Phasor Diagram of an RC Circuit

\[ V(t) = V_m \sin(\omega t) \]

- Current is a reference in *series* circuit

**KVL:**  
\[ V_m = V_R + V_C \]

![Diagram of RC Circuit with phasors](image)
Phasor Diagram of an *RL* Circuit

\[ V(t) = V_m \sin(\omega t) \quad V_R(t) \]

\[ V_i(t) \quad R \quad L \quad V_o(t) \]

**KVL:** \[ V_m = V_R + V_L \]

\[ V_L \quad \phi \quad V_m \]

\[ V_R \quad I_m \]
Phasor Diagram of a Series $RLC$ Circuit

KVL: $V_m = V_C + V_L + V_R$

- Voltages across capacitor and inductor *compensate each other*
\[
\overline{V_i} = \overline{V_R} + \overline{V_C} \quad \text{or} \quad \left| V_i \right| = \sqrt{\left| V_R \right|^2 + \left| V_C \right|^2}
\]

- Assume high frequency \(1/\omega C \ll R\) then \(V_R \gg V_C\)

\[
V_R \approx V_i
\]

\[
V_o = \frac{1}{C} \int_0^t I_i dt \approx \frac{1}{RC} \int_0^t V_i dt
\]

\[
V_o \propto \int V_i dt
\]

- Accurate integrating can be obtained at high frequency that leads to a low output signal
Differentiating Circuit

\[ V_i(t) \]

\[ V_o(t) \]

\[ V_C \]

\[ V_R \]

\[ C \]

\[ R \]

\[ V_{i} \approx V_{i} \]

\[ V_{c} \approx V_{i} \]

\[ V_{i} \approx \frac{1}{C} \int_{0}^{t} I_{i} dt \Rightarrow I_{i} = C \frac{d}{dt} V_{i} \]

\[ V_{o} = V_{R} = I_{i} R \approx RC \frac{d}{dt} V_{i} \]

\[ V_{o} \propto \frac{d}{dt} V_{i} \]

- Assume *low* frequency \( 1/\omega C >> R \) then \( V_{R} << V_{C} \)

- An accurate result can be achieved at low frequency.
Transient Processes in Passive Circuits

RC Circuit without a Source

- The circuit response is due only to the energy stored in the capacitor

\[ t=0 \]

\[ C \quad R \]

- The capacitor \( C \) is precharged to the voltage \( V_0 \)

*Applying KCL:*

\[ C \frac{dV}{dt} + \frac{V}{R} = 0 \]

- This is the 1\(^{st}\) order differential equation
Solution of the Differential Equation

\[ \frac{dV}{dt} + \frac{V}{RC} = 0 \]

\[ \frac{dV}{V} = -\frac{dt}{RC} \]

\[ \ln V = -\frac{t}{RC} + a \]

\[ V = A e^{-t/RC}, \quad A = e^a \]

- \(e\) is the base of the natural logarithms, \(e = 2.718\ldots\)
- \textit{Continuity} requires the initial condition:

\[ \text{At } t = 0 \quad V(0) = V_0 \]

\[ V(t) = V_0 e^{-t/RC} \]

- The response governed by the elements themselves without external force is called a \textit{natural response}
• The time constant $\tau = RC$ is the rate at which the natural response decays to zero

• Time $\tau$ is required to decay by a factor of $1/e=0.368$

• After the time period of $3\tau$ the transient process is considered to be completed
Determination of the Time Constant

1) From the solution of the equation
By definition: as the time when the signal decreases by 1/e

\[ V(t+\tau)/V(t) = 1/e \]

The time \( t \) can be chosen arbitrarily

2) From the slope of the line at \( t=0 \)
Differentiating the solution

\[ dV/dt = -\{V_0/\tau \} e^{-t/\tau} \]

\[ V_1(t) = -\{V_0/\tau \} t + V_0 \quad \text{V}_1(\tau) = 0 \]
Transient Response of an Integrating RC Circuits

KVL: \( V_i = V_R + V_C \)
Transient Response of a Differentiating RC Circuits

KVL: \( V_i = V_C + V_R \)
Transfer Function

\[ V_i(t) = V_{im} \cos(\omega t) \]

\[ V_o(t) = V_{om} \cos(\omega t + \phi) \]

\[ V_{om}(\omega) \quad \phi(\omega) \]

\[ V_o(t) = \text{Re} \left\{ V_o e^{j\omega t} \right\}, \quad V_o = V_{om} e^{j\phi} \]

\[ T(\omega) = \frac{V_o(\omega)}{V_i} \]

Amplitude response
\[ |T(\omega)| = \frac{V_{om}(\omega)}{V_{im}} \]

Phase response
\[ \angle T(\omega) = \phi(\omega) \]
The Transfer Function of an *Integrating* RC Circuit

\[ \bar{T}(\omega) = \frac{\bar{V}_o(\omega)}{\bar{V}_i} = \frac{1}{j\omega C} \left( \frac{1}{R + \frac{1}{j\omega C}} \right) = \frac{1}{1 + j\omega RC} \]

\[ \bar{T}(\omega) = \frac{1}{1 + j\omega \tau} \quad \tau = RC \]
Amplitude Response of the Integrating RC Circuit

- The magnitude of the transfer function is

\[ |\bar{T}(\omega)| = \frac{1}{\sqrt{1 + \omega^2 \tau^2}} \]

- The Integrating RC circuit is a LOW-PASS filter
Phase Response of the Integrating RC Circuit

The angle of the transfer function is

$$\angle \bar{T}(\omega) = \arctan \frac{\text{Im}[T(\omega)]}{\text{Re}[T(\omega)]}$$

$$\angle \bar{T}(\omega) = \angle \frac{1 - j\omega\tau}{(1 + j\omega\tau)(1 - j\omega\tau)} = -\arctan(\omega\tau)$$
The Transfer Function of a Differentiating RC Circuit

\[ \overline{T}(\omega) = \frac{\overline{V_o}(\omega)}{\overline{V_i}(\omega)} = \frac{\overline{I}(\omega) \cdot R}{\overline{I}(\omega) \cdot X_{RC}(\omega)} \]

\[ \overline{T}(\omega) = \frac{R}{X_{RC}(\omega)} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC} \]

\[ \overline{T}(\omega) = \frac{j\omega \tau (1 - j\omega \tau)}{1 + \omega^2 \tau^2} = \frac{\omega \tau (\omega \tau + j)}{1 + \omega^2 \tau^2} \]

\[ \tau = RC \]
Amplitude response of the Differentiating RC Circuit

- Amplitude response of the RC circuit is the magnitude of $|T(\omega)|$

$$|T(\omega)| = \left| \frac{\omega \tau (\omega \tau + j)}{1 + \omega^2 \tau^2} \right| = \frac{\omega \tau}{\sqrt{1 + \omega^2 \tau^2}}$$

- The Differentiating RC circuit is a HIGH-PASS filter
Phase response of the Differentiating RC Circuit

- Phase response of the RC circuit is the phase angle of $\bar{T}(\omega)$

$$\angle \bar{T}(\omega) = \angle \left\{ \frac{\omega \tau (\omega \tau + j)}{1 + \omega^2 \tau^2} \right\}$$

$$\angle \bar{T}(\omega) = \arctan \frac{1}{\omega \tau}$$
Frequency Response of a Series $RLC$ Circuit

KVL: \[ V_i = V_C + V_L + V_R \]

- Voltages across capacitor and inductor compensate each other
Amplitude Response of a Series RLC Circuit

\[
|V_0/V_{IN}| = \left| \frac{R}{j\omega L + \frac{1}{j\omega C} + R} \right| = \frac{R}{\sqrt{R^2 + \left(\frac{\omega L - \frac{1}{\omega C}}{\omega C}\right)^2}}
\]
Amplitude and Phase Response of the Ladder RC Circuit

\[ V_o = \frac{\left| V_o/V_{in} \right|}{\omega} \]

slope: -40 dB/dec
Amplitude and Phase Response of a Wien-Bridge Circuit

\[ T(\omega) = \frac{Z_{\text{OUT}}}{Z_{\text{TOTAL}}} = \frac{\left( \frac{1}{R} + j\omega C \right)^{-1}}{\left( \frac{1}{R} + j\omega C \right)^{-1} + R + \frac{1}{j\omega C}} \]

\[ |V_o/V_{\text{in}}| \]

\[ V_o = 0.707 \: V_{\text{max}} \]