'_{2i+1} exists. Note that x_i *separates* the two segments s'_{2i-1}, s'_{2i+1} for the second user. From Remark 3, s'_{2i-1} and s'_{2i+1} have to be two distinct messages provided that both of them are (at least partly) active for the second user. On the other hand, there are n_2 such segments $\{s'_1, s'_3, \ldots, s'_{2n_2-1}\}$, whereas the number of messages is upper bounded by $L_2 \leq n_2 - 1$. Consequently, $\exists 1 \leq i_2 \leq n_2$, such that $f_2(x) = 0, \forall x \in s_{2i_2-1}$. In other words, for the second user, there must be a segment with an odd index that is *fully inactive*. By Lemma 9, in this case, $R^{\text{sum}} \leq 1$. ■

Similarly to the symmetric case, we conclude that if the number of messages used for *either* user is fewer than the number used in the optimal scheme (76), the maximum achievable sum-rate drops to 1.

Appendix C
Proof of Lemma 4 and 5

Proofs of Lemma 4: At the first receiver, the message $x_1^{(1)}$ is decoded by treating all other messages $x_1^{(2)}, \ldots, x_1^{(L)}, x_2^{(1)}, \ldots, x_2^{(L)}$ as noise, and has an SNR_1 of $\frac{p^{(1)}}{(\bar{p}-p^{(1)})+g_{21}\bar{p}+1}$.

At the second receiver, $x_2^{(1)}$ is first decoded and peeled off. Suppose $x_1^{(1)}$ is also decoded at the second receiver (by treating $x_1^{(2)}, \ldots, x_1^{(L)}$, $x_2^{(2)}, \ldots, x_2^{(L)}$ as noise) it has an SNR_2 of $\frac{g_{12}p^{(1)}}{g_{12}(\bar{p}-p^{(1)})+(\bar{p}-p^{(1)})+1}$. To equalize the rate constraints for $x_1^{(1)}$ at both receivers, we need

$$\text{SNR}_1 = \text{SNR}_2 \Rightarrow p^{(1)} = 1 - g_{12} + (1 - g_{12}^2)\bar{p}.$$

Note that $p^{(1)} < \bar{p}$ requires that $\bar{p} > \frac{1-g_{12}}{g_{12}^2}$. Otherwise, $\bar{p} \leq \frac{1-g_{12}}{g_{12}^2}$, and the above $1-g_{12}+(1-g_{12}^2)\bar{p} \geq \bar{p}$. It implies that we should not decode $x_1^{(1)}$ at the second receiver, i.e., $x_i^{(1)} (i = 1, 2)$ is the only message $(L = 1)$ of the ith user, which is treated as noise at the other receiver. ■

Proof of Lemma 5: At the second receiver, the message $x_1^{(1)}$ is decoded by treating all other messages $x_1^{(2)}, \ldots, x_1^{(L)}, x_2^{(1)}, \ldots, x_2^{(L)}$ as noise, and has an SNR_2 of $\frac{g_{12}p^{(1)}}{g_{12}(\bar{p}-p^{(1)})+\bar{p}+1}$.

At the first receiver, $x_2^{(1)}$ is first decoded and peeled off. Next, $x_1^{(1)}$ is decoded by treating $x_1^{(2)}, \ldots, x_1^{(L)}, x_2^{(2)}, \ldots, x_2^{(L)}$ as noise, and has an SNR_1 of $\frac{p^{(1)}}{(\bar{p}-p^{(1)})+g_{21}(\bar{p}-p^{(1)})+1}$. To equalize the rate constraints for $x_1^{(1)}$ at both receivers, we need

$$\text{SNR}_1 = \text{SNR}_2 \Rightarrow p^{(1)} = \frac{g_{12} - 1 + (g_{12}^2 - 1)\bar{p}}{g_{12}^2}.$$

Note that $p^{(1)} < \bar{p}$ requires that $\bar{p} > g_{12} - 1$. Otherwise, $\bar{p} \leq g_{12} - 1$, and the above $\frac{g_{12}-1+(g_{12}^2-1)\bar{p}}{g_{12}^2} \geq \bar{p}$. It implies that, even if the common message $x_1^{(1)} (x_2^{(1)})$ is allocated with all the power $\bar{p}$, it still has a higher rate constraint at the second (first) receiver than at the first (second) receiver. ■

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